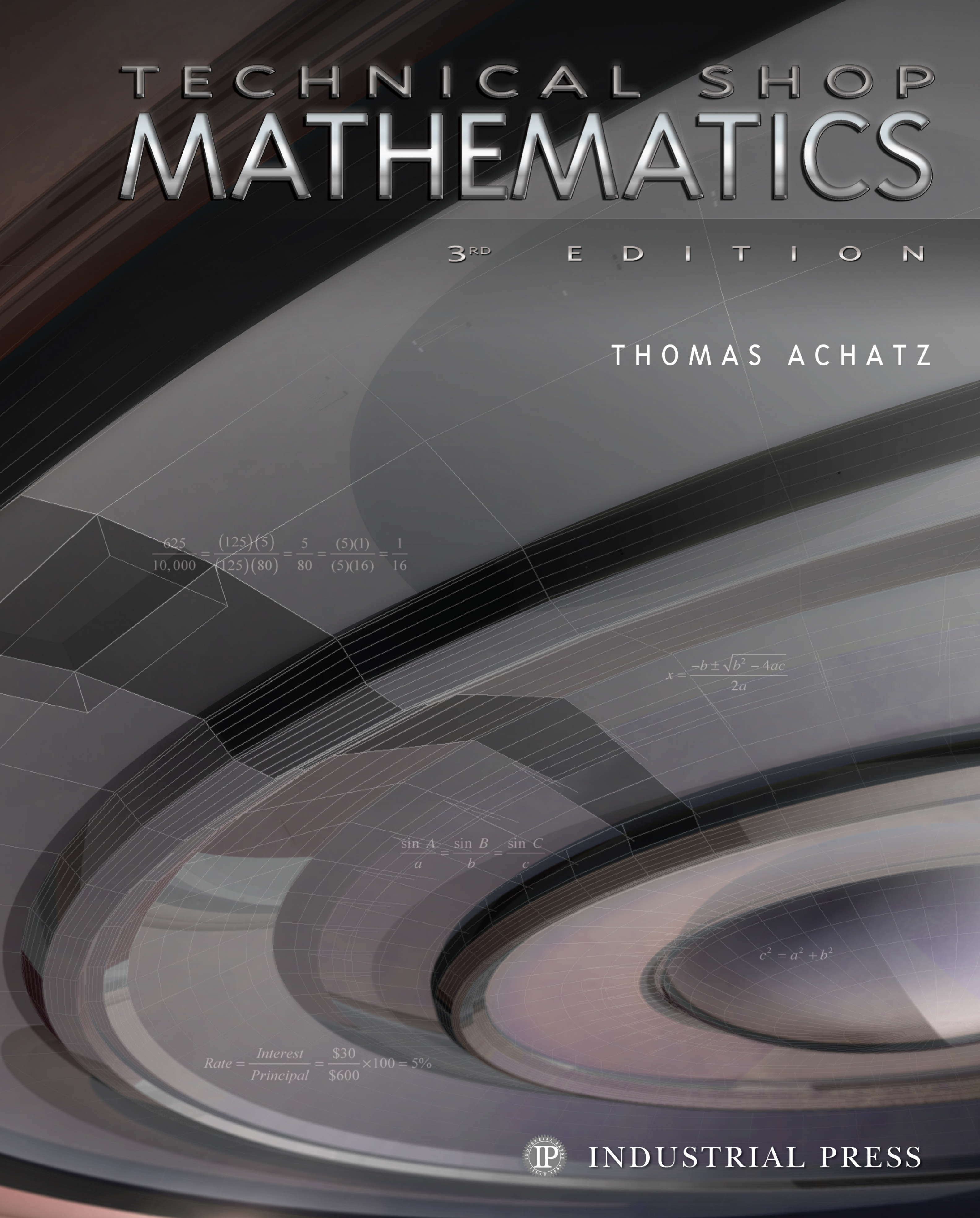
TECHNICAL SHOP MATHEMATICS

3RD EDITION

THOMAS ACHATZ


$\frac{625}{10,000} = \frac{(125)(5)}{(125)(80)} = \frac{5}{80} = \frac{(5)(1)}{(5)(16)} = \frac{1}{16}$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
$c^2 = a^2 + b^2$
$Rate = \frac{Interest}{Principal} = \frac{\$30}{\$600} \times 100 = 5\%$

INDUSTRIAL PRESS

Technical Shop Mathematics

Third Edition

THOMAS ACHATZ, P.E.

WITH

JOHN G. ANDERSON
Contributing Author

KATHLEEN MCKENZIE
Contributing Editor

2005
INDUSTRIAL PRESS INC.
NEW YORK

Library of Congress Cataloging-in-Publication Data

Achatz, Thomas.
Technical shop mathematics / Thomas Achatz and John G. Anderson – 3rd ed.
p. cm.
Previous eds. by John G. Anderson
ISBN 0-8311-3086-5
TJ1165.A56 2004
510'.246--dc22
2004056762

TECHNICAL SHOP MATHEMATICS
Third Edition

INDUSTRIAL PRESS, INC.
989 AVENUE OF THE AMERICAS
NEW YORK, NY 10018

FIRST PRINTING, NOVEMBER 2005
SECOND PRINTING, AUGUST 2007

COPYRIGHT © 1974, © 1988, © 2006 by Industrial Press Inc., New York, NY.

All rights reserved. This book or parts thereof may not be reproduced, stored in a retrieval system, or transmitted in any form without permission of the publishers.

Printed and bound in the United States of America by Edwards Bros. Inc., Ann Arbor, MI

10 9 8 7 6 5 4

To John, Ruth, Diane, and Elizabeth

PREFACE

Technical Shop Mathematics, 3rd edition, is a major revision with many new topics added, old topics updated, illustrations improved, and a larger, cleaner format. The use of two colors offers easier reading and better delineation of key points, while margin notes include historical information, caveats, and other useful references. Building on the strengths of the original editions, the 3rd edition delivers an expanded number of real-world exercises in a consistent manner using straightforward language throughout.

This versatile edition may be used as a classroom textbook, a self-study refresher or a convenient on-the-job reference. Community colleges, high school vocational programs, and trade schools will appreciate the systematic organization of topics that are well suited for a thorough two-semester course or an accelerated one-semester course. For those who are pursuing higher education, this edition serves as an excellent review of fundamental mathematical skills or as a primer for advanced algebra, trigonometry, or calculus. Industry professionals such as machinists, HVAC technicians, mechanics, electricians, surveyors, and others interested in the practical application of mathematics will find that the individual topics are comprehensive and clearly identified, thereby allowing for easy navigation and quick reference.

The fundamental areas of arithmetic, algebra, geometry, and trigonometry are further divided into chapters concentrating on particular topics. This format allows for either a cumulative, sequential approach to learning new subjects, or for use as a reference on specific points of interest. The review of arithmetic includes signed number operations, place values, Roman numerals, fractions, percents, rounding, and measurement systems. The algebra topics build a strong foundation by extending arithmetic to include exponents, logarithms, ratio and proportion, Cartesian coordinates, graphing linear functions, solving equations and word problems, manipulating literal variables, working with radicals, factoring, and finding quadratic roots. Covered within the geometry topics are practical applications of Euclid's axioms, postulates, and theorems while proofs are presented as a motivation for solving problems from a series of reasoned steps. The final chapters provide a structured approach to right angle and oblique trigonometry. Graphing trigonometric functions is emphasized to build mathematical intuition.

The math skills required to solve technical problems are the foundation of critical thinking. Conceptual understanding, practical application, and the ability to adapt and extend underlying principles are far more valuable in the work environment than mere memorization. My hope is that everyone who uses this book, regardless of prior mathematical skills or experiences, gains an increased ability to solve practical mathematics problems, develops an appreciation for the study of technical mathematics, and finds improved career prospects.

ACKNOWLEDGMENTS

This edition of *Technical Shop Mathematics* has been a long time in the making. Many people, far too numerous to mention, have contributed one way or another, directly or indirectly, to this project. They have my gratitude even if their contributions are not explicitly recognized here. Among them are professors from my years at the University of Michigan and Rensselaer Polytechnic Institute, as well as former mathematics and engineering students who have inspired my goals of clear presentation and meaningful application of concepts.

One of the earliest participants in the effort to produce this book was Kathleen McKenzie from Radical X Editing Services who provided constructive criticism, content suggestions, and final proofreading. Countless versions of the manuscript in various formats were rendered by Elena Godina who tirelessly revised chapter layouts for optimal visual appearance. Robert Weinstein provided meticulous and expert copy editing. Many of the illustrations from the second edition were recast electronically by Michigan Technological University engineering student James Kramer. My colleagues, Dennis Bila and James Egan from Washtenaw Community College, and Debi Cohoon from General Motors University, provided encouragement to complete the project and offered many opportunities to develop my presentation style and teaching skills. The attractive cover was designed and produced by William Newhouse, *Split3Studio.com*. Lisa Patishnock, Mary Walker, and Mary Best furnished technical guidance, typing assistance, and cross-referencing services. I am particularly grateful for the timely support from Charlie Achatz, who carefully read the manuscript multiple times, corrected errors, and re-worked all of the exercises. Lastly, special recognition is due to John G. Anderson whose legacy continues through many of the exercises preserved from earlier editions.

The staff at Industrial Press, especially John Carleo, have provided support and encouraging feedback throughout this project. Christopher McCauley, Riccardo Heald, and Janet Romano made the painstaking details of getting from manuscript to finished product achievable and enjoyable.

Notwithstanding the able and dedicated efforts of so many, errors and omissions may nonetheless be present. Please provide suggestions for improvement by visiting the home page for this book at www.industrialpress.com and clicking on the link to “Email the Author.” Your feedback would be greatly appreciated.

Thomas Achatz, PE

TABLE OF CONTENTS

1 THE LANGUAGE OF MATHEMATICS

Symbols — The Alphabet of Mathematics 1
Properties of Real Numbers 5
Real Number Set and Subsets 6
The Multiplication Table 9
Operations in Arithmetic 10
Order of Mixed Operations 14

2 SIGNED NUMBER OPERATIONS

Addition and Subtraction on the Number Line 17
Absolute Value 20
Combining More than Two Numbers through Addition and Subtraction 22
Multiplication and Division 24
Combining All Signed Number Operations 26

3 COMMON FRACTIONS

Common Fractions as Division 27
Converting Improper Fractions and Mixed Numbers 29
Raising a Common Fraction to Higher Terms 31
Reducing a Common Fraction to Lowest Terms 32
Addition and Subtraction of Common Fractions 37
Addition and Subtraction of Mixed Numbers 43
Multiplication and Division of Common Fractions 48
Multiplication and Division of Mixed Numbers 52
Complex Fractions 55

4 DECIMAL FRACTIONS

Meaning of a Decimal Fraction **57**

Converting Common Fractions to Decimal Fractions **59**

Converting Decimal Fractions to Common Fractions **61**

Addition and Subtraction of Decimal Fractions **63**

Multiplication of Decimal Fractions **65**

Division of Decimal Fractions **67**

Place Value and Rounding **69**

Measurement Arithmetic **73**

Decimal Tolerances **76**

5 OPERATIONS WITH PERCENTS

Working with Percents **83**

Solving Percent Problems **90**

Simple Interest **95**

List Price and Discounts **96**

6 EXPONENTS: POWERS AND ROOTS

Powers of Positive and Negative Bases **99**

Exponent Rules Part 1 **101**

Exponent Rules Part 2 **104**

Scientific Notation **108**

Logarithms **111**

7 MEASUREMENT

Systems of Measurement **115**

Measures of Length, Area, and Volume **118**

Angle Measure **126**

Weight and Mass Measure **131**

Measures of Temperature and Heat **132**

Measures of Pressure **135**

Strain **140**

8 ALGEBRAIC EXPRESSIONS

Working with Algebraic Expressions **141**

Operations on Expressions — Exponents **146**

Operations on Expressions — Radicals **149**

Operations on Expressions — Rationalizing the Denominator **151**

Operations on Expressions — Combining Like Terms **152**

9 SOLVING EQUATIONS AND INEQUALITIES IN X

Solving Linear Equations in One Variable **155**

Solving Inequalities in x **163**

10 GRAPHING LINEAR EQUATIONS

The Cartesian Plane **169**

Graphing Points of a Line **172**

The Slope of a Line **175**

Applying Linear Equation Forms to Graphs **180**

11 TRANSFORMING AND SOLVING SHOP FORMULAS

Literal Equations 187
Applications of Literal Equations in Shop Mathematics 190

12 RATIO AND PROPORTION

Statements of Comparison 213
Mixture Proportions 223
Tapers and Other Tooling Calculations Requiring Proportions 225
Variation 233

13 OPERATIONS ON POLYNOMIALS

Expanding Algebraic Expressions 243
Factoring Polynomials 248
Binomial Factors of a Trinomial 251
Special Products 256
Algebraic Fractions 259

14 SOLVING QUADRATIC EQUATIONS

Solving Quadratic Equations of Form x^2 = Constant 271
The Quadratic Formula 276

15 LINES, ANGLES, POLYGONS, AND SOLIDS

Points, Lines, and Planes 279
Polygons 286
Polyhedrons and Other Solid Figures 294

16 PERIMETER, AREA, AND VOLUME

Perimeter 297
Area of a Polygon 301
Surface Area and Volume of a Solid 312

17 AXIOMS, POSTULATES, AND THEOREMS

Axioms and Postulates 321
Theorems About Lines and Angles in a Plane 325

18 TRIANGLES

Special Lines in Triangles 341
Similar Triangles 344
Pythagorean Theorem 353
Congruent Triangles 360
The Projection Formula 365
Hero's Formula 369

19 THE CIRCLE

Definitions 375
Theorems Involving Circles 381

20 TRIGONOMETRY FUNDAMENTALS

Some Key Definitions Used in Trigonometry 401
Solving Sides of Triangles Using Trigonometric Functions 410
Special Triangles and the Unit Circle 427
Graphing the Trigonometric Functions 433

21 OBLIQUE ANGLE TRIGONOMETRY

Solving Oblique Triangles Using Right Triangles 439
Special Laws of Trigonometry 445

22 SHOP TRIGONOMETRY

Sine Bars and Sine Plates **463**
Hole Circle Spacing **468**
Coordinate Distances **470**
Solving Practical Shop Problems **476**
Trigonometric Shop Formulas **485**

A APPENDIX

Greek Letters and Standard Abbreviations **489**
Factors and Prefixes for Decimal Multiples of SI Units **489**
Linear Measure Conversion Factors **490**
Square Measure Conversion Factors **491**
Cubic Measure Conversion Factors **492**
Circular and Angular Measure Conversion Factors **493**
Mass and Weight Conversion Factors **494**
Pressure and Stress Conversion Factors **495**
Energy Conversion Factors **495**
Power Conversion Factors **496**
Heat Conversion Factors **496**
Temperature Conversion Formulas **496**
Gage Block Sets — Inch Sizes **497**
Gage Block Sets — Metric Sizes **498**

B ANSWERS TO SELECTED EXERCISES

Chapter 1 Exercises **499**
Chapter 2 Exercises **501**
Chapter 3 Exercises **502**
Chapter 4 Exercises **506**
Chapter 5 Exercises **509**
Chapter 6 Exercises **510**
Chapter 7 Exercises **512**
Chapter 8 Exercises **515**
Chapter 9 Exercises **517**
Chapter 10 Exercises **519**
Chapter 11 Exercises **524**
Chapter 12 Exercises **525**
Chapter 13 Exercises **527**
Chapter 14 Exercises **531**
Chapter 15 Exercises **533**
Chapter 16 Exercises **534**
Chapter 17 Exercises **537**
Chapter 18 Exercises **538**
Chapter 19 Exercises **539**
Chapter 20 Exercises **540**
Chapter 21 Exercises **551**
Chapter 22 Exercises **552**

INDEX **557**

1

THE LANGUAGE OF MATHEMATICS

Mathematics is a universal language that has evolved over thousands of years. It draws on contributions from every civilization and corner of the world—from the ancient worlds of the Middle East, Greece, and Rome, to India, China, Russia, Africa, and pre-Columbian Mayan culture.

Mathematics is used all over the world to solve problems in economics, engineering, manufacturing, construction, electronics, social science and myriad other disciplines. Through mathematics, people can communicate abstract ideas with each other even though they may speak different languages and may come from different cultures.

The language of mathematics consists of many dialects, or subdisciplines. These include arithmetic, algebra, geometry, trigonometry, and statistics, to name a few. This book concentrates on the rudimentary skills needed to study mathematics and solve practical problems encountered in technical fields.

1.1 Symbols — The Alphabet of Mathematics

As with any language, mathematics has established rules and terminology. These are written with symbols—a sort of mathematical alphabet that is used to construct complicated expressions and convey abstract concepts in a compact, unambiguous form. Unlike the English alphabet, which has twenty-six symbols, the mathematical language has numerous symbols and is not recited in any particular order.

Many common mathematical symbols are listed in Table 1.1. Become familiar with these symbols and refer to them throughout this book.

Greek letters, such as π (pi) in the familiar circle formulas, are sometimes used to represent operations, constants, or variables. Part of the Greek alphabet that is commonly used in mathematics is given in Table 1.1.

Greek letters are often used to represent angles.

TABLE 1.1: Common Mathematical Symbols

Symbol		Symbol	
$+$	Plus (sign of addition), or positive	$\cup$	Union of sets
$-$	Minus (sign of subtraction), or negative	$\subset, \subseteq$	Subset of
$\pm$ ($\mp$)	Plus or minus (minus or plus)	$\varnothing$	Empty set
$\times, \cdot$	Multiplication	$\cap$	Intersection of sets
$\div, /$	Division	α	Alpha
$:$	Is to (ratio or proportion)	λ	Lamda (wavelength)
$=$	Is equal to	μ	Mu (coefficient of friction)
$\neq$	Is *not* equal to	π	Pi (3.1416…)
$\equiv$	Is identical to	Σ	Sigma (sign of summation)
$\approx$	Approximately equals	β	Beta
$\cong$	Is congruent to	$\triangle$	Triangle
$\sim$	Is similar to	sin	Sine
$>$	Is greater than	cos	Cosine
$<$	Is less than	tan	Tangent
$\geq$	Is greater than or equal to	cot	Cotangent
$\leq$	Is less than or equal to	sec	Secant
$\propto$	Varies directly as	csc	Cosecant
∞	Infinity	$\sin^{-1}$	Inverse sine
$\therefore$	Therefore	a_n	a sub n
$\sqrt{\ }$	Square root	a'	a prime
$\sqrt[3]{\ }$	Cube root	a''	a double prime
$\sqrt[n]{\ }$	nth root	a_1	a sub one
i (or j)	Imaginary number ($\sqrt{-1}$)	$\angle$	Angle
a^2	a squared (second power of a)	$\parallel$	Is parallel to
a^3	a cubed (third power of a)	$\perp$	Is perpendicular to
a^n	nth power of a	$^\circ$	Degree (circular or temperature)
$\frac{1}{n}$	Reciprocal value of n	$'$	Minutes or feet
$\lvert n \rvert$	Absolute value of n	$''$	Seconds or inches
$\int$	Integral (in calculus)	$\overline{AB}$	Segment AB
( ), { }, [ ]	Parentheses / Braces / Brackets	$\overrightarrow{AB}$	Ray AB
!	Factorial ($5! = 5 \times 4 \times 3 \times 2 \times 1$)	$\overleftrightarrow{AB}$	Line AB
log	Logarithm (base 10)		
ln	Natural logarithm (base e)		

Roman Numerals

There are many symbols used to represent actions and operations; even numerals themselves are symbols. Roman numerals, for example, are seen on the cornerstones of some public buildings, on clocks and watches, in outlines, in tables of contents, and many other places.

The Roman numeral system is not suited to the work of complex mathematics. Nonetheless, knowledge of the basic Roman numeral system is useful. The basic elements of the Roman numeral system are provided in the sidebar. Notice that these elements are not *digits* in the sense of our familiar Arabic number system, and they do not have *place* value.

In the Roman numeral system, individual elements are combined to build numbers in such a way that when added together, they result in a value. One generally writes the element of highest value first and decreases the value of elements from left to right. For example, 29 is XXVIIII. Of course this can result in very long strings of numbers. To get around this problem and write numbers more compactly, a subtractive rule was devised. The subtractive rule states that when an element of smaller value appears before one of larger value, the individual values are subtracted. For example, 9 can be written as VIIII using the addition rule or as IX using the subtractive rule. Both rules are used in some cases, as in MDCCIX whose Arabic equivalent is 1709.

Other examples of Roman numerals in comparison to their Arabic equivalents are provided in Table 1.2.

TABLE 1.2: Comparison of Arabic and Roman Numerals

Arabic Numerals	Roman Numerals
7	VII
14	XIV
19	XIX
38	XXXVIII
44	XLIV
80	LXXX
99	XCIX
150	CL
627	DCXXVII
1234	MCCXXXIV
2000	MM

Some historians say that the Roman civilization fell because it lacked mathematical science.

Common Roman Numerals
I = 1
II = 2
III =3
IV = 4
V =5
VI = 6
VII = 7
VIII = 8
IX = 9
X = 10
L = 50
C =100
D = 500
M =1000

Years are sometimes written in Roman numerals. For example, the year 2000 is written as MM.

Arabic Numerals

Our number system came to us many centuries ago. We call its symbols Arabic numerals, but they really first came from ancient India where the Hindu people originated them. Arab merchants and traders of the Middle or Dark Ages adopted the Hindu number system to help them in commerce. While people of Western Europe were still struggling with the Latin language and the Roman numerals in their schools and universities, science was waiting for a breakthrough in communications, particularly in mathematics; Arabic numerals saved the day.

dig·it *(dĭj′ĭt)*
A human finger or toe.

Arabic numerals have a great advantage over Roman numerals because they are built on the *base 10 number system*. In this system, the magnitude of a number is based on the place values of its *digits*, so named because fingers are so often used for counting.

The Arabic system has ten distinct digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Alone, the value of each digit is a quantity that can be counted. For any number in the Arabic system, a digit has *place value* equal to a power of 10 as determined by the digit's location in the number.

Consider this simple illustration of how powers of 10 are generated:

$10^0 = 1$	*ones or units*
$10^1 = 10$	*tens*
$10^2 = 10 \times 10 = 100$	*hundreds*
$10^3 = 10 \times 10 \times 10 = 1000$	*thousands*
$10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$	*ten thousands*
$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$	*hundred thousands*
$10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1{,}000{,}000$	*millions*

The small number above and to the right of each 10 is called an *exponent*. It indicates the number of factors of 10 are multiplied to produce a corresponding *power* of 10. Table 1.3 shows these powers of 10 in a grid over the number 52,307.

TABLE 1.3: **Powers of 10**

Ten thousands	Thousands	Hundreds	Tens	Ones or Units
$10^4 = 10{,}000$ 5	$10^3 = 1000$ 2	$10^2 = 100$ 3	$10^1 = 10$ 0	$10^0 = 1$ 7

The 5 has a value of 5 × 10,000, or 50,000. The 2 has a value of 2 × 1000 or 2000. The 3 is really 3 × 100 or 300. The 0 is 0 × 10, and the 7 is 7 × 1. Writing these out in this way is called *expanded notation* of a number.

To read the number 4271 we say "four thousand two hundred seventy-one." We would not say "four two seven one" as a rule unless we were reading certain kinds of numbers, such as a telephone number.

We read large numbers by reading one group at a time: thousands, millions, billions, and so on. Hence, for the number 7,952,024 we say "seven million nine-hundred fifty-two thousand twenty-four." Notice that for the thousands group, we say "nine hundred fifty-two," then the word "thousand." We do *not* say "seven million nine hundred fifty-two thousand *and* twenty-four." The word "and" is reserved for the decimal point. We review how to read decimal numbers in Chapter 4.

EXERCISES

1.1 Referring to Table 1.1 write the indicated symbol.

a) Is greater than
b) Is less than or equal to
c) Square root
d) Is parallel to
e) Alpha
f) Beta
g) Is perpendicular to
h) Angle

1.2 Change the indicated Arabic numeral to a Roman numeral.

a) 3
b) 18
c) 160
d) 133
e) 524
f) 1001
g) 2005
h) 4262

1.3 Change the indicated Roman numeral to an Arabic numeral.

a) IV
b) XXIV
c) CCLI
d) CDXIV
e) CMVI
f) CCCXXXIII
g) DCCVI
h) MMXVII

1.4 Write and say in words the indicated number.

a) 67,000
b) 3,880,131
c) 205,009
d) 27,393
e) 7,123,226
f) 18,323,516
g) 104,362,456
h) 932,418,207

1.2 Properties of Real Numbers

Having established a number system, we note that the structure of mathematics is built upon properties, definitions, and operations of *real numbers*. We begin by stating several properties, or axioms, of the set of real numbers as given in Table 1.4.

An ***axiom*** *is a statement whose truth is accepted without proof.*

For the stated properties, the letters *a, b,* and *c* represent real numbers.

The symbol $\Rightarrow$ means "implies."

TABLE 1.4: Properties of Real Numbers

$a = a$	Reflexive property
$a = b \Rightarrow b = a$	Symmetric property
$a = b,\ b = c \Rightarrow a = c$	Transitive property
$a + b = b + a$	Commutative property of addition
$ab = ba$	Commutative property of multiplication
$(a + b) + c = a + (b + c)$	Associative property of addition
$(ab)c = a(bc)$	Associative property of multiplication
$a(b + c) = ab + ac$ $(a + b)c = ac + bc$	Distributive property of multiplication over addition
$a + 0 = 0 + a = \mathrm{a}$	Additive identity
$a + (-a) = 0$	Additive inverse
$a \cdot 1 = 1 \cdot a = a$	Multiplicative identity
$a \cdot \frac{1}{a} = 1$	Multiplicative inverse
$a \cdot 0 = 0 \cdot a = 0$	Zero property of multiplication

EXERCISES

1.5 Name the property illustrated in the example.

a) $x + 1 = x + 1$

b) $x = 7 \Rightarrow 7 = x$

c) $a = 2,\ 2 = x \Rightarrow a = x$

d) $1448 \times 1 = 1448$

e) $14 = 14$

f) $52 + 0 = 52$

g) $4 + 11 = 11 + 4$

h) $(1)(77) = 77$

i) $(6)(7 \cdot 9) = (6 \cdot 7)(9)$

j) $x(2 + 4) = x \times 2 + x \times 4 = 2x + 4x$

k) $(15)(27) = (27)(15)$

l) $(3 + 12)5 = 3 \times 5 + 12 \times 5$

1.3 Real Number Set and Subsets

Real numbers form a *set*. Basic definitions and properties of sets are necessary for a clear discussion of the real numbers and the various subsets of real numbers.

Definitions Relating to Sets

Set A collection of objects, called *elements*, often related in some obvious way. Usually symbolized by an uppercase letter, for example, set S. The elements are often represented by lowercase letters, such as a, b.

A set within a set. Every set is a subset of itself and the set containing no elements is also a subset of any set. Symbolized by $A \subset S$ or $A \subseteq S$ and read, "A is a subset of S." This means A is contained in S. Another way of saying this: "All elements in A are also in S."	***Subset of a set***
A set whose elements are *all* the members of both sets A and B. Symbolized by $A \cup B$ and read "A union B." Sets are joined in a set union, but no elements are listed more than once.	***Union of sets A and B***
A set whose elements are in *both* A and B. Symbolized by $A \cap B$. The sets overlap, forming a new set containing those elements the intersected sets have in common.	***Intersection of sets A and B***
A relative term meaning the set of all elements from which subsets under consideration are drawn. The set of real numbers is, generally, the universal set in a discussion of real numbers.	***Universal set***
The set having no elements. Certain set intersections produce empty sets (i.e., when the sets intersected have no elements in common, an empty set results). The empty set is symbolized as $\varnothing$ or $\{\ \}$.	***Empty or null set***

The set of real numbers contains various subsets.

EXAMPLE 1.1: ***Union and Intersection of Sets***

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 5, 7, 11\}$
then $A \cup B = \{1, 2, 3, 4, 5, 7, 11\}$ and $A \cap B = \{2, 3, 5\}$.
If $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$
then $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and $A \cap B = \varnothing$ or $\{\ \}$.

If P = {prime numbers} and O = {odd numbers} then $P \cup O$ = {odd numbers and 2}. This is the case because the set of prime numbers is included in the set of odd numbers, with the exception of the number 2, an element not found in the odd numbers.

An integer greater than 1 is called a prime number if its only positive divisors are 1 and itself.

If A = {Democrats in Baltimore} and B = {Republicans in Baltimore}
then $A \cap B = \varnothing$.

Real Number Subsets

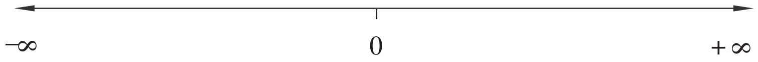
The set of real numbers is the universal set in the discussion that follows. It contains various subsets, all defined below. These include *integers*, which is the set we will use to illustrate the rules of signed number operations.

The set of real numbers, represented by the symbol $\mathbb{R}$, is depicted by the *real number line*. This unbroken line symbolizes the union of the real number subsets, the *rational* and *irrational* numbers. These two subsets of real numbers

A repeating pattern is indicated by three dots, called an ellipsis, or by a bar over the repeating digits.

have no elements in common. Thus, *a real number is either rational or irrational, but it cannot be both,* as the following definitions show:

Real numbers, $\mathbb{R}$ $\mathbb{R}$ = The set of rational numbers and irrational numbers.



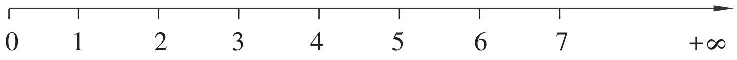
The real numbers make up the real number line.

Natural numbers, $\mathbb{N}$ Natural numbers are the counting numbers.

$\mathbb{N} = \{1, 2, 3, \ldots\}$

Whole numbers, $\mathbb{W}$ Whole numbers are the natural numbers and zero.

$\mathbb{W} = \{0, 1, 2, 3, \ldots\}$

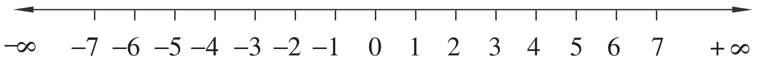


Natural numbers ⊂ Whole numbers

$\mathbb{N} \subset \mathbb{W}$

Integers, $\mathbb{Z}$ Integers are the whole numbers and their negative counterparts.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$



Whole numbers ⊂ Integers

$\mathbb{W} \subset \mathbb{Z}$

Rational numbers, $\mathbb{Q}$ $\mathbb{Q}$ = the set of numbers whose decimal form repeats or terminates. Alternately, the set of numbers that can be represented in fraction form. Note that this includes integers, as they can be written with a denominator of one. Writing an integer this way illustrates its rational form $\frac{n}{1}$.

Examples of rational numbers:

$$\frac{6}{13}, \frac{7}{1}, 0.333..., 5, -8.\overline{341}$$

Irrational numbers, $\mathbb{Q}'$ $\mathbb{Q}'$ = The set of numbers whose decimal form does not repeat or terminate.

The sets of rational and irrational numbers have no element in common.

Examples of irrational numbers:

$$\pi = 3.141592..., \ e = 2.7182..., \ \sqrt{2}, \ -\sqrt{11}$$

An interesting fact to note is that the square root of any number that is not a perfect square is irrational.

The scheme of real number subset inclusion is illustrated in the following diagram; subsets nest in other subsets contained in the universal set, the real numbers.

Natural numbers	$\subset$	Whole numbers	$\subset$	Integers	$\subset$	Rational numbers	$\subset$	Real numbers

$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Rational $\cup$ Irrational = Real numbers

$\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$

Rational $\cap$ Irrational = $\varnothing$

$\mathbb{Q} \cap \mathbb{Q}' = \varnothing$

EXERCISES

1.6 Let A = {Dog, Cat, Bird, Bear} and B = {Dolphin, Goat, Fish}. Find the indicated sets.

a) $A \cup B$ **b)** $A \cap B$

1.7 Let $A = \{0, 3, 12, 131\}$ and $B = \{3, 10, 23, 99\}$. Find the indicated sets.

a) $A \cup B$ **b)** $A \cap B$

1.8 True or false:

a) All rational numbers are integers.
b) All integers are natural numbers.
c) All whole numbers are integers.
d) All irrational numbers are real numbers.
e) The real number line is completely composed of integers and rational numbers.
f) All integers are real numbers, but not all real numbers are integers.

1.4 The Multiplication Table

The best way to start learning technical shop math is to memorize and master the multiplication table given in Table 1.5.

To find the *product* of any two numbers, themselves called *factors* of the product, determine where the column for the first number intersects the row for the second number. Once the multiplication table is committed to memory, any other product of two numbers can be easily calculated. The table can also be used in reverse to learn division.

Make a set of flash cards to help memorize the products in the multiplication table. Use 3 by 5 index cards with the multiplication on one side of the card and the answer on the reverse side. Shuffle the cards and try to answer each multiplication. Check the answer by looking at the back of the card. Practice until the multiplication table becomes second nature.

TABLE 1.5: **Multiplication Table**

	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

EXAMPLE 1.2: ***Using the Multiplication Table***

Using the multiplication table, multiply 7 by 8.

Solution:

STEP 1: Locate 7 along the top row.

STEP 2: Locate 8 down the left side column.

STEP 3: Read answer, 56, at intersection of the 7 column and the 8 row.

EXAMPLE 1.3: ***Using the Multiplication Table for Division***

Divide 24 by 6.

Solution:

STEP 1: Locate 6 along the top row.

STEP 2: Locate 24 down the column identified in Step 1.

STEP 3: Read answer, 4, in the left column along the row identified in Step 2.

EXERCISES

1.9 Use the multiplication table to find the indicated products.

a) 3×4 **b)** 7×6 **c)** 6×7 **d)** 9×2
e) 5×3 **f)** 2×8 **g)** 4×3 **h)** 8×4

1.5 Operations in Arithmetic

Addition and subtraction are operations in the *additive process*. Multiplication and division are closely related to addition and subtraction, and are therefore also part of the additive process.

Addition and Subtraction

When numbers are added, the sequence of addition may be taken in any order. For example, $3 + 2 + 7 + 8 = 20$. Rearranging the sequence does not change the result: $2 + 3 + 8 + 7 = 20$ and $8 + 3 + 7 + 2 = 20$.

EXAMPLE 1.4: ***Addition and Subtraction Operations***

Solve using steps: $90 - 15 + 10 - 5$

Solution:

STEP 1: $90 - 15 = 75$

STEP 2: $75 + 10 = 85$

STEP 3: $85 - 5 = 80$

Series Multiplication and Division

A series of multiplications may be performed in any sequence.

For instance: $2 \times 5 \times 7 \times 3 = 210$; also, $7 \times 2 \times 3 \times 5 = 210$.

However, a series of divisions *must be done in the sequence given:*

For instance: $90 \div 15 \div 3 = ?$

by steps: $90 \div 15 = 6$

$6 \div 3 = 2.$

If the sequence is not followed, an error will be made; for instance:

$15 \div 3 = 5$

$90 \div 5 = 18,$

which is not the correct answer to the original problem.

Short and Long Division

Two methods of division are used in arithmetic. The first, called "short division," is used when the divisor has only one digit. The second, called "long division," is used when the divisor has two or more digits. Examples 1.5 and 1.6 illustrate these methods.

Example 1.5: ***Short Division***

Divide 636 by 6.

Step 1: Determine whether the divisor 6 will divide the first digit of the dividend 636. It will since it is not greater than this digit. The result of division is 1, which is placed under the 6.

$$\begin{array}{r|l} 6 & \underline{636} \\ & 1 \end{array}$$

Step 2: Determine whether the divisor 6 will divide the second digit of the dividend. Since 6 will not divide 3, a zero is placed under the 3.

$$\begin{array}{r|l} 6 & \underline{636} \\ & 10 \end{array}$$

Step 3: The 3 is now taken with the third digit 6 to become 36. The divisor 6 divides 36 and the quotient 6 is placed under the 6. The answer is 106.

$$\begin{array}{r|l} 6 & \underline{636} \\ & 106 \end{array}$$

Example 1.6: ***Long Division***

Divide 6048 by 56.

Solution:

Step 1: Set up the problem by placing a division symbol over the dividend 6048 with the divisor 56 to the left.

$$56\overline{)6048}$$

Step 2: Start from the left of the dividend and find the smallest string of digits that the divisor will divide. In this case, the number is 60. Place the quotient 1 above the division symbol directly over the 0 of 60.

$$\begin{array}{r} 1 \\ 56\overline{)6048} \end{array}$$

Step 3: Multiply the divisor 56 by the quotient 1 and place the answer under the dividend found in Step 2.

$$\begin{array}{r} 1 \\ 56\overline{)6048} \\ 56 \end{array}$$

Step 4: Subtract the product, 56, from the first two digits of the number above it, 60.

$$\begin{array}{r} 1 \\ 56\overline{)6048} \\ \underline{56} \\ 4 \end{array}$$

Step 5: Bring down the next digit in the dividend to form a partial remainder, 44.

$$\begin{array}{r} 1 \\ 56\overline{)6048} \\ \underline{56} \\ 44 \end{array}$$

Step 6: Divide the divisor 56 into the partial remainder 44 and place the quotient above the division symbol. Since 56 cannot divide 44, a zero is placed to the right of the 1.

$$\begin{array}{r} 10 \\ 56\overline{)6048} \\ \underline{56} \\ 44 \end{array}$$

Example 1.6: *Long Division (Continued)*

Step 7: Bring down the next digit in the dividend to form a new partial remainder, 448.

$$\begin{array}{r} 10 \\ 56\overline{)6048} \\ \underline{56} \\ 448 \end{array}$$

Step 8: Divide 56 into the partial remainder 448 and place the quotient above the division symbol.

$$\begin{array}{r} 108 \\ 56\overline{)6048} \\ \underline{56} \\ 448 \end{array}$$

Step 9: Multiply the divisor 56 by the quotient 8 and place the product below the partial remainder.

$$\begin{array}{r} 108 \\ 56\overline{)6048} \\ \underline{56} \\ 448 \\ 448 \end{array}$$

Step 10: Subtract the product, 448, from the previous partial remainder, 448.

$$\begin{array}{r} 108 \\ 56\overline{)6048} \\ \underline{56} \\ 448 \\ \underline{448} \\ 0 \end{array}$$

In many problems the answer may have a *remainder;* that is, the divisor is not a factor of the dividend. The remainder is handled as shown in Example 1.7.

A factor of a number divides the number without a remainder.

EXAMPLE 1.7: ***Division with a Remainder***

Divide 4789 by 25.

The answer is:

$$\begin{array}{r} 191 \\ 25\overline{)4789} \\ \underline{25} \\ 228 \\ \underline{225} \\ 39 \\ \underline{25} \\ 14 \end{array}$$

Since we have used all of the digits given in the problem and are not left with a zero at the bottom, the leftover 14 is the remainder. The final result is written as $191\frac{14}{25}$.

EXERCISES

1.10 Perform the indicated operations.

a) $12 + 18 - 10 - 4$
b) $6 - 1 + 8 - 3$
c) $13 - 4 + 6 - 3$
d) $7 - 2 - 3 + 17$
e) $9 + 3 - 4 + 6$
f) $7 + 3 + 10 - 12$
g) $19 - 10 - 6 - 1$
h) $21 + 4 + 11 - 30$

1.11 Perform the indicated operations.

a) $9 \times 6 \times 2$
b) $4 \times 2 \times 6 \times 8 \times 4$
c) $45 \div 3 \div 5$
d) $11 \times 24 \times 4$
e) $128 \div 4 \div 4 \div 2$
f) $98 \div 7 \div 7$
g) $54 \div 3 \div 6$
h) $8 \times 6 \times 3 \times 7$
i) $6 \times 5 \times 10 \times 2$
j) $75 \div 5 \div 5$

1.12 Perform the indicated division:

a) $390 \div 13$
b) $9134 \div 17$
c) $35{,}000 \div 128$
d) $1000 \div 33$
e) $50{,}412 \div 24$
f) $1357 \div 19$

1.6 Order of Mixed Operations

In most technical mathematics problems several different operations need to be performed. To arrive at the correct answer, the operations must be performed in the proper sequence. The rules for the proper sequence are known as the *order of operations*.

As a simple example of applying the correct order of operations to a problem, consider this: $7 + 2 \times 4 - 6 + 15 \div 3 - 2$.

The correct way to group the operations is $7 + (2 \times 4) - 6 + (15 \div 3) - 2$, wherein the portions in ( ) are done first, working from left to right.

$$\begin{aligned}
7+(2\times4)-6+(15\div3)-2 &= 7+8-6+(15\div3)-2 && \textit{multiplication}\\
&= 7+8-6+5-2 && \textit{division}\\
&= 15-6+5-2 && \textit{addition}\\
&= 9+5-2 && \textit{subtraction}\\
&= 14-2 && \textit{addition}\\
&= 12 && \textit{subtraction}
\end{aligned}$$

Operations are performed in a specific sequence: Multiplication and division are done first, in the order they appear, left to right. Then additiona and subtraction are done in the order they appear, left to right.

This is called the **MDAS** rule for **M**ultiplication, **D**ivision, **A**ddition, **S**ubtraction.

We often combine numbers with several operations somewhat automatically. Usually parentheses are included, but if no order is intended other than **MDAS**, the parentheses may be left out. That is why knowing the order of operations is so important.

When an operational order is intended other than the order provided by the MDAS rule, *grouping symbols* are necessary. Grouping symbols include parentheses, brackets, and braces. Accordingly, we must add "**P**—Parentheses" to the memory device to get **PMDAS**. Operations in parentheses are always taken care of first.

For example, if the previous problem were written with parentheses inserted as shown, the answer would have been different:

$$(7 + 2) \times 4 - 6 + 15 \div (3 - 2) = 9 \times 4 - 6 + 15 \div 1 = 36 - 6 + 15 = 30 + 15 = 45$$

Grouping symbols may be "nested," that is, one set may appear within another set. Often when this happens, other symbols—namely, brackets [ ] and braces { }—further define the order. For example,

$$\begin{aligned}
\{7 - [3 \times (4 - 2)] \div 2\} + 1 &= \{7 - [3 \times 2] \div 2\} + 1\\
&= \{7 - 6 \div 2\} + 1\\
&= \{7 - 3\} + 1\\
&= 4 + 1\\
&= 5
\end{aligned}$$

Finally, we insert "**E**—Exponents" in the phrase, so that if any term inside or outside the parentheses is raised to an exponent, the exponent is taken before the other operations. Exponents are discussed in Chapter 6. The phrase is now **PEMDAS**, which stands for: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, and **S**ubtraction.

*A mnemonic device for remembering the order of operations is the phrase, "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally."*

EXERCISES

1.13 Perform according to the order of operations:

a) $24 \div 3 + 4 \times 5 - 6$

b) $60 - 3 \times 8 + 6 \times 5 - 14 \div 2 + 33 \div 11$

c) $148 - 34 \times 2 - 37$

d) $25 \div 5 + 3 \times 6$

e) $24 \div 8 + 6 \times 4 - 10 \div 2$

f) $5 \times 7 + 6 - 4 \times 7$

g) $32 \div 4 + 8 \times 3 - 5$

h) $25 \div (2 + 3) \times 5$

i) $[(7 \times 2) + 3] \times 2$

j) $6 + [(9 \times 2) + 1] \times 3$

2

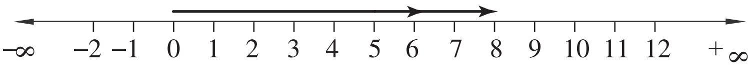
SIGNED NUMBER OPERATIONS

In technical shop mathematics, as well as in every day activities, we encounter all real numbers, *positive* and *negative*, *rational* and *irrational*. Quantities less than zero are everyday occurrences— a temperature of extreme cold, for example, or a negative reading on a voltmeter.

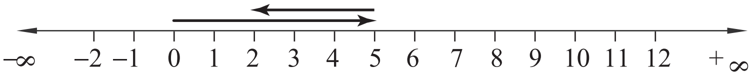
Positive and negative numbers are often referred to as *signed numbers*. When learning how to work with signed numbers it is best to look at examples of *integer* operations and to sketch the problem on a number line. Be aware that anything said about “numbers” or “integers” in this chapter applies to all real numbers, but for simplicity, integers alone will be used to explain the rules of real number operations. Also notice that a sign precedes only negative numbers. Positive signs are implied on positive numbers.

2.1 Addition and Subtraction on the Number Line

Addition of a positive number to another number is shown on the number line by an arrow pointing to the right of the starting number. “Starting number” means the first number in the operation or string of operations. Subtraction of a positive number is represented on the number line by an arrow pointing to the left of the starting number. This familiar convention is used in elementary arithmetic. For example, in the cases of $5 + 3$ and $5 - 3$, the starting number is 5 and the operations are represented on the number line as:



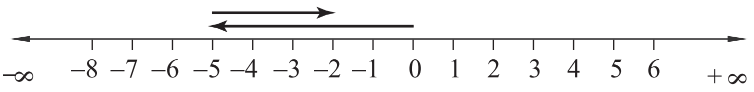
5 + 3 = 8



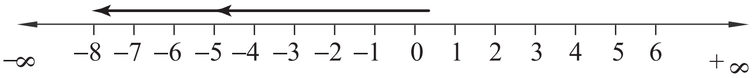
5–3 = 2

The answers are 8 and 2, respectively. In the above examples our starting number was positive. The same principle applies to a negative starting number. For example, $(-5) + 3$ and $(-5) - 3$ can be shown as

(–5) + 3 = –2

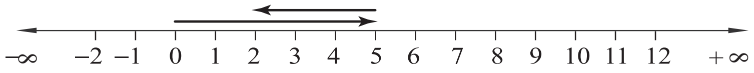


(–5) – 3 = –8



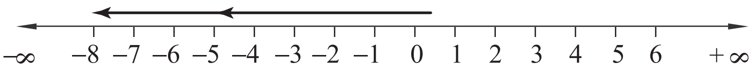
Now look at the next diagram for *addition of negative* numbers.

5 + (– 3) = 2



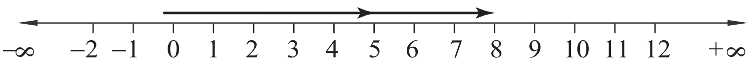
Notice that *adding a negative number is the same as subtracting a positive number*. This is the case whether adding a negative number to a positive number or to another negative number. Here is (–5) + (–3):

–5 + (– 3) = –8

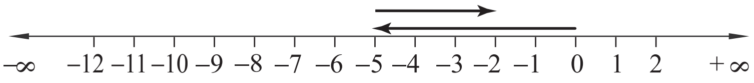


Finally, consider *subtraction of a negative number* from either a positive number or a negative number.

5 – (–3) = 8



–5 – (–3) = –2



Subtracting a negative number is the same as adding a positive number. This is true whether we subtract a negative number from a positive number or from another negative number.

Based on these observations, we state two rules of signed number addition and subtraction.

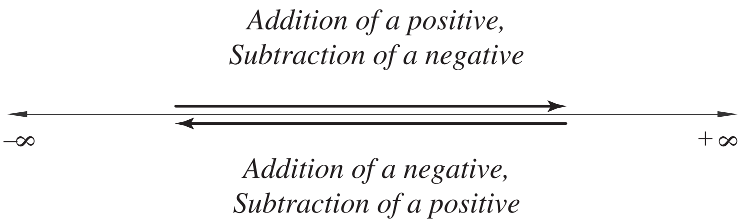
Rule 1

Addition of a positive number is the same as subtraction of a negative number. Both are shown by movement to the right on the number line.

Rule 2

Subtraction of a positive number is the same as addition of a negative number. Both are shown by movement to the left on the number line.

The key to solution of signed number addition and subtraction is to visualize direction of movement on the number line.



Placing parentheses around negative numbers is not necessary; however, they can help us keep the sign of a negative number separate from the operation on the number.

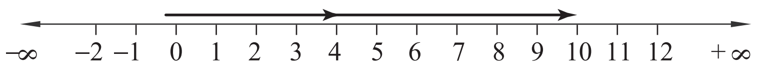
EXAMPLE 2.1: ***Using the Number Line***

Use a number line to illustrate each equation.

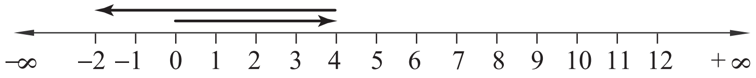
$4 + 6 = 10$	$4 - 6 = -2$	$4 + (-6) = -2$	$4 - (-6) = 10$
$(-4) + 6 = 2$	$(-4) - 6 = -10$	$(-4) + (-6) = -10$	$(-4) - (-6) = 2$

Solution: Sketching the problems on the number lines shows that the equations are correct. Notice that only four answers result by taking all possible combinations of signs and operations on 4 and 6. Problems with identical answers are combined on one number line.

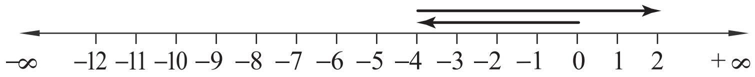
$4 + 6 = 10$ $4 - (-6) = 10$



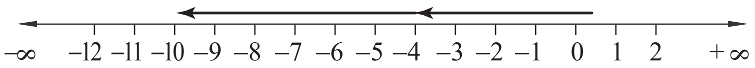
$4 - 6 = -2$ $4 + (-6) = -2$



$(-4) + 6 = 2$ $(-4) - (-6) = 2$



$(-4) - 6 = -10$ $(-4) + (-6) = -10$



An addition operation and a negative sign combine to form a subtraction. ***Rule 3***

A subtraction operation and a negative sign combine to form an addition. ***Rule 4***

One final note about real number addition and subtraction is that any real number and its negative counterpart are *additive inverses*. This means that their sum is zero. Thus, among real number properties we have the property of additive inverse, namely:

Additive inverse

$$a + (-a) = 0$$

where a is any real number. For example, 2 and −2 are additive inverses, as are 141 and −141. Every real number has an additive inverse, except of course zero, which has no sign. Note again that addition of a negative number is equivalent to subtraction of the number's positive counterpart. Symbolically,

$$a + (-a) = a - a = 0$$

So while a and $-a$ are additive inverses, addition and subtraction are *inverse operations*. Informally, this means that one "undoes" what the other does. Inverse operations become very important later in this book, when algebraic equations are solved.

EXERCISES

2.1 Draw number lines to perform the indicated addition or subtraction.

a) $3 + 9$
b) $3 - 9$
c) $-3 + 9$
d) $-3 - 9$
e) $5 + 7$
f) $5 - 7$
g) $-5 + 7$
h) $-5 - 7$

2.2 Find the additive inverse of the indicated number.

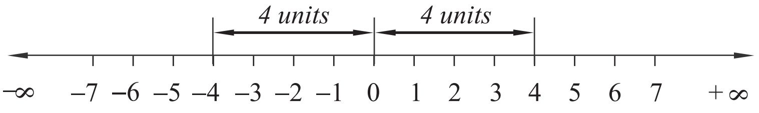
a) -4
b) 6
c) 8
d) -10
e) 0
f) -3
g) 3
h) 7

2.2 Absolute Value

Using a number line is usually not practical. Fortunately, we can generalize the rules so any problem can be solved as if we were working with addition alone! We have already seen the essence of this idea in the preceding examples. To develop the concept further, we need another aspect of integers—*absolute value*.

The absolute value of zero is zero.

Notice on the number line that −4 and 4 are an equal distance from zero, namely, four units.



This "distance from zero" quality of a number is known as *absolute value*. The absolute value of a number is always positive. The absolute value of 4 is 4; the absolute value of −4 is also 4; the absolute value of 10 is 10; the absolute value of −10 is also 10; and so on. The symbol for absolute value is $|\ \ |$. Thus,

$$|\,4\,| = 4 \text{ and } |-4| = 4$$

Roughly speaking, we calculate the absolute value of a number by making its sign positive if it is not already positive.

$$|-8| = 8 \qquad |\,8\,| = 8$$

$$|-0.172| = 0.172 \qquad |0.172| = 0.172$$

Addition and Subtraction Using Absolute Value

We now form general rules of signed number addition and subtraction using the concept of absolute value rather than assigning each kind of operation a direction of movement on the number line.

Rules for signed number addition and subtraction

Rule 1. Adding a positive number to a negative number

To add a positive number and a negative number, subtract the absolute values of the numbers; give the result the sign of the number with the larger absolute value.

Rule 2. Adding two negative numbers

To add two negative numbers, add the absolute values of these numbers; give the result a negative sign.

Rule 3. Subtracting a number from either a positive or negative number

Restate the problem as an addition and continue with either the first or second rule.

EXAMPLE 2.2: ***Adding a Positive Number to a Negative Number***

−94 + 56 = ?

Solution:

The absolute values of −94 and 56 are 94 and 56, respectively. These absolute values are subtracted to give 38 and the answer takes the sign of the number with larger absolute value. It will be negative because −94 has a larger absolute value than 56:

$$-94 + 56 \rightarrow 94 - 56 = 38 \rightarrow -38$$

EXAMPLE 2.3: ***Adding Two Negative Numbers***

(−94) + (−56) = ?

Solution:

The absolute values of −94 and −56 are 94 and 56, respectively. Add 94 to 56 and give the answer a negative sign:

$$-94 + -56 \rightarrow 94 + 56 = 150 \rightarrow -150$$

EXAMPLE 2.4: ***Subtracting a Positive Number from a Number***

$-94 - 56 = ?$

Solution: Use the fact that subtracting a positive is equivalent to adding a negative:

$$-94 - 56 = (-94) + (-56) = -150$$

from the previous example.

EXAMPLE 2.5: ***Subtracting a Negative Number from a Number***

$-94 - (-56) = ?$

Solution: Use the fact that subtracting a negative is equivalent to adding a positive:

$$-94 - (-56) = (-94) + 56 = -38$$

An obvious rule of signed number operations is that subtraction of a positive number from another positive number smaller than itself results in a negative answer. For example, $19 - 23 = -4$.

This kind of problem could be approached by writing it as $19 + (-23)$ then subtracting absolute values to get $23 - 19 = 4$ and finally giving the answer the sign of the number with largest absolute value (-23) to arrive at -4.

EXERCISES

2.3 Find the indicated absolute value.

a) $|-4|$ **b)** $|6|$ **c)** $|8|$ **d)** $|-10|$

e) $|0|$ **f)** $|-3|$ **g)** $|3|$ **h)** $|-7|$

2.4 Perform the indicated addition and subtraction.

a) $2 + (-2)$ **b)** $(-3) - (-3)$ **c)** $(-4) + 6$ **d)** $4 - (6)$

e) $16 - (-12)$ **f)** $(-10) + 2$ **g)** $(-14) + (-7)$ **h)** $(-8) - (-9)$

2.3 Combining More than Two Numbers through Addition and Subtraction

Addition is a *binary operation.* This means that only two numbers are operated upon at once. The same is true of subtraction, multiplication, and division. Although this may seem obvious, it is an essential point of arithmetic theory and when working on complex problems with many numbers involved, it helps to remember this binary nature of addition and subtraction. Consider the following example.

EXAMPLE 2.6: *Combining More than Two Numbers*

$$(-12)+51+(-28)-19-(-32)+7 \ = ?$$

Two methods for solving this problem and ones like it are provided.

Solution Method 1:

In this method and the next we begin by eliminating all instances where a sign and an operation appear next to each other, replacing with one sign according to the rules just learned. This will also eliminate the need for parentheses.

$$(-12)+51+(-28)-19-(-32)+7=-12+51-28-19+32+7$$

By discarding the parentheses all negative signs are viewed as subtraction operations and all positive signs are viewed as addition operations. Problems in combining signed numbers are thus simplified considerably.

Working on two numbers at a time in the order they are encountered, we proceed, using Rules 1, 2, and 3 as needed.

$$-12+51=51-12=39$$

$$39-28=11$$

$$11-19=-8$$

$$-8+32=32-8=24$$

$$24+7=31$$

Solution Method 2:

In this method we again eliminate all instances where a sign and an operation appear next to each other, according to the rules just learned. Thus,

$$(-12)+51+(-28)-19-(-32)+7=-12+51-28-19+32+7$$

Consider the right side of the last equation again:

$$-12+51-28-19+32+7$$

Look at the sign or operation immediately to the left of each number. Starting with minus signs, draw a triangle around sign and number:

$$\boxed{-12}+51\boxed{-28}\boxed{-19}+32+7$$

Add these as negative numbers to get the negative sum, −59.

The sum of the positive numbers remaining is 51 + 32 + 7 = 90.

Combining positive and negative through addition again, we get:

$$90+-59=31$$

EXAMPLE 2.6: ***Combining More than Two Numbers (Continued)***

This method of surrounding like-signed numbers with little triangles is sometimes called the “tortilla chip method,” since these triangles look like tortilla chips.

Check for the presence of additive inverses when combining signed numbers. These can be canceled before any other operation is performed, making the problem easier to solve.

EXAMPLE 2.7: ***Eliminating Additive Inverses***

Simplify $-4+21+(-9)+16+(-21)-(-4)$

Solution: First eliminate parentheses.

$$-4+21+(-9)+16+(-21)-(-4)=-4+21-9+16-21+4$$

Cancel additive inverses: $-\cancel{4}+\cancel{21}-9+16-\cancel{21}+\cancel{4}$

Combine the numbers that remain: –9 + 16 = 7

EXERCISES

2.5 Perform the indicated operations.

a) 25 + 34 – 22 – 7
b) –36 – 58 + 27 + 82
c) 33 – 22 + 25 + 6
d) 45 + 34 – 42 – 32
e) –33 – 22 + 13 + 45
f) 36 + 14 –28 – 8
g) 18 – 12 – 9 – 27
h) –31 + 35 – 79
i) –5 – (–1) + (–2) – 4
j) 20 + (–14) –20 – (–14)

2.4 Multiplication and Division

The rules of operations for multiplication and division of signed real numbers are far simpler than the rules for signed number addition and subtraction.

Rules for Signed Number Multiplication and Division

When two numbers of the same sign are multiplied or divided, the result is positive. When two numbers of opposite sign are multiplied or divided, the result is negative. Thus,

$$\textit{positive} \times \textit{positive} = \textit{positive}$$

$$\textit{positive} \times \textit{negative} = \textit{negative}$$

$$\textit{negative} \times \textit{positive} = \textit{negative}$$

$$\textit{negative} \times \textit{negative} = \textit{positive}$$

EXAMPLE 2.8: *Multiplying Signed Numbers*

$$(7)(5) = 35 \qquad (0.5)(0.03) = 0.015 \qquad \left(\frac{1}{9}\right)\left(\frac{3}{7}\right) = \frac{1}{21}$$

$$(-7)(-5) = 35 \qquad (-0.5)(-0.03) = 0.015 \qquad \left(-\frac{1}{9}\right)\left(-\frac{3}{7}\right) = \frac{1}{21}$$

$$(7)(-5) = -35 \qquad (0.5)(-0.03) = -0.015 \qquad \left(-\frac{1}{9}\right)\left(\frac{3}{7}\right) = -\frac{1}{21}$$

$$(-7)(5) = -35 \qquad (-0.5)(0.03) = -0.015 \qquad \left(\frac{1}{9}\right)\left(-\frac{3}{7}\right) = -\frac{1}{21}$$

When more than two numbers are involved, we work with only two numbers at a time. This is because, like addition and subtraction, multiplication and division are *binary operations*.

EXAMPLE 2.9: *Multiplying Signed Numbers*

$$(3)(-8)(-5)(4)(-2) = (-24)(-5)(4)(-2) = (120)(4)(-2) = 480(-2) = -960$$

Now we consider division. Signed number division follows the same pattern as multiplication.

positive ÷ *positive* = *positive*

positive ÷ *negative* = *negative*

negative ÷ *positive* = *negative*

negative ÷ *negative* = *positive*

Division can also be represented by the fraction bar. We will use this form in the examples.

EXAMPLE 2.10: *Dividing Signed Numbers*

$$\frac{35}{5} = 7 \qquad \frac{35}{-5} = -7 \qquad \frac{-35}{5} = -7 \qquad \frac{-35}{-5} = 7$$

EXERCISES

2.6 Perform the indicated operations.

a) 9 × 12
b) 9 × (–12)
c) (–9) × 12
d) (–108) ÷ 12
e) 108 ÷ 9
f) 15 × 6
g) 15 × (–6)
h) (–15) × (–6)
i) 90 ÷ (–6)
j) 90 ÷ 15
k) 10 × (–3)
l) 10 × 3
m) (–10) × 3
n) 30 ÷ 3
o) 30 ÷ 10

2.5 Combining All Signed Number Operations

Applying the rules for order of operations, we can practice combined operations with signed numbers as we did with positive number operations.

Consider first $90 + (-120) \div 3 \times 10 - 6 \times (-2)$

***PEMDAS**:*
Parentheses
Exponents
Multiplication
Division
Addition
Subtraction

The correct order in which to perform operations is *PEMDAS,* as learned in Chapter 1. The only parentheses here are those around the negative numbers. However, these parentheses are not a grouping device. They simply separate the negative sign from the sign of the operation preceding it. Therefore, we can go first to multiplication and division, followed by addition and subtraction:

$$\begin{aligned} 90 + (-120) \div 3 \times 10 - 6 \times (-2) &= 90 + (-40) \times 10 - 6 \times (-2) \\ &= 90 - 400 + 12 \\ &= -310 + 12 \\ &= -298 \end{aligned}$$

Negative signs and subtraction signs combine to give addition.

EXAMPLE 2.11: ***Combining All Signed Number Operations***

$$\begin{aligned} (-8) \div (-2) + (-6)(-1) - (7 + 31) &= (-8) \div (-2) + (-6)(-1) - 38 \\ &= 4 + 6 - 38 \\ &= 10 - 38 \\ &= -28 \end{aligned}$$

EXERCISES

2.7 Perform the indicated operations

a) $24 \times 2 \div 8$
b) $(-32) \div (-4) + 6$
c) $80 \div 5 \times 2$
d) $14 \times (-3) + 4 - 24 \div 6 + 5 - (-7)$
e) $(-6) \times (-8) \times (-3)$
f) $57 \div 3 - 5 \times (-5)$
g) $8 \times 3 - 16 \div 4 + 5$
h) $32 \div 4 + 7 \times 3 - (-4)$
i) $12 \times 3 \div (4 + 2) - 3 \times 2$
j) $25 \div 5 + 3 \times 4 - (-2)$
k) $36 - 14 + 12 \times 3 \div (-4)$
l) $22 - 11 + 12 \div 4 - 18 \times 2$

3

COMMON FRACTIONS

Familiarity with fractions is essential for success in any trade. This chapter reviews how to work with fractions.

3.1 Common Fractions as Division

A common fraction represents *division* of one number (the numerator) by another number (the *denominator*). For example, a number represented by the letter *a* divided by another number represented by the letter *b*, is written mathematically as:

$$\frac{a}{b} \equiv \frac{\textit{numerator}}{\textit{denominator}}$$

The denominator of a common fraction designates the number of equal-sized parts into which some complete entity—one whole—is split or divided. The numerator of a fraction designates the number of parts being considered.

Examples of common fractions are:

$$\frac{2}{3}, \frac{11}{16}, \frac{5}{6}, \frac{21}{7}, \text{ and } \frac{1}{8}$$

Fractions are sometimes written using a slash (/) rather than a horizontal bar. This style should be avoided in technical work so that fractions are always seen in terms of division, hence and the order of operations is clear.

Technically, the fraction bar indicates a division operation wherein the numerator (top number) is divided by the denominator (bottom number). Although division is not actually performed every time a fraction is encountered, thinking of a fraction as a division problem is helpful when converting a common fraction to its decimal or percent form.

Say "topple it over" to help remember the order of division in a fraction.

$\frac{a}{b} = a \div b$ or $b\overline{)a}$

The fraction $\frac{2}{3}$ indicates that 2 parts of an entity split into 3 equal parts is under consideration. For example, the first circle in Figure 3.1 is split into three equal sized pieces. In the second circle, 2 of the 3 pieces are shaded. We could say that the circle is $\frac{2}{3}$ shaded. If the circle represents an apple pie we have been eating, the shaded region may mean that $\frac{2}{3}$, or two pieces of a pie that had been cut into three equal pieces, were eaten.

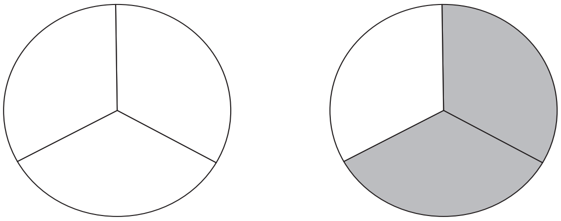


FIGURE 3.1

Proper and Improper Fractions

Fractions can be classified as *proper fractions* or *improper fractions.* A fraction with value less than 1 is called a *proper* fraction. Its numerator is smaller than its denominator. Mathematically,

*In a proper fraction, when **a** is divided by **b**, the result is a decimal number less than 1.*

$$\frac{a}{b} < 1 \text{ because } a < b$$

A fraction whose value is greater than one is called an *improper* fraction. Its numerator is larger than its denominator. Mathematically,

*In an improper fraction, when **a** is divided by **b**, the result is a decimal number greater than 1.*

$$\frac{a}{b} > 1 \text{ because } a > b$$

A fraction whose numerator and denominator are equal signifies *unity*, the number one. For example, $\frac{7}{7}$ or $\frac{a}{a}$ are cases of *unity*, because the result would equal 1 if the actual division were carried out. Note that we do not have to know the value of *a* since anything divided by itself must equal 1. However, *a* may not be zero.

TABLE 3.1: **Examples of Proper, Improper, and Unity Fractions**

Proper fractions				Improper fractions			Unity fractions		
$\frac{6}{7}$	$\frac{3}{5}$	$\frac{233}{654}$	$\frac{6}{24}$	$\frac{53}{4}$	$\frac{4}{3}$	$\frac{284}{25}$	$\frac{8}{8}$	$\frac{65}{65}$	$\frac{999}{999}$

Mixed Numbers

A *mixed number* consists of a whole number and a proper fraction. The implied operation between the two parts of a mixed number is addition. For example, $1\frac{2}{3}$ means $1 + \frac{2}{3}$. Since improper fractions are greater than one, they are often changed to mixed numbers.

EXERCISES

3.1 Draw shaded circles to illustrate the indicated fraction.

a) $\frac{3}{8}$ **b)** $\frac{4}{16}$ **c)** $\frac{2}{2}$ **d)** $\frac{3}{12}$

e) $\frac{9}{16}$ **f)** $\frac{2}{32}$ **g)** $\frac{3}{6}$ **h)** $\frac{6}{8}$

3.2 State whether the indicated fraction is a proper fraction, improper fraction, or mixed number.

a) $\frac{10}{9}$ **b)** $\frac{9}{10}$ **c)** $1\frac{1}{10}$ **d)** $\frac{2}{3}$

e) $\frac{3}{2}$ **f)** $\frac{7}{16}$ **g)** $4\frac{2}{3}$ **h)** $\frac{12}{7}$

3.2 Converting Improper Fractions and Mixed Numbers

Converting an Improper Fraction to a Mixed Number

An improper fraction is converted to a mixed number by dividing the numerator by the denominator and retaining the remainder in a proper fraction. For example, the improper fraction $\frac{7}{2}$ is converted to a mixed number as follows:

$$\frac{7}{2} = 7 \div 2 = 3 \text{ R } 1, \text{which equals } 3\frac{1}{2}$$

R stands for "remainder."

Notice that the remainder is written in the numerator of the fraction part of the final answer. The denominator is unchanged.

EXAMPLE 3.1A: ***Converting an Improper Fraction to a Mixed Number***

Change $\frac{53}{8}$ to a mixed number.

Write as a long division: $8\overline{)53}$

Divide 53 by 8:
$$\begin{array}{r} 6 \\ 8\overline{)\ 53} \\ -48 \\ \hline 5 \end{array}$$

Write answer as whole number and a fraction: $6\frac{5}{8}$.

EXAMPLE 3.1B: ***Converting an Improper Fraction to a Mixed Number***

Change $\frac{284}{25}$ to a mixed number:

Write as a long division problem and solve:

$$\begin{array}{r} 11 \\ 25\overline{)284} \\ -\underline{25} \\ 34 \\ \underline{-25} \\ 9 \end{array}$$

Write answer as whole number and a fraction: $11\frac{9}{25}$.

Converting a Mixed Number to an Improper Fraction

A mixed number is converted to an improper fraction using three steps:

STEP 1: Multiply whole number part by the denominator of the fractional part.

STEP 2: Add result of Step 1 to numerator of fractional part.

STEP 3: Place result of Step 2 over the denominator of the fractional part.

EXAMPLE 3.2: ***Converting a Mixed Number to an Improper Fraction***

$$3\frac{1}{2} = \frac{3\times 2+1}{2} = \frac{7}{2}$$

This is the reverse of the process for converting improper fractions to mixed numbers.

EXERCISES

3.3 Convert the indicated improper fraction to a mixed number.

a) $\frac{12}{5}$ **b)** $\frac{38}{5}$ **c)** $\frac{148}{12}$ **d)** $\frac{1605}{40}$

e) $\frac{97}{8}$ **f)** $\frac{58}{3}$ **g)** $\frac{32}{8}$

3.4 Convert the indicated mixed number to an improper fraction

a) $6\frac{1}{2}$ **b)** $2\frac{3}{4}$ **c)** $9\frac{7}{8}$

d) $12\frac{9}{12}$ **e)** $25\frac{4}{5}$ **f)** $19\frac{11}{13}$

3.3 Raising a Common Fraction to Higher Terms

Any fraction has an infinite number of *equivalent* forms.

Equivalent fractions are equal in value but different in form.

To illustrate, consider the following: $\frac{3}{4}=\frac{6}{8}=\frac{9}{12}=\frac{12}{16}=\ldots=\frac{75}{100}=\ldots$

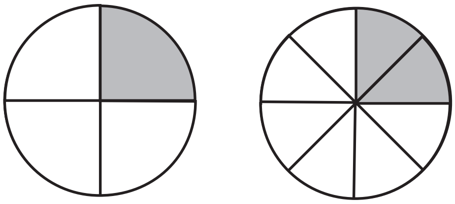


FIGURE 3.2 ***Equivalent Fractions***

A glance at the circles cut into fourths and eighths in Figure 3.2 illustrates why the first two fractions are equivalent. However, since a fraction has an infinite number of equivalent forms, a single drawing does not prove the point for all equivalent forms. Even if we sought to prove it in this fashion for a few more forms, drawing the diagrams would be very tedious. Instead, consider that the fourth fraction, $\frac{12}{16}$, is obtained from the first, $\frac{3}{4}$, by multiplying both the numerator and denominator of $\frac{3}{4}$ by 4:

$$\frac{3\times 4}{4\times 4}=\frac{12}{16}$$

Multiplying both numerator and denominator by the same number is the same as multiplying the entire fraction by 1, the *multiplicative identity*. Remember that multiplying a number by 1 does not change its value. Multiplication of $\frac{3}{4}$ by 1 in the form of $\frac{25}{25}$ gives the last equivalent form shown:

$$\frac{3\times 25}{4\times 25}=\frac{75}{100}$$

This process is called *raising a fraction to higher terms*. It is used to find equivalent forms for a fraction. Equivalent fractions always have denominators which are *multiples* of the original denominator. Thus, a fraction with a denominator of 7 can only be raised to equivalent fractions with denominator values of 14, 21, 28, 35, and so on.

Recall that a multiple of a given number is a number that results when the given number is multiplied by any whole number.

EXAMPLE 3.3A: ***Raising a Fraction to Higher Terms***

Raise $\frac{6}{11}$ to an equivalent fraction with a denominator of 88.

Solution: First ask what number 11 must be multiplied by to give 88. The answer is 8. Hence, multiply both numerator and denominator by 8:

$$\frac{6}{11}\times\frac{8}{8}=\frac{48}{88}$$

EXAMPLE 3.3B: ***Raising a Fraction to Higher Terms***

$$\frac{13}{20}=\frac{?}{60}\rightarrow\frac{13}{20}\times\frac{3}{3}=\frac{39}{60}$$

$$\frac{9}{32}=\frac{?}{128}\rightarrow\frac{9}{32}\times\frac{4}{4}=\frac{36}{128}$$

Notice that for a mixed number, the process is the same but is performed on just the fractional part of the mixed number:

$$12\frac{1}{2}=12\frac{?}{10}\rightarrow12\left(\frac{1}{2}\times\frac{5}{5}\right)=12\frac{5}{10}$$

EXERCISES

3.5 Raise the indicated fraction to an equivalent fraction.

a) $\frac{2}{3}$ to $\frac{?}{12}$ **b)** $\frac{5}{6}$ to $\frac{?}{12}$ **c)** $\frac{3}{4}$ to $\frac{?}{24}$ **d)** $\frac{11}{16}$ to $\frac{?}{64}$

3.6 Raise the indicated fraction to an equivalent fraction.

a) $\frac{2}{3}$ to $\frac{4}{?}$ **b)** $\frac{5}{6}$ to $\frac{15}{?}$ **c)** $\frac{3}{4}$ to $\frac{36}{?}$ **d)** $\frac{11}{16}$ to $\frac{22}{?}$

3.7 The fraction $\frac{5}{8}$ is how many 64ths?

3.8 The fraction $\frac{4}{5}$ is how many 85ths?

3.4 Reducing a Common Fraction to Lowest Terms

It is usually convenient when working with a fraction to *reduce it to lowest terms*. A fraction in lowest terms is one whose numerator and denominator cannot be divided by the same whole number. In other words, the numerator and denominator have no *common factor.*

People typically say, "A number that divides another number *evenly* is a factor of that number." This is not strictly correct terminology. The phrase in mathematics "*a* divides *b*" is *defined* to mean that $b \div a$ gives a whole number quotient, that is, there is no remainder. Hence, the correct definition of factor:

A number that divides another number without a remainder is a *factor* of the other number.

Fractions can be reduced to lowest terms if the numerator and denominator contain common factors. Common factor pairs divide to give 1 and thereby cancel each other to give a fraction in lowest terms.

EXAMPLE 3.4: ***Reducing a Fraction to Lowest Terms***

$$\frac{27}{51}=\frac{\overset{1}{\cancel{3}}\times 9}{\underset{1}{\cancel{3}}\times 17}=\frac{9}{17} \qquad \frac{36}{40}=\frac{\overset{1}{\cancel{2}}\times 18}{\underset{1}{\cancel{2}}\times 20}=\frac{\overset{1}{\cancel{2}}\times 9}{\underset{1}{\cancel{2}}\times 10}=\frac{9}{10} \qquad \frac{120}{220}=\frac{\overset{1}{\cancel{10}}\times 12}{\underset{1}{\cancel{10}}\times 22}=\frac{6\times\overset{1}{\cancel{2}}}{11\times\underset{1}{\cancel{2}}}=\frac{6}{11}$$

Notice that in these examples the obvious common factors are canceled first, then other cancellations are made. To reduce a fraction in one step, we need to cancel the *greatest common factor (GCF)* of the numerator and denominator. For example,

$$\frac{36}{40}=\frac{\overset{1}{\cancel{4}}\times 9}{\underset{1}{\cancel{4}}\times 10}=\frac{9}{10}$$

Sometimes the GCF of two numbers is obvious. When the GCF is not obvious, a method called *prime factorization* can be used to find it.

The ***GCF*** *is the greatest common factor of two or more numbers.*

Reducing Fractions Using Prime Factorization

Before we can discuss how to reduce fractions using prime factorization, we must learn to recognize prime numbers. A prime number is a number whose only factors are 1 and itself. The smallest prime number is 2. The prime numbers from 2 to 31 are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31.

Any other number between 2 and 31 has factors other than itself and 1. For example:

The number 2 is the only even prime.

$4 = 2 \times 2$ $\quad 6 = 2 \times 3$ $\quad 8 = 2 \times 2 \times 2$ $\quad 9 = 3 \times 3$

$10 = 2 \times 5$ $\quad 12 = 2 \times 2 \times 3$ $\quad 14 = 2 \times 7$ $\quad 15 = 3 \times 5$

A number's prime factors are easily determined by performing a series of short divisions on the number, starting with the lowest prime number that divides it and continuing in this manner until no prime factors remain. The following observations are worth noting:

- Even numbers have a factor of 2.
- Numbers whose digits add to a number divisible by 3 are also divisible by 3.
- Numbers that end in 0 or 5 are divisible by 5.

Example 3.5: ***Finding Prime Factors of a Number***

Find the prime factors of 36:

$$36 = 4 \times 9 = (2 \times 2) \times (3 \times 3)$$

The prime factors of 36 are therefore 2, 2, 3, and 3.

Similarly, the prime factors of 48 are 2, 2, 2, 2, and 3 because:

$$48 = 4 \times 12 = (2 \times 2) \times (2 \times 6) = (2 \times 2) \times (2) \times (2 \times 3)$$

Notice that 2 divides both 36 and 48 because they are even numbers. Also, 3 divides 36 and 48 because each of them has digits that add to numbers divisible by 3.

In Example 3.5 we found the prime factors for two numbers, 36 and 48. It makes sense now to ask what the GCF of these two numbers is. In most cases, the GCF can be found by inspection. When inspection does not readily reveal the GCF, the GCF can be found methodically using the procedure described next.

Procedure for Finding GCF

To find the GCF of two or more numbers use the following four step procedure:

Step 1: Write the prime factorization of each number.

Step 2: Compare the factors. Note how many times each common factor repeats.

Step 3: For each common factor, choose the least number of repeats.

Step 4: Multiply only the prime factors identified in Step 3 and the result is the GCF for the original pair of numbers

Example 3.6a: ***Finding the GCF***

Find the GCF of 36 and 48. Then reduce the fraction $\frac{36}{48}$.

Solution:

Prime factors of 36: $2 \times 2 \quad 3 \times 3$

Prime factors of 48: $2 \times 2 \times 2 \times 2 \quad 3$

In this case, only the factors 2 and 3 appear. They are common to both 36 and 48. Looking at the factors of 2, we can see that 36 has the fewest, so we keep both of them and discard the factors of 2 for 48. Likewise, looking at the factors of 3, we can see that 48 has only a single 3 so we keep it and discard the factors of 3 for 36. Finally, we arrive at:

$2 \times 2 \times 3 = 12$ to be the GCF for 36 and 48

EXAMPLE 3.6A: *Finding the GCF (Continued)*

To reduce $\frac{36}{48}$ using the GCF, we rewrite the numerator and denominator as a multiplication using the GCF, 12. We can then see that the 12 in the numerator cancels the 12 in the denominator and we are left with $\frac{3}{4}$ as our answer.

$$\frac{36}{48} = \frac{3 \times \cancel{12}}{4 \times \cancel{12}} = \frac{3}{4}$$

Note that since we reduced the fraction using the GCF, the answer cannot be reduced any further and it is said to be *in lowest terms*.

EXAMPLE 3.6B: *Finding the GCF*

Find the GCF of 108 and 150. Then reduce the fraction $\frac{108}{150}$.

Solution: Prime factors of 108 are 2, 2, 3, 3, 3 because:

$$108 = 2 \times 54 = (2) \times (2 \times 27) = (2) \times (2) \times (3 \times 9) = (2) \times (2) \times (3) \times (3 \times 3)$$

Prime factors of 150 are 2, 3, 5, 5 because:

$$150 = 2 \times 75 = (2) \times (3 \times 25) = (2) \times (3) \times (5 \times 5)$$

The common factors are 2 and 3. Among the factors of 2, the 150 has the fewest so we keep one 2; for factors of 3, the 150 has the fewest so we keep one 3; and for factors of 5, the 108 does not have any so we do not include it in finding the GCF.

Finally, multiplying the factors we kept, we find the GCF to be:

$$\text{GCF} = 2 \times 3 = 6$$

Rewriting the original fraction with the 6 as a factor in the numerator and denominator, we find the lowest term result:

$$\frac{108}{150} = \frac{\cancel{6} \times 18}{\cancel{6} \times 25} = \frac{18}{25}$$

EXAMPLE 3.7: *Reducing a Fraction Using Prime Factorization*

Reduce to lowest terms using prime factorization $\frac{18}{35}$.

Solution:

Prime factors of 18 are 2, 3, 3.

Prime factors of 35 are 5, 7.

None of the prime factors of 18 matches the prime factors of 35, so no cancellation is possible and thus $\frac{18}{35}$ is already in lowest terms.

EXAMPLE 3.8A: *Reducing a Fraction to Lowest Terms Using Prime Factorization*

Reduce $\frac{25}{60}$ to lowest terms using prime factorization.

Solution:

Prime factors of 25 are 5, 5.

Prime factors of 60 are 2, 2, 3, 5.

The only common factor is 5 and the fewest times it occurs is once so:

$$\frac{25}{60}=\frac{\cancel{5}\times 5}{\cancel{5}\times 12}=\frac{5}{12}$$

Reducing Mixed Numbers

To reduce a mixed number, reduce only the fractional part. For example:

$$15\frac{25}{30}=15\frac{\cancel{5}\times 5}{\cancel{5}\times 6}=15\frac{5}{6}$$

Therefore, $15\frac{25}{30}=15\frac{5}{6}$

EXERCISES

3.9 List the prime numbers from 30 to 100.

3.10 Why is 2 the only even prime number?

3.11 Does 3 divide 803? 1704?

3.12 Determine the prime factors of the indicated numbers:

a) 74 **b)** 66 **c)** 99 **d)** 168 **e)** 84 **f)** 55 **g)** 91 **h)** 621

3.13 Reduce the indicated fractions to lowest terms.

a) $\frac{7}{35}$ **b)** $\frac{104}{120}$ **c)** $\frac{105}{115}$ **d)** $\frac{68}{85}$

e) $\frac{133}{152}$ **f)** $\frac{12}{16}$ **g)** $\frac{15}{30}$ **h)** $\frac{28}{32}$

i) $19\frac{12}{36}$ **j)** $5\frac{3}{15}$ **k)** $7\frac{19}{38}$ **l)** $9\frac{6}{27}$

3.5 Addition and Subtraction of Common Fractions

The skills learned to convert fractions in one form to their equivalent forms are used in fraction addition and subtraction.

Fractions with the Same Denominators

Fractions with the same denominator are said to have a *common denominator*. To add or subtract such fractions, we simply add or subtract the individual numerators and place the sum over the common denominator. If necessary, the result is reduced to lowest terms or converted to mixed number.

EXAMPLE 3.9A: ***Adding Fractions with Same Denominator***

Add: $\frac{2}{5}+\frac{4}{5}+\frac{1}{5}$

Solution: Since each of the fractions has a common denominator of 5, we simply add the numerators and retain the common denominator:

$$\frac{2}{5}+\frac{4}{5}+\frac{1}{5}=\frac{2+4+1}{5}=\frac{7}{5}=1\frac{2}{5}$$

Do not add or subtract denominators.

EXAMPLE 3.9B: ***Adding Fractions with Same Denominator***

Add: $\frac{3}{8}+\frac{5}{8}+\frac{7}{8}+\frac{1}{8}=\frac{3+5+7+1}{8}=\frac{16}{8}=2$

$$\frac{16}{8}=\frac{2\times\cancel{8}}{\cancel{8}}=2$$

EXAMPLE 3.10: ***Subtracting Fractions with Same Denominator***

Subtract: $\frac{7}{9}-\frac{4}{9}=\frac{7-4}{9}=\frac{3}{9}=\frac{1}{3}$

Fractions with Different Denominators

To add or subtract fractions whose denominators are not the same, a *common denominator* must be found and each fraction in the problem must be converted to an equivalent fraction having this denominator.

A common denominator of a group of fractions is a *common multiple* of the fractions' denominators. In a group of fractions, the product of all of the denominators is always a common denominator. However, this common denominator is frequently a very large number that is unwieldy to work with. Often a smaller common denominator can be found.

The smallest possible common denominator that can be used is one into which all the original denominators can be divided. This is called the *Least Common Denominator,* which is abbreviated LCD. The arithmetic of adding and subtracting fractions can be greatly reduced by using the LCD.

Often, the LCD can be found by inspection. Otherwise, a procedure similar to the procedure used to find the GCF can be applied, as shown in the following examples.

EXAMPLE 3.11A: ***Finding the Least Common Denominator***

Find the LCD (Least Common Denominator) of the following group of fractions: $\frac{1}{20} \quad \frac{7}{18} \quad \frac{4}{15} \quad \frac{1}{36}$

Solution:

Note that the GCF takes the lowest number of common prime factors, while LCD takes the greatest number of all prime factors.

STEP 1: Under each denominator place its prime factors:

$$\begin{array}{cccc} \frac{1}{20} & \frac{7}{18} & \frac{4}{15} & \frac{1}{36} \\ 2,2,5 & 2,3,3 & 3,5 & 2,2,3,3 \end{array}$$

STEP 2: Determine the greatest frequencies of *all* primes in any one factorization:

2 occurs at most 2 times
3 occurs at most 2 times
5 occurs at most 1 time

STEP 3: Find the LCD by multiplying each prime by itself according to its greatest frequency. Hence,

$$\text{LCD} = (2\times2)\times(3\times3)\times(5) = 180$$

The calculated LCD can be proved correct by dividing it by each of the original denominators to get whole number answers:

$$\frac{180}{20} = 9 \quad \frac{180}{18} = 10 \quad \frac{180}{15} = 12 \quad \frac{180}{35} = 5$$

Therefore, 180 is the LCD.

EXAMPLE 3.11B: ***Finding the Least Common Denominator***

Add the fractions: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$, and $\frac{1}{8}$

Solution:

STEP 1: Under each denominator place its prime factors:

$$\begin{array}{ccccc} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{5}{6} & \frac{1}{8} \\ 1,2 & 1,3 & 2,2 & 2,3 & 2,2,2 \end{array}$$

STEP 2: Determine frequencies of primes:

2 occurs at most 3 times
3 occurs at most 1 time
1 occurs at most 1 time

EXAMPLE 3.11B: ***Finding the Least Common Denominator (Continued)***

STEP 3: Multiply primes according to their most frequent occurrences to get LCD:

$$2 \times 2 \times 2 \times 3 = 24$$

When the LCD is determined, the next step in the addition and subtraction of fractions with different denominators is to convert the different fractions into equivalent fractions, each with the LCD as its denominator.

STEP 4: Convert to equivalent fractions using the LCD:

$$\frac{1}{2}=\frac{12}{24},\quad \frac{2}{3}=\frac{16}{24},\quad \frac{3}{4}=\frac{18}{24},\quad \frac{5}{6}=\frac{20}{24},\quad \frac{1}{8}=\frac{3}{24}$$

STEP 5: Add the equivalent fractions:

$$\frac{12}{24}+\frac{16}{24}+\frac{18}{24}+\frac{20}{24}+\frac{3}{24}=\frac{12+16+18+20+3}{24}=\frac{69}{24}$$

STEP 6: Reduce to lowest terms:

$$\frac{69}{24}=2\frac{21}{24}=2\frac{7}{8}$$

Generalized Procedures for Adding and Subtracting Fractions with Different Denominators

Now that we know how to find the least common denominator, we can generalize the procedures for adding and subtracting fractions with different denominators into 4 steps:

STEP 1: Find the least common denominator (LCD).

STEP 2: Convert all fractions to equivalent fractions using the LCD.

STEP 3: Add or subtract the converted fractions.

STEP 4: Reduce answer if possible.

Combined Addition and Subtraction of Fractions

As we have seen, whenever both addition and subtraction operations appear in a problem, the operations are performed in the order in which they appear, unless parentheses are present.

PEMDAS
Parentheses
Exponents
Multiplication
Division
Addition
Subtraction

EXAMPLE 3.12: ***Combined Addition and Subtraction of Fractions***

Perform the indicated operations: $\frac{2}{3}-\frac{1}{2}+\frac{3}{4}+\frac{5}{6}-\frac{5}{8}$

Solution: The LCD is 24, as before. Convert and perform operations:

$$\frac{16}{24}-\frac{12}{24}+\frac{18}{24}+\frac{20}{24}-\frac{15}{24}=\frac{16-12+18+20-15}{24}=\frac{27}{24}=1\frac{3}{24}=1\frac{1}{8}$$

EXERCISES

3.14 Find the LCD of each group of fractions.

a) $\frac{7}{8}, \frac{3}{4}, \frac{3}{16}$ **b)** $\frac{5}{18}, \frac{2}{3}, \frac{11}{14}, \frac{5}{9}$ **c)** $\frac{3}{8}, \frac{3}{5}, \frac{17}{20}$

d) $\frac{1}{15}, \frac{3}{10}, \frac{4}{5}$ **e)** $\frac{7}{8}, \frac{4}{9}, \frac{2}{3}$ **f)** $\frac{1}{6}, \frac{3}{8}, \frac{5}{9}, \frac{2}{3}$

3.15 Perform the indicated addition and subtraction.

a) $\frac{3}{2}+\frac{5}{2}$ **b)** $\frac{3}{4}-\frac{5}{8}+\frac{1}{2}$ **c)** $\frac{3}{16}+\frac{5}{32}-\frac{5}{8}$

d) $\frac{9}{16}-\frac{1}{4}$ **e)** $\frac{7}{9}+\frac{5}{6}-\frac{1}{18}$ **f)** $\frac{3}{32}+\frac{3}{8}-\frac{3}{64}$

g) $\frac{1}{2}-\frac{1}{6}+\frac{1}{4}-\frac{1}{8}$ **h)** $\frac{1}{2}+\frac{3}{4}-\frac{5}{8}$ **i)** $\frac{2}{5}+\frac{3}{4}-\frac{1}{6}$

j) $\frac{3}{4}+\frac{7}{9}-\frac{2}{9}-\frac{6}{9}$ **k)** $\frac{21}{24}+\frac{11}{12}-\frac{3}{8}$ **l)** $\frac{1}{3}+\frac{3}{8}-\frac{4}{7}$

m) $\frac{11}{15}-\frac{2}{9}+\frac{4}{5}-\frac{1}{3}$ **n)** $\frac{4}{5}-\frac{3}{4}+\frac{2}{3}-\frac{1}{2}$ **o)** $\frac{5}{12}-\frac{4}{5}+\frac{2}{3}$

p) $\frac{13}{14}+\frac{11}{21}-\frac{9}{14}-\frac{5}{7}$ **q)** $\frac{5}{8}+\frac{7}{8}-\frac{3}{4}+\frac{5}{8}-\frac{9}{16}$

3.16 A tube has an inside diameter of $\frac{5}{16}$ inch and a wall thickness of $\frac{1}{8}$ inch. What is the outside diameter (O.D.)?

3.17 A bolt is used to fasten two $1\frac{5}{8}$-inch planks. The nut for the bolt is $\frac{1}{4}$ inch thick. What is the minimum bolt length needed if bolts come in length increments of $\frac{1}{4}$ inch?

3.18 A special laminated wooden beam is made up of two pieces of planking, each $1\frac{5}{8}$ inch thick, and three pieces of lumber, each $\frac{13}{16}$ inch thick. What is the thickness of the beam?

3.19 A special jet fuel is made in a laboratory in small quantities for testing. If the formula calls for $\frac{3}{4}$ lb of chemical A, $\frac{1}{8}$ lb of chemical B, and $\frac{2}{3}$ lb of chemical C, what is the total weight of the experimental batch of jet fuel?

3.20 A carpenter is nailing some $\frac{3}{8}$-inch plywood onto furring strips that are $\frac{13}{16}$ inch thick. If nails come in $\frac{1}{8}$-inch incremental lengths, how long a nail must be used to insure maximum penetration but not go completely through the furring strip?

3.21 A drafter's scale (ruler) has the inches divided into halves, quarters, eighths, and sixteenths. How long a line would result if one-half, three-quarters, five-eighths and eleven-sixteenths were added together?

3.22 The dimensions in Figure 3.3 are given in inches. Find *A, B, C, D, E*, and *F*.

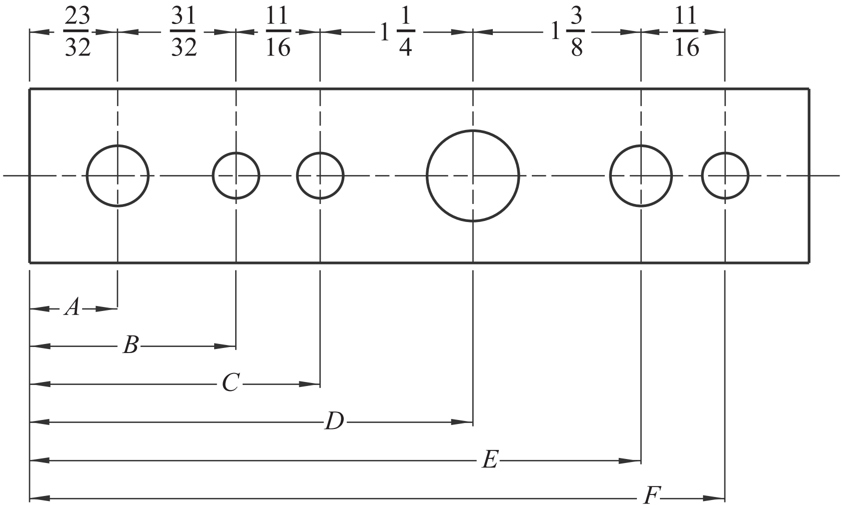


FIGURE 3.3

3.23 The dimensions in Figure 3.4 are given in inches. Determine the overall dimension, *A*, of the locating pin shown**.**

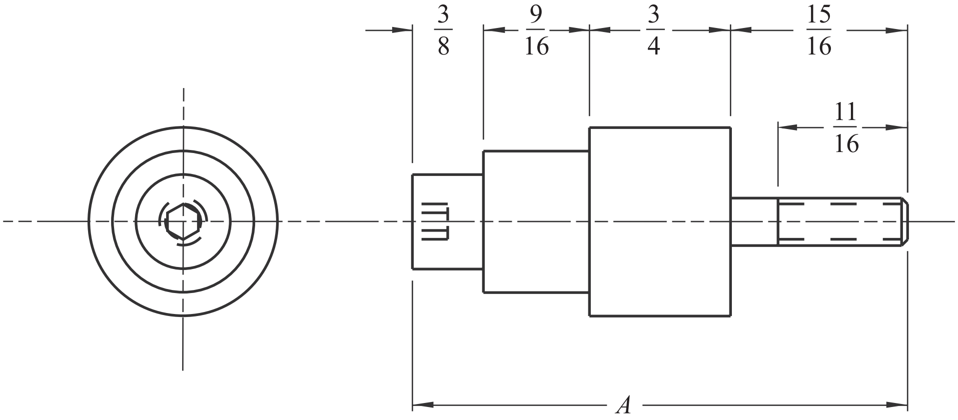


FIGURE 3.4

3.24 The dimensions in Figure 3.5 are given in inches. Determine horizontal dimensions, *A, B, C*, and *D*. Also determine vertical dimensions, *E, F*, *G*, and *H*.

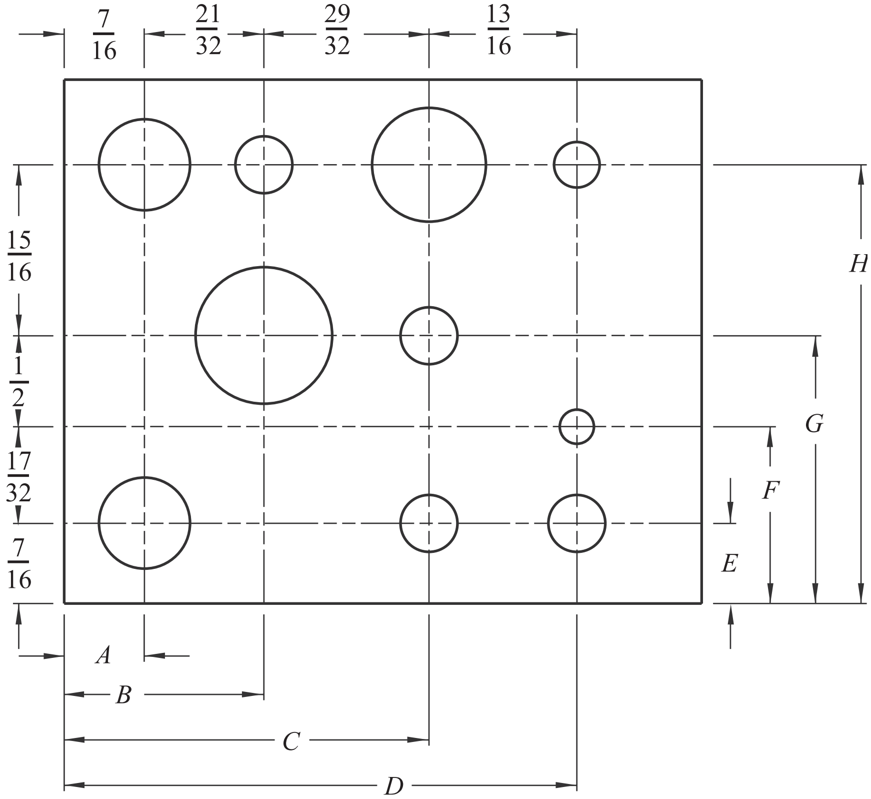


FIGURE 3.5

3.6 Addition and Subtraction of Mixed Numbers

Mixed numbers can be added by either of two methods. In one method, the mixed numbers are converted into improper fractions, which are then added by means of the LCD. In the other method, the fractional parts are added independently from the whole number parts. This method also usually requires that an LCD be found, and it is the technique used most often, so we will review it in depth. Consider mixed numbers whose fractional parts have like denominators:

$$9\frac{2}{3}+5\frac{1}{3}$$

Since mixed numbers themselves have an addition operation implied between the whole number part and the fraction part, the problem could be represented as $9 + \frac{2}{3} + 5 + \frac{1}{3}$. From this representation it is logical to add whole number to whole number and fraction to fraction. Thus,

$$9+\frac{2}{3}+5+\frac{1}{3}=14+\frac{3}{3}=14+1=15$$

We would not normally write out the addition between the mixed number parts; we only do so here to clarify the process. Notice that the fraction parts add to $\frac{3}{3}$, which reduces to 1. Final answers must always be reduced to simplest form.

The process is a little more involved for mixed numbers whose fractions have unlike denominators. Consider the problem $2\frac{1}{5} + 1\frac{2}{3}$. We write this vertically and show all steps toward the solution:

$$\begin{array}{r}
2\frac{1}{5} = 2\frac{1}{5}\cdot\frac{3}{3} = 2\frac{3}{15} \\
+1\frac{2}{3} = 1\frac{2}{3}\cdot\frac{5}{5} = 1\frac{10}{15} \\
\hline
3\frac{13}{15}
\end{array}$$

Notice that we had to convert the fractional parts to their equivalent forms with LCD of 15. The final answer required no reducing.

Now we show the other method of mixed number addition, namely, converting the mixed numbers to improper fractions before adding them. Using the same problem, we again show all steps toward the solution.

$$\begin{array}{r} 2\frac{1}{5} = \frac{11}{5} = \frac{11}{5}\cdot\frac{3}{3} = \frac{33}{15} \\ +\ 1\frac{2}{3} = \frac{5}{3} = \frac{5}{3}\cdot\frac{5}{5} = \frac{25}{15} \\ \hline \frac{58}{15} = 3\frac{13}{15} \end{array}$$

Notice that this method requires some extra steps. This is why it is the least used of the two techniques for mixed number addition.

Example 3.13: ***Mixed Number Addition***

9 = 3 · 3
12 = 3 · 4

Perform the indicated addition using prime factorization: $7\frac{5}{9}+6\frac{7}{12}$

Solution: Using prime factorization, we find the LCD is 3 × 3 × 4 = 36. Thus,

$$\begin{array}{r} 7\frac{5}{9} = 7\frac{5}{9}\cdot\frac{4}{4} = 7\frac{20}{36} \\ +\ 6\frac{7}{12} = 6\frac{7}{12}\cdot\frac{3}{3} = 6\frac{21}{36} \\ \hline 13\frac{41}{36} \end{array}$$

Notice that $13\frac{41}{36}$ can be reduced to $13 + 1\frac{5}{36}$ for a final answer of $14\frac{5}{36}$.

The improper fraction in the answer is converted to a whole number and proper fraction. The whole number is then added to the whole number part of the answer.

Subtracting mixed numbers can be more complicated than adding them. Consider the problem $5\frac{1}{4}-2\frac{3}{4}$. As with mixed number addition, the whole number parts are subtracted separately from the fraction parts. However, the subtraction $\frac{1}{4}-\frac{3}{4}$ cannot be solved without negative numbers. We are interested in only nonnegative numbers at this point in our review of arithmetic so even though we could resort to using negative numbers to get the answer, it is not necessary to do so. Write the problem in vertical fashion:

$$\begin{array}{r} 5\frac{1}{4} \\ -2\frac{3}{4} \\ \hline \end{array}$$

Math Trivia: The minuend is the first number in the subtraction.The subtrahend is the second number.

Although $\frac{1}{4}$ is smaller then $\frac{3}{4}$, the entire subtrahend $2\frac{3}{4}$, is smaller than the minuend $5\frac{1}{4}$, so the problem is certainly manageable in the nonnegative numbers.

To solve, we increase $\frac{1}{4}$ by "borrowing" from 5. Taking one whole (1) from 5 and adding it to $\frac{1}{4}$ gives an improper fraction. The rest of the problem is straightforward.

$$\begin{array}{r}
5\frac{1}{4} \;=\; 4 + 1\frac{1}{4} \;=\; 4\frac{5}{4} \\
-2\frac{3}{4} \\
\hline
2\frac{2}{4} \;=\; 2\frac{1}{2}
\end{array}$$

In this kind of problem, it is sometimes easier to use the other method of mixed number addition and subtraction, namely, changing the mixed numbers to improper fractions first.

$$\begin{array}{r}
5\frac{1}{4} \;=\; \frac{21}{4} \\
-2\frac{3}{4} \;=\; -\frac{11}{4} \\
\hline
\frac{10}{4} \;=\; 2\frac{2}{4} \;=\; 2\frac{1}{2}
\end{array}$$

When unlike denominators are involved in the subtraction, the LCD must be found. This adds to the steps, as the next example shows.

EXAMPLE 3.14: ***Subtraction of Mixed Numbers with Unlike Denominators***

Perform the indicated subtraction: $8\frac{4}{7} - 3\frac{2}{3}$

Solution: First the LCD is found ($3 \cdot 7 = 21$). Then the fractional parts are converted to equivalent fractions with a denominator of 21:

$$8\frac{4}{7} = 8\frac{4}{7} \cdot \frac{3}{3} = 8\frac{12}{21}$$

$$3\frac{2}{3} = 3\frac{2}{3} \cdot \frac{7}{7} = 3\frac{14}{21}$$

This problem still requires that the fractional portion of the minuend be increased by borrowing from the whole number portion:

$$\begin{array}{r}
8\frac{12}{21} \;=\; 7 + 1\frac{12}{21} \;=\; 7\frac{33}{21} \\
-3\frac{14}{21} \\
\hline
4\frac{19}{21}
\end{array}$$

EXERCISES

3.25 Perform indicated operations:

a) $3\frac{1}{2}+2\frac{1}{8}-\frac{5}{16}$ **b)** $6\frac{7}{8}+3\frac{5}{6}$ **c)** $\frac{19}{32}+6\frac{1}{4}$

d) $10\frac{2}{32}+\frac{7}{16}-1\frac{1}{8}$ **e)** $12\frac{1}{8}-2\frac{3}{4}-\frac{5}{6}$ **f)** $6\frac{7}{8}+2\frac{2}{3}-5\frac{3}{5}$

g) $45\frac{3}{5}-22\frac{7}{10}-11\frac{14}{15}$ **h)** $22\frac{8}{9}+11\frac{5}{6}-8\frac{1}{3}$

3.26 The dimensions of the detail in Figure 3.6 are given in inches. Calculate the overall length.

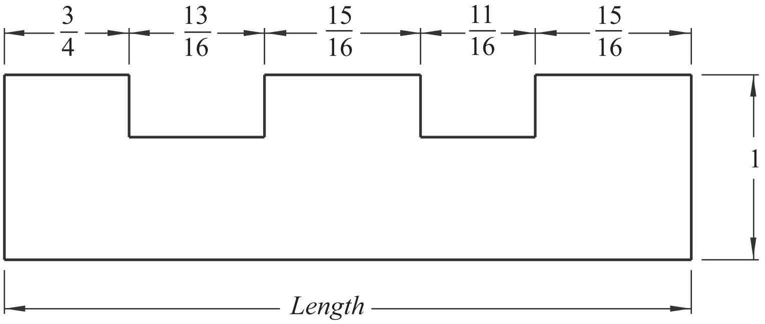


FIGURE 3.6

3.27 The dimensions of the detail in Figure 3.7 are given in inches. Determine the indicated length and width.

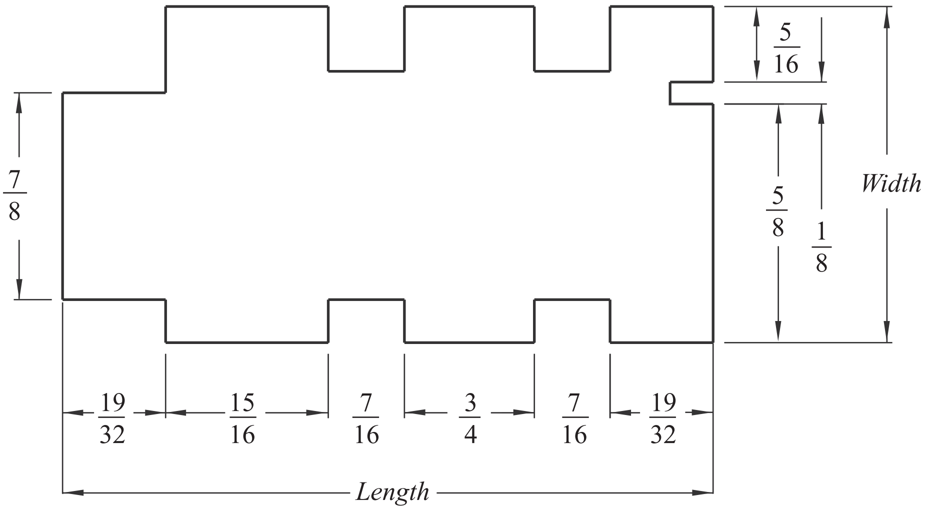


FIGURE 3.7

3.28 The dimensions of the detail in Figure 3.8 are given in inches. Determine the indicated length and width.

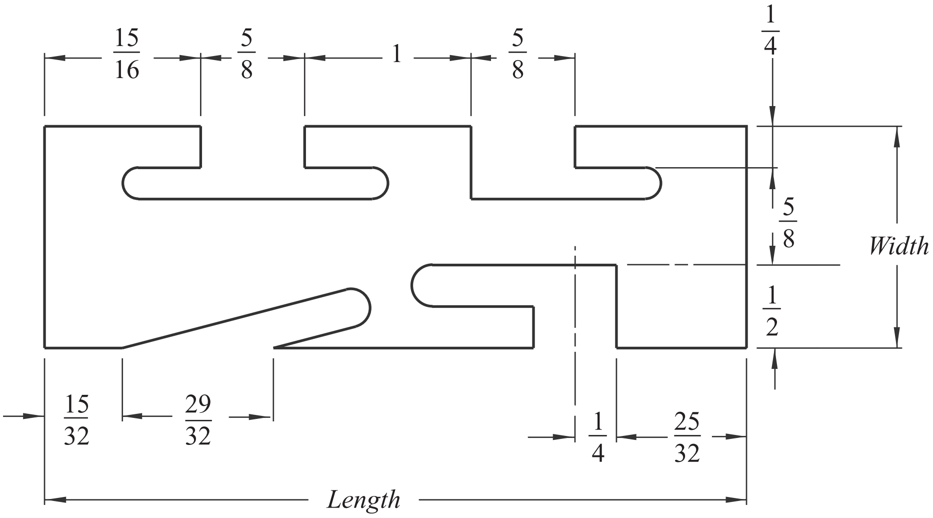


FIGURE 3.8

3.29 Six sheets of steel have the following thickness in inches: $\frac{1}{8}$, $\frac{3}{32}$, $\frac{3}{16}$, $\frac{1}{4}$, $\frac{1}{16}$ and $\frac{11}{64}$. What is the total thickness of the six sheets?

3.30 The dimensions of the detail in Figure 3.9 are given in inches. Determine dimensions *A, B, C*, and *D*.

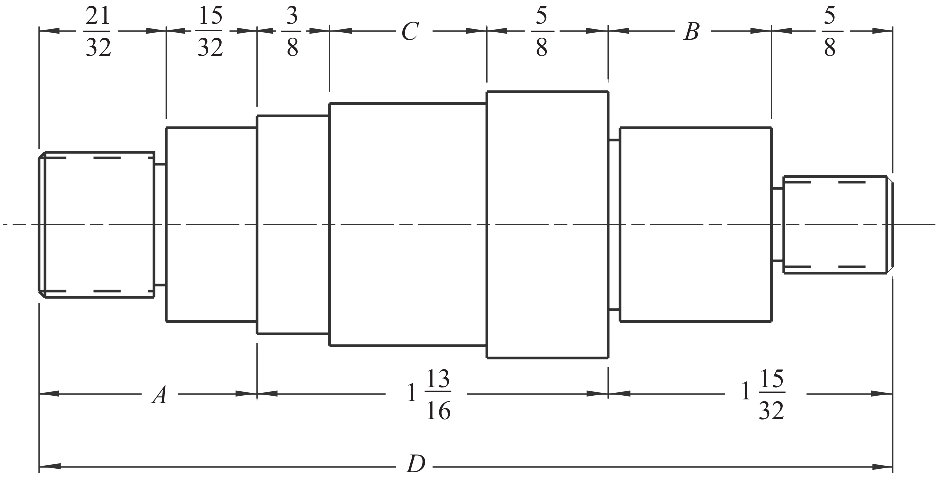


FIGURE 3.9

3.31 Eight 2 by 4 inch studs, 8 feet long are nailed end to end with each joint having an overlap of $6\frac{3}{8}$ inches. What is the total length of the assembly?

3.32 A rectangular field is 450 feet, 8 inches long by 235 feet, 11 inches wide. How many feet of fencing are needed to enclose the field?

3.7 Multiplication and Division of Common Fractions

Raising a fraction to its equivalent form is done to produce fractions with common denominators in order to add or subtract them. This process requires fraction multiplication, as we discussed earlier in this chapter. Now we look at fraction multiplication for all purposes. The rule for multiplication of fractions is simple:

Fraction Multiplication

- Multiply all numerators to obtain the numerator of the answer.
- Multiply all denominators to obtain the denominator in the answer.
- Reduce answer to lowest terms.

EXAMPLE 3.15: ***Fraction Multiplication***

Solve: $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}$

Solution:

$$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1\times2\times3}{2\times3\times4}=\frac{6}{24}=\frac{1}{4}$$

As seen in Example 3.15, the result in fraction multiplication is often not in lowest terms. However, the process of reducing the result to lowest terms can be simplified if cancellation of common factors is done *before* multiplication takes place. Cancellation is the dividing of the numerator and denominator of a fraction by a common factor. Note that if all possible cancellation is performed before the multiplication takes place, no reduction will be necessary at the end, as the result will already be in lowest terms.

EXAMPLE 3.16: ***Fraction Multiplication using Cancellation***

Solve: $\frac{30}{48}\times\frac{12}{15}\times\frac{4}{6}$

Solution:

Cancellation is shown on the original problem. The result of the canceling is written over the original number. Then the fractions are multiplied.

Remember, a fraction in lowest terms has no common factors in its numerator and denominator.

$$\frac{\overset{2}{\cancel{30}}}{\underset{4}{\cancel{48}}}\times\frac{\overset{1}{\cancel{12}}}{\underset{1}{\cancel{15}}}\times\frac{\overset{2}{\cancel{4}}}{\underset{3}{\cancel{6}}}=\frac{2}{4}\times\frac{1}{1}\times\frac{2}{3}=\frac{4}{12}=\frac{1}{3}$$

We could have canceled thoroughly before multiplying (the two 2s with the one 4) and then the final answer would have been in lowest terms, leaving no need to reduce further.

An optional cancellation is: $\frac{\overset{5}{\cancel{30}}}{\underset{4}{\cancel{48}}}\times\frac{\overset{1}{\cancel{12}}}{15}\times\frac{4}{\underset{1}{\cancel{6}}}=\frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{4}}}\times\frac{1}{\underset{3}{\cancel{15}}}\times\frac{\overset{1}{\cancel{4}}}{1}=\frac{1}{3}$

Notice that the cancellation can be done between any factor in the numerator and any factor in the denominator.

The method of dividing fractions is also quite simple, as it follows from the rule for fraction multiplication. The rule of division is:

Invert ("flip over") the fraction following the division sign, change the division sign to a multiplication sign, and proceed as in multiplication of fractions. The inverted fraction is called the *reciprocal* of the fraction.

Fraction Division

EXAMPLE 3.17A: ***Division of Fractions***

Solve: $\frac{5}{6} \div \frac{2}{3}$

Solution:

STEP 1: Invert second fraction and change to multiplication:

$$\frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \times \frac{3}{2}$$

STEP 2: Perform cancellation and reduce:

$$\frac{5}{6} \times \frac{3}{2} = \frac{5}{\underset{2}{\cancel{6}}} \times \frac{\overset{1}{\cancel{3}}}{2} = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4} = 1\frac{1}{4}$$

EXAMPLE 3.17B: ***Division of Fractions***

Solve: $\frac{2}{3} \div \frac{1}{2} \times \frac{1}{4} \div \frac{5}{6}$

Solution:

Invert fractions to right of division signs: $\frac{2}{3} \div \frac{1}{2} \times \frac{1}{4} \div \frac{5}{6} = \frac{2}{3} \times \frac{2}{1} \times \frac{1}{4} \times \frac{6}{5}$

Cancel: $\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{2}}}{1} \times \frac{1}{\underset{1}{\cancel{4}}} \times \frac{\overset{2}{\cancel{6}}}{5}$

Multiply: $\frac{1}{1} \times \frac{1}{1} \times \frac{1}{1} \times \frac{2}{5} = \frac{2}{5}$

Do not invert the fraction left of the division sign.

Cancellation must not be done before inversion.

EXERCISES

3.33 Perform the indicated operations:

a) $\frac{2}{9}\times\frac{3}{8}$ **b)** $\frac{7}{8}\times\frac{3}{21}$ **c)** $\frac{2}{9}\div\frac{1}{3}$ **d)** $\frac{4}{5}\div\frac{2}{15}$

e) $\frac{3}{7}\div\frac{12}{21}$ **f)** $\frac{1}{2}\times\frac{3}{4}$ **g)** $\frac{4}{3}\div 2$ **h)** $\frac{5}{9}\times\frac{2}{3}$

i) $\frac{3}{8}\times 4$ **j)** $2\div\frac{1}{2}$

3.34 Perform the indicated operations:

a) $\frac{91}{99}\times\frac{6}{13}\times\frac{22}{49}$ **b)** $\frac{18}{35}\times\frac{77}{99}\times\frac{55}{66}$

c) $\frac{22}{27}\times\frac{9}{11}\times\frac{3}{4}$ **d)** $\frac{15}{32}\times\frac{16}{45}\times\frac{8}{13}\times\frac{52}{64}$

e) $\frac{144}{165}\times\frac{15}{24}\times\frac{32}{65}\times\frac{25}{36}$ **f)** $\frac{3}{4}\times\frac{12}{9}\times\frac{16}{3}\times\frac{18}{24}$

g) $\frac{5}{8}\times\frac{3}{10}\times\frac{8}{9}$ **h)** $\frac{6}{10}\times\frac{24}{18}\times\frac{3}{36}\div\frac{36}{18}$

i) $5\times\frac{4}{3}\times\frac{7}{8}\div\frac{5}{6}$ **j)** $\frac{32}{49}\div\frac{4}{7}\times\frac{16}{25}\times\frac{1}{2}$

3.35 Perform the indicated operations:

a) $\frac{7}{11}\div\frac{14}{22}\times\frac{1}{2}\div\frac{1}{2}$ **b)** $\frac{8}{9}\times\frac{3}{4}\div\frac{1}{3}\times\frac{4}{5}\times\frac{1}{8}$

c) $\frac{21}{32}\times\frac{8}{9}\times\frac{7}{8}\div\frac{3}{4}$ **d)** $\frac{25}{36}\times\frac{5}{9}\times\frac{2}{7}\times\frac{3}{4}\div\frac{1}{4}$

e) $\frac{7}{8}\div\frac{35}{48}\times\frac{6}{7}\div\frac{36}{70}\div\frac{2}{3}$ **f)** $\frac{3}{7}\times\frac{5}{36}\div\frac{2}{3}\times\frac{7}{16}\times\frac{8}{25}$

g) $\frac{94}{100}\times\frac{5}{8}\div\frac{5}{24}\times\frac{2}{15}$ **h)** $\frac{3}{4}\times\frac{7}{9}\times\frac{21}{35}\times\frac{5}{7}\times\frac{27}{42}$

i) $\frac{4}{7}\times\frac{7}{9}\times\frac{3}{14}\times\frac{2}{3}\div\frac{2}{21}$ **j)** $\frac{3}{2}\times\frac{6}{5}\times\frac{22}{7}\times\frac{8}{3}\div\frac{44}{70}$

3.36 Multiply the difference of $\frac{6}{15}$ and $\frac{3}{20}$ by their sum.

3.37 A special nonferrous alloy is composed of $\frac{5}{12}$ tin, $\frac{1}{3}$ copper, and $\frac{1}{4}$ zinc. How many pounds of each metal will be used to pour a casting weighing 224 lb?

3.38 A pipefitting weighs $\frac{2}{3}$ lb. How many fittings are in a box weighing 122 lb if the empty box weighs 32 lb?

3.39 How many holes, spaced on $\frac{9}{16}$ inch centers, can be drilled in the detail shown in Figure 3.10?

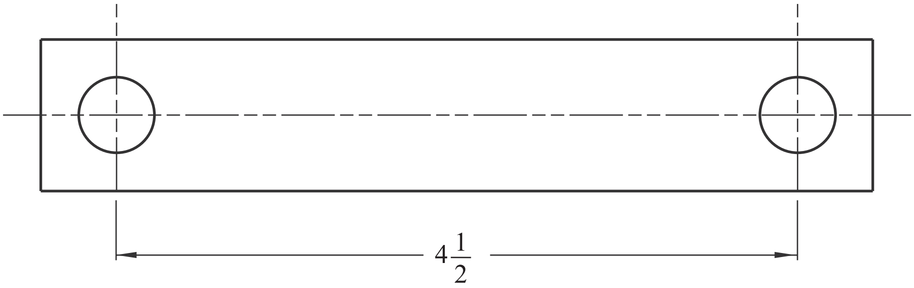


FIGURE 3.10

3.40 Determine the number of discs that can be blanked from a 24-foot roll of #18-gauge cold rolled steel if each blank is $1\frac{1}{8}$ inches in diameter and there is a $\frac{3}{32}$-inch web between each disc. See Figure 3.11.

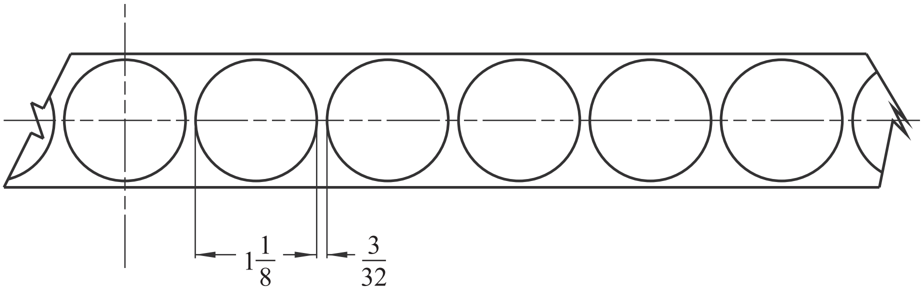


FIGURE 3.11

3.41 A millwright must use a rough wooden timber measuring 8 by 12 inches. If a timber that is $7\frac{13}{16}$ inches thick is required, and equal thickness are cut from each rough side, how thick are the pieces cut off each side? See Figure 3.12.

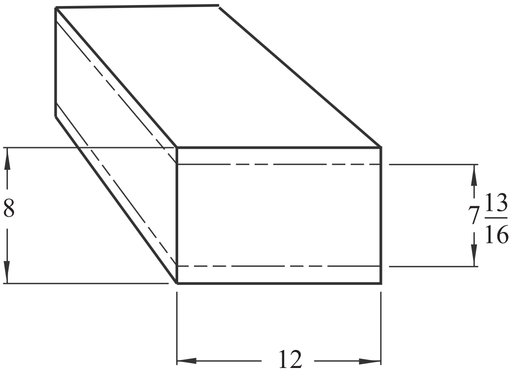


FIGURE 3.12

3.42 Masonry walls of cement blocks are made from a standard size block which is $15\frac{5}{8}$ inches long, $7\frac{5}{8}$ inches high, and 8 inches thick. The mortar joint allowance for each block is $\frac{3}{8}$ of an inch. How many blocks will be needed to build a wall that is 8 feet high and 160 feet long?

3.8 Multiplication and Division of Mixed Numbers

Multiplication of mixed numbers proceeds as shown in Example 3.18, in which the mixed numbers are converted to improper fractions. Multiplication of mixed numbers can also be done by another method; however, the process is cumbersome and so is not presented here. The division of mixed numbers, on the other hand, must be done by conversion to improper fractions.

EXAMPLE 3.18: ***Multiplying Mixed Numbers***

Multiply: $8\frac{1}{3}$ by $3\frac{3}{5}$

Solution:

STEP 1: Convert to improper fractions: $\frac{25}{3}\times\frac{18}{5}$

STEP 2: Cancel: $\frac{\overset{5}{\cancel{25}}}{\underset{1}{\cancel{3}}}\times\frac{\overset{6}{\cancel{18}}}{\underset{1}{\cancel{5}}}=\frac{5}{1}\times\frac{6}{1}=30$

EXAMPLE 3.19: ***Dividing Mixed Numbers***

Divide $4\frac{1}{3}$ by $3\frac{5}{7}$

EXAMPLE 3.19: ***Dividing Mixed Numbers (Continued)***

Solution:

STEP 1: Convert to improper fractions: $\frac{13}{3}\div\frac{26}{7}$

STEP 2: Invert, cancel and multiply:

$$\frac{\overset{1}{\cancel{13}}}{3}\times\frac{7}{\underset{2}{\cancel{26}}}=\frac{1}{3}\times\frac{7}{2}=\frac{7}{6}$$

Reduce to lowest terms: $\frac{7}{6}=1\frac{1}{6}$

Combined multiplication and division of mixed numbers is accomplished in the same manner as with fractions.

Combined Multiplication and Division

EXAMPLE 3.20: ***Combined Multiplication and Division of Mixed Numbers***

Perform the following operations: $13\frac{3}{4}\div 6\frac{7}{8}\times 2\frac{13}{16}\div 4\frac{1}{2}$

Solution:

STEP 1: Convert to improper fractions:

$$\frac{55}{4}\div\frac{55}{8}\times\frac{45}{16}\div\frac{9}{2}$$

STEP 2: Invert fraction to right of each division and change division to multiplication:

$$\frac{55}{4}\times\frac{8}{55}\times\frac{45}{16}\times\frac{2}{9}$$

STEP 3: Cancel common factors from numerators and denominators:

$$\frac{\overset{1}{\cancel{55}}}{\underset{1}{\cancel{4}}}\times\frac{\overset{\overset{1}{\cancel{2}}}{\cancel{8}}}{\underset{1}{\cancel{55}}}\times\frac{\overset{5}{\cancel{45}}}{\underset{\underset{4}{\cancel{8}}}{\cancel{16}}}\times\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{9}}}$$

STEP 4: Carry out multiplication:

$$\frac{1}{1}\times\frac{1}{1}\times\frac{5}{4}\times\frac{1}{1}=\frac{5}{4}$$

Change improper fraction to a mixed number: $1\frac{1}{4}$

EXERCISES

3.43 Perform the indicated operations:

a) $6\frac{7}{8}\div 2\frac{3}{4}\times 3\frac{2}{3}$

b) $33\frac{11}{24}\times 2\frac{5}{8}\div 6\frac{11}{12}$

c) $14\frac{5}{6}\times 21\frac{15}{18}\div\frac{2}{3}$

d) $45\times 2\frac{5}{8}\times\frac{3}{4}\div 5$

e) $22\frac{9}{10}\times 14\frac{7}{10}\div\frac{3}{5}$

f) $\frac{17}{12}\times 15\div 6\frac{7}{8}+\frac{3}{8}$

g) $8\frac{13}{25}\div\frac{9}{15}\times\frac{3}{5}\div\frac{19}{25}$

h) $6\frac{21}{32}\times\frac{17}{24}\div 5\frac{11}{16}\times 4\frac{7}{8}$

i) $14\frac{3}{7}\times 3\frac{18}{21}\div\frac{11}{14}$

j) $14\frac{15}{16}\times 8\frac{5}{8}\div 2\frac{1}{2}$

k) $9\frac{7}{9}\times 31\frac{3}{18}\div 6\frac{25}{27}$

l) $7\frac{1}{6}\times 2\frac{3}{4}\div\frac{5}{8}\times\frac{3}{4}$

m) $\frac{13}{20}\div 5\frac{1}{5}$

n) $3\frac{1}{4}\times 20\times\frac{5}{32}$

o) $\left(10\frac{1}{2}+3\frac{4}{9}\right)\div\frac{18}{3}$

p) $15\frac{2}{3}\times 8\frac{5}{8}\div 4\frac{3}{4}$

q) $23\frac{4}{6}\div\left(2\frac{4}{7}+3\frac{1}{4}\right)$

3.44 An I-beam weighs 35 lb per foot. What does a piece 3 feet, 4 inches weigh?

3.45 How long a piece of $\frac{3}{4}$-inch hexagon stock is needed to produce 13 pieces, each $1\frac{7}{8}$-inches long if each cut requires $\frac{1}{32}$-inch waste?

3.46 A hexagon bar of steel is placed between centers on a lathe. The bar is $23\frac{1}{2}$ inches long. If the bar is machined for $\frac{1}{3}$ of its length, how many inches of unmachined hexagon bar remain?

3.47 A gasoline storage tank with a capacity of $134\frac{3}{4}$ gallons is $\frac{3}{8}$ full. How many gallons will be required to fill the tank?

3.48 How many $6\frac{1}{2}$-inch nipples can be cut from a 14-foot length of galvanized pipe? Allow $\frac{1}{16}$ inch per cut.

3.49 Gasoline is selling for $99\frac{9}{10}$ cents per gallon. How much will it cost to fill a 20-gallon gas tank if there are already $6\frac{1}{3}$ gallons in the tank?

3.50 Calculate the weight of 24 brass bars, each $8\frac{1}{2}$ feet long, if brass weighs $39\frac{4}{5}$ lb per foot.

3.51 A cement contractor has four piles of reinforcing bars. In the first pile there are thirteen $\frac{1}{2}$-inch bars, 14 feet 6 inches long, weighing $\frac{3}{4}$ pounds per foot. The second pile contains twenty-seven $\frac{1}{2}$-inch bars, $12\frac{1}{2}$ foot long. The third pile has nine $\frac{3}{4}$-inch bars, weighing $1\frac{1}{2}$ pounds per foot, $14\frac{1}{2}$ feet long. The fourth pile has thirty-five $\frac{3}{4}$-inch bars, 28 feet 4 inches long. How many pounds of steel does the contractor have?

3.9 Complex Fractions

Complex fractions are "built-up" fractions, like these: $\dfrac{\frac{1}{4}}{\frac{2}{3}}, \dfrac{\frac{1}{9}}{3}, \dfrac{5}{\frac{1}{2}}$.

The fractions within the complex fraction may be either proper or improper. It is also possible that one part of the complex fraction is a whole number. A complex fraction may also be composed of mixed numbers.

To solve a complex fraction means to *simplify* it. As illustrated in the examples, there are several ways to simplify a complex fraction. Each method relies on a rule of fraction reduction, forming equivalent fractions, and fraction division learned in this chapter. We may use whichever method is appropriate as the examples show.

EXAMPLE 3.21: ***Complex Fractions***

Simplify the complex fraction: $\dfrac{\frac{4}{9}}{\frac{2}{3}}$

Solution: Three methods are shown.

Method 1. Reducing by canceling within the fraction.

$$\frac{\frac{4}{9}}{\frac{2}{3}} = \frac{\frac{\cancel{4}^{2}}{9}}{\frac{\cancel{2}^{1}}{3}} = \frac{\frac{2}{\cancel{9}^{3}}}{\frac{1}{\cancel{3}^{1}}} = \frac{\frac{2}{3}}{\frac{1}{1}} = \frac{\frac{2}{3}}{1} = \frac{2}{3}$$

Cancel numerator of top fraction with numerator of bottom fraction; cancel denominator of top fraction with denominator of bottom fraction.

Method 2. Rewriting the fraction as a division problem.

$$\frac{\frac{4}{9}}{\frac{2}{3}} = \frac{4}{9} \div \frac{2}{3} = \frac{\overset{2}{\cancel{4}}}{\underset{3}{\cancel{9}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}} = \frac{2}{3}$$

Method 3. Equivalent fraction method.

A complex fraction may also be simplified by rewriting it as an equivalent complex fraction whose denominator part is reduced to 1. Multiplying the fraction in the numerator and in the denominator by the reciprocal of the denominator fraction does this.

Recall that any number divided by itself equals 1.

$$\frac{\frac{4}{9}}{\frac{2}{3}} \times \frac{\frac{3}{2}}{\frac{3}{2}}$$

EXAMPLE 3.21: ***Complex Fractions (Continued)***

Recall that the reciprocal of a number is the multiplicative inverse of the number: Simply "flip" the number over.

Since $\frac{\frac{3}{2}}{\frac{3}{2}}$ equals 1, the value of the complex fraction is not changed. Only its form is different.

The denominator now becomes 1, as $\frac{2}{3}\times\frac{3}{2}=\frac{6}{6}=1$.

Thus, $\frac{\frac{4}{9}\times\frac{3}{2}}{1}=\frac{\frac{12}{18}}{1}=\frac{\overset{2}{\cancel{12}}}{\underset{3}{\cancel{18}}}=\frac{2}{3}$

EXERCISES

3.52 Simplify:

a) $\frac{\frac{7}{8}}{\frac{3}{4}}$ **b)** $\frac{\frac{3}{8}}{\frac{1}{2}}$ **c)** $\frac{\frac{2}{3}}{\frac{5}{6}}$ **d)** $\frac{\frac{1}{4}}{\frac{2}{3}}$

e) $\frac{\frac{13}{16}}{\frac{9}{16}}$ **f)** $\frac{\frac{5}{6}}{\frac{3}{4}}$ **g)** $\frac{5}{\frac{3}{4}}$ **h)** $\frac{\frac{9}{32}}{\frac{4}{5}}$

i) $\frac{4}{\frac{3}{16}}$ **j)** $\frac{\frac{31}{64}}{\frac{1}{4}}$ **k)** $\frac{\frac{\frac{4}{2}\times\frac{1}{3}}{3}}{\frac{2}{3}\times\frac{3}{4}\times\frac{5}{6}}$ **l)** $\frac{3\frac{1}{3}\times5\frac{3}{8}}{8\frac{1}{8}\times4\frac{2}{3}}$

3.53 Determine the center-to-center distance of the equally spaced holes in Figure 3.13. The dimensions are in inches.

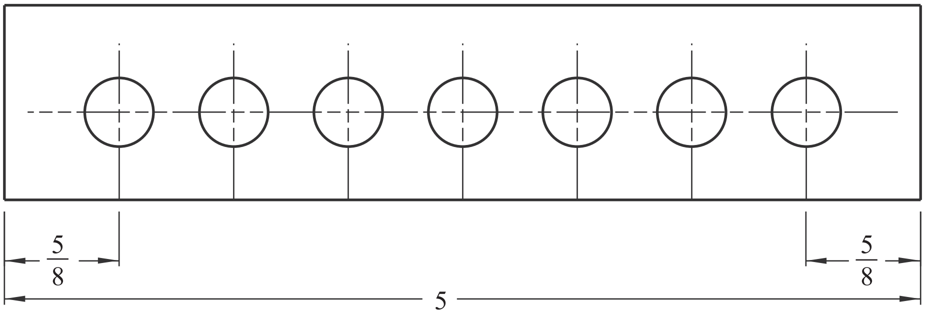


FIGURE 3.13

4

DECIMAL FRACTIONS

In Chapter 3 we discussed common fractions. This chapter deals with decimal fractions, commonly called decimals. Numbers that fall between any two whole numbers, such as 0.42 and 17.001, are decimal fractions. Whole numbers are also decimal fractions; their decimal places contain zeros, which are often not shown, unless they are needed to imply a certain degree of accuracy in a measurement. For example, 3.00 cm implies greater accuracy than simply 3 cm written without the zeros.

Notice that a whole number is both a decimal fraction and an improper fraction whose denominator is one. For example, the whole number 5 can be thought of as 5.0 or $\frac{5}{1}$.

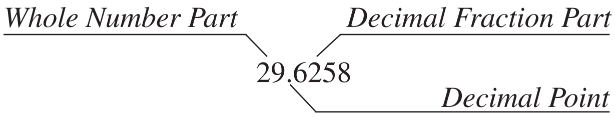
The use of decimals in machine details and assembly drawings came into wide use during World War I. The automobile, aircraft, and machine tool industries pioneered the use of the decimal system in American manufacturing. Some industries, for example, the building trades, still give dimensions in common fractions but these fractions usually cannot communicate the accuracy needed in tooling and machine use. Decimal fractions are necessary for these jobs and in the majority of science and engineering computations.

4.1 Meaning of a Decimal Fraction

As we saw in Chapter 1, Arabic numerals are built on the base 10 number system. We also call the Arabic number system the decimal system from the Latin word *decimus* meaning "tenth."

dec·i·mal *(dès´e-mel) Medieval Latin decimâlis, of tenths or tithes, from Latin decima, a tenth part or tithe, from decem, ten.*

In the decimal system, whole numbers are written to the left of a period called the *decimal point* and decimal fractions are written to the right of the decimal point. Consider the number 29.6258.



Although the denominator is not shown in a decimal fraction, we can readily figure out what it is by counting the number of digits to the right of the decimal point. The denominator increases by a power of 10 for each digit to the right of the decimal point. For example,

$$0.3 = \frac{3}{10} \qquad 0.02 = \frac{2}{100} \qquad 0.252 = \frac{252}{1000} \qquad 0.3946 = \frac{3946}{10{,}000}$$

Decimal place values correlate to equivalent common fractions. Thus,

TABLE 4.1: Equivalent Common and Decimal Fractions

Common Fraction	Decimal Fraction	Spoken As
1	1.0	one
$\frac{1}{10}$	0.1	one tenth
$\frac{1}{100}$	0.01	one hundredth
$\frac{1}{1000}$	0.001	one thousandth
$\frac{1}{10{,}000}$	0.0001	one ten-thousandth
$\frac{1}{100{,}000}$	0.00001	one hundred-thousandth
$\frac{1}{1{,}000{,}000}$	0.000001	one millionth

Reading a Decimal

To read a decimal fraction simply read the number as if it were a whole number followed by the name of the decimal place held by the last digit. Thus, 0.67 is read *sixty-seven* ***hundredths***, 0.08 is read *eight* ***hundredths***, and 0.104 is read *one hundred four* ***thousandths***.

When a decimal number has both whole number and a fractional parts, the decimal point is read as the word *and*. Thus the number 6.4 is read *six* ***and*** *four tenths.*

EXAMPLE 4.1: ***Reading Decimal Numbers***

Write the indicated decimal fraction as it would be spoken:

99.35	ninety-nine and thirty-five hundredths
28.462	twenty-eight and four hundred sixty-two thousandths

EXAMPLE 4.1: ***Reading Decimal Numbers (Continued)***

17.009	seventeen and nine thousandths
0.3895	three thousand eight hundred ninety-five ten-thousandths

Decimal fractions and common fractions are equivalent expressions of the same quantity. Whether one or the other is used depends on a job's specifications. In the next section we will discuss how to convert between the two systems.

Sometimes machinists refer to ten-thousandths as tenths. When tenths is spoken, make sure to find out if ten-thousandths is meant.

EXERCISES

4.1 Write the indicated decimal fraction as it would be spoken.

a) 3.46 **b)** 0.0693 **c)** 425.356
d) 53.963 **e)** 237.2562

4.2 Express the written number in numerals.

a) two hundred and eleven thousandths
b) two hundred sixteen and one ten-thousandth
c) five hundred twenty-five and three hundred nineteen ten-thousandth
d) three and fifteen millionths
e) eighty thousandths

4.2 Converting Common Fractions to Decimal Fractions

The method for converting a common fraction to its decimal equivalent is based on the fact that a fraction signifies division. Carrying out the division of a numerator by a denominator results in a decimal fraction. Remember that a fraction signifies division.

For example, to convert the common fraction $\frac{7}{8}$ to its decimal fraction equivalent, simply divide 7 by 8 to get 0.875. A good way to visualize this is to say that the fraction is "toppled over" to give the long division of 7 by 8:

$$\frac{7}{8} \Rightarrow 8\overline{)7.000}$$

$$\begin{array}{r} 0.875 \\ 8\overline{)7.000} = 0.875 \\ -\underline{64} \\ 60 \\ -\underline{56} \\ 40 \\ -\underline{40} \\ 0 \end{array}$$

Example 4.2: *Converting a Common Fraction to a Decimal Fraction*

Convert the common fraction $\frac{9}{16}$ to a decimal fraction.

Solution:

$$\frac{9}{16} \Rightarrow 16\overline{)9.0000}$$

$$\begin{array}{r} 0.5625\phantom{{}=0.5625} \\ 16\overline{)9.0000} = 0.5625 \\ \underline{-8\,0}\phantom{000{}=0.5625} \\ 100\phantom{00{}=0.5625} \\ \underline{-96}\phantom{00{}=0.5625} \\ 40\phantom{0{}=0.5625} \\ \underline{-32}\phantom{0{}=0.5625} \\ 80\phantom{{}=0.5625} \\ \underline{-80}\phantom{{}=0.5625} \\ 0\phantom{{}=0.5625} \end{array}$$

Converting fractions with denominators of 10, 100, 1000, and so on is easily accomplished by simply moving the numerator's decimal point to the left as many places as there are zeros in the denominator.

For example, $\frac{234}{1000} = 0.234$.

Notice that because the denominator of the fraction is 1000, the last place filled is the *thousandths* place. If the denominator had been 10,000, then the last place filled would have been the *ten-thousandths* place, etc.

In some cases zeros must be inserted so that the final decimal fraction is correct. Consider $\frac{25}{1000} = 0.025$.

In this case the decimal point in the numerator moves three places to the left, even though only two places seem available. Inserting a zero before the first digit resolves the apparent difficulty. Likewise, if the denominator had been 10,000 then 2 zeros would have been inserted to yield 0.0025.

.

Example 4.3: *Converting a Mixed Number to a Decimal Fraction*

Convert $5\frac{7}{10,000}$ to a decimal fraction.

Solution: The whole number part of the mixed number remains.

$$5\frac{7}{10,000} = 5.0007$$

In this problem, three zeros are inserted to make the last decimal place ten-thousandths.

The decimal fraction equivalents for many common fractions, such as those given in Table 4.2, are worth memorizing.

TABLE 4.2: Decimal Fraction Equivalents

$\frac{1}{2} = 0.5$

$\frac{1}{3} = 0.333...$ $\frac{2}{3} = 0.666...$

$\frac{1}{4} = 0.25$ $\frac{2}{4} = 0.5$ $\frac{3}{4} = 0.75$

$\frac{1}{5} = 0.20$ $\frac{2}{5} = 0.4$ $\frac{3}{5} = 0.6$ $\frac{4}{5} = 0.8$

$\frac{1}{8} = 0.125$ $\frac{2}{8} = \frac{1}{4} = 0.25$ $\frac{3}{8} = 0.375$ $\frac{4}{8} = \frac{1}{2} = 0.5$ $\frac{5}{8} = 0.625$ $\frac{6}{8} = \frac{3}{4} = 0.75$ $\frac{7}{8} = 0.875$

EXERCISES

4.3 Convert the indicated common fractions to decimal fractions.

a) $\frac{3}{8}$ $\frac{16}{64}$ $\frac{4}{100}$ $\frac{25}{125}$ $\frac{1}{25}$ $5\frac{3}{64}$

b) $\frac{3}{4}$ $\frac{5}{8}$ $\frac{3}{16}$ $\frac{1}{4}$ $\frac{11}{32}$ $\frac{7}{8}$ $\frac{21}{64}$

c) $\frac{1}{3}$ $\frac{2}{3}$ $\frac{7}{9}$ $\frac{3}{13}$ $\frac{15}{32}$ $\frac{25}{64}$ $\frac{37}{64}$ $\frac{19}{32}$

4.4 Convert the indicated numbers to decimal fractions.

a) $3\frac{7}{8}$ **b)** $6\frac{15}{64}$ **c)** $2\frac{11}{64}$ **d)** $9\frac{17}{32}$

e) $\frac{77}{64}$ **f)** $6\frac{3}{8}$ **g)** $\frac{3}{64}$ **h)** $12\frac{5}{8}$

4.3 Converting Decimal Fractions to Common Fractions

The method for converting a decimal fraction to its common fraction equivalent is based on the fact that the implied denominator of a decimal fraction is always a power of 10. Since the implied denominator of a decimal fraction is always a multiple of 10, the only possible choices for common factors are 2, 5, and their multiples.

Decimal fractions are converted to common fractions by first writing the decimal with its correct tenth, hundredth, thousandth, etc., denominator. Then the fraction is reduced, if possible, by dividing both numerator and denominator by a common factor.

EXAMPLE 4.4A: ***Converting a Decimal Fraction to a Common Fraction***

Convert the decimal fraction 5.068 to a common fraction.

Solution: Observing that the place value of the last digit in the decimal fraction is *thousandths*, we can rewrite the decimal fraction as $5\frac{68}{1000}$.

Noting that $\frac{68}{1000}$ can be reduced, our final answer becomes

$$5\frac{68}{1000} = 5\frac{\cancel{4}\cdot 17}{\cancel{4}\cdot 250} = 5\frac{17}{250}$$

Fractions are reduced by dividing both numerator and denominator by common factors until none remain.

EXAMPLE 4.4B: ***Converting a Decimal Fraction to a Common Fraction***

Convert the decimal fraction 0.28 to a common fraction.

Solution: Zero and twenty-eight ***hundredths*** $0.28 = \frac{28}{100} = \frac{7\cdot\cancel{4}}{25\cdot\cancel{4}} = \frac{7}{25}$

EXAMPLE 4.4C: ***Converting a Decimal Fraction to a Common Fraction***

Convert 18.050 to a common fraction.

Solution: Eighteen and fifty ***thousandths*** $\rightarrow\ 18\frac{50}{1000} = 18\frac{\cancel{50}\cdot 1}{\cancel{50}\cdot 20} = 18\frac{1}{20}$

Notice that 18.050 = 18.05.

However, the zero in the thousandths place implies an accuracy greater than if the number were written with the 5 in the hundredths place as the last digit.

EXAMPLE 4.5: ***Converting a Decimal Fraction with Many Decimal Places***

Convert 3.46875 to a common fraction.

Solution: Three and forty-six thousand eight hundred seventy-five *hundred thousandths* $\rightarrow 3\frac{46,875}{100,000}$.

To reduce this fraction, cancel common factors until no pair remains. Looking at just the fractional part of the mixed number:

$$\frac{46,875}{100,000} = \frac{\cancel{25}\cdot 1875}{\cancel{25}\cdot 4000} = \frac{\cancel{25}\cdot 75}{\cancel{25}\cdot 160} = \frac{\cancel{5}\cdot 15}{\cancel{5}\cdot 32} = \frac{15}{32}$$

Now add the fraction to the whole number: $3 + \frac{15}{32} = 3\frac{15}{32}$

Sometimes a repeating decimal has to be converted to a common fraction. Many repeating decimals are forms of familiar fractional equivalents, for example, 0.333…, which has a common fraction equivalent of $\frac{1}{3}$.

Repeating digits often occur in the decimal fraction form of many common fractions. Such numbers are rounded to whatever accuracy is desired. Rounding is discussed later in this chapter.

Other repeating decimals, such as 0.111…, are not as familiar. The way to convert such decimals relies on algebra and is discussed later.

EXERCISES

4.5 Change the indicated decimal fraction to a common fraction.

a) 0.8 **b)** 0.04 **c)** 0.375
d) 1.27 **e)** 15.007 **f)** 6.415
g) 0.09 **h)** 121.875 **i)** 0.0045
j) 8.06 **k)** 2.75 **l)** 18.008

4.6 Change the indicated decimal fraction to a common fraction.

a) 0.015625 **b)** 6.4375 **c)** 2.859375
d) 0.8125 **e)** 3.750 **f)** 0.125
g) 5.96875 **h)** 0.40625 **i)** 301.0
j) 77.036 **k)** 3.333 **l)** 0.003175

4.4 Addition and Subtraction of Decimal Fractions

Decimal fractions are much easier to add and subtract than common fractions. Manual addition of decimal fractions is accomplished by writing addends with decimal points aligned and proceeding as with addition of whole numbers.

EXAMPLE 4.6: ***Adding Decimal Fractions***

$$\begin{array}{r} 25.327 \\ 0.0672 \\ 0.0032 \\ +1.6 \\ \hline 26.9974 \end{array} \qquad \begin{array}{r} 0.00002 \\ 12.001 \\ 1.6003 \\ +2.0202 \\ \hline 15.62152 \end{array} \qquad \begin{array}{r} 2.635 \\ 4.5734 \\ 0.0032 \\ +1.4837 \\ \hline 8.6953 \end{array}$$

When a whole number is given without the decimal point, be careful to write the number with its decimal point to the right rather than the left of the number. Thus, 3 = 3. as a decimal.

Subtraction of decimals is set up exactly as addition of decimals, aligning decimal points first.

EXAMPLE 4.7: ***Subtracting Decimal Fraction***

$$\begin{array}{r} 16.3715 \\ -12.2504 \\ \hline 4.1211 \end{array} \qquad \begin{array}{r} \overset{8}{\cancel{9}}.6\overset{1}{\cancel{2}}73 \\ -8.7183 \\ \hline 0.9090 \end{array} \qquad \begin{array}{r} 12.\overset{8}{\cancel{9}}\overset{6}{\cancel{7}}\overset{4}{\cancel{5}}3 \\ -6.4876 \\ \hline 6.4877 \end{array}$$

EXERCISES

4.7 Add:

a) 0.762 and 0.883
b) 1.192 and 0.035
c) 1.234 and 0.567
d) 4.62 and 0.0025

4.8 Subtract:

a) 2.765 from 5.326
b) 10.0234 from 11.0000

4.9 Perform the indicated operations:

a) $1.2634 + 2.6379 + 1.23 + 0.6324$
b) $2.678 - 1.032 - 0.275 - 0.659$
c) $\frac{3}{8} - 0.03125$
d) $3\frac{1}{8} - 0.125$
e) $8\frac{3}{16} + 0.6875$
f) $4.3986 + 2\frac{11}{16} - 0.736$
g) $1.7986 + 2.3333 + 3.4567 + 1.0012$
h) $0.666 + 0.334 + 1.5 - 2.25$
i) $12.653 - \frac{15}{16}$
j) $3.625 - 2\frac{7}{32}$
k) $6\frac{15}{32} + 2\frac{7}{24} - 0.90625$
l) $\frac{7}{16} - \frac{3}{32}$

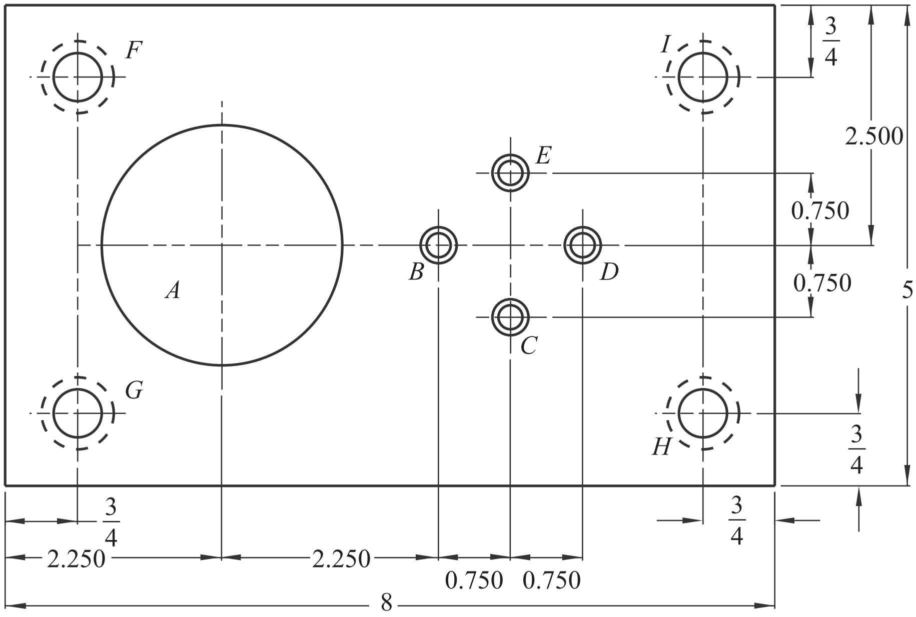


FIGURE 4.1 ***Punch plate detail (dimensions in inches)***

4.10 In Figure 4.1, centers *A–H* are shown. Calculate the distances between these centers:

a) *A* to *B*
b) *A* to *D*
c) *E* to *C*
d) *B* to *D*
e) *H* to *I*
f) *F* to *G*

4.11 In Figure 4.1, calculate the distances from the left side of the punch plate to the centers of the following holes:

a) B
b) *C* and *E*
c) D
d) *H* and *I*

4.12 In Figure 4.1, calculate the distances from the upper side of the punch plate to the centers of the following holes:

a) E
b) *A, B* and *D*
c) C
d) *G* and *H*

4.13 Perform the indicated operations:

a)
$$\begin{array}{r} 236.02 \\ 62.34 \\ 19.10 \\ +221.62 \\ \hline \end{array}$$

b)
$$\begin{array}{r} 102.632 \\ 1.023 \\ 3.142 \\ +10.001 \\ \hline \end{array}$$

c)
$$\begin{array}{r} 14.16 \\ 71.23 \\ 45.11 \\ +0.015 \\ \hline \end{array}$$

4.14 Express as decimals and add:

a) $9\frac{3}{4}+2.19+1\frac{1}{2}+3.33+2\frac{7}{8}$

b) $3\frac{2}{3}+1.33+9.60+2\frac{1}{8}+3.875$

c) $10.5+\frac{1}{4}+\frac{5}{8}+\frac{1}{8}+1.375$

d) $19.78+1\frac{5}{8}+2\frac{1}{3}+6.75+3\frac{2}{3}$

4.15 Subtract:

a) 0.6239 from 1.0000
b) 8.23 – 2.72
c) $10.5-0.5275$
d) $9.375-8.125$

4.16 Perform the indicated operations:

a) $9.62-2.76-3.67-0.23$
b) $12.36+4.68-1.23-6.74$

4.5 Multiplication of Decimal Fractions

The method of multiplying decimal fractions is the same as the multiplication of whole numbers, except the position of the decimal point in the product must be considered.

Procedure for determining the location of the decimal point

Step 1: Count the number of digits to the right of the decimal in the first number.

Step 2: Count the number of digits to the right of the decimal in the second number.

Step 3: Add the results of steps 1 and 2 for the total number of digits to the right of the decimal point in the product. Place the decimal point accordingly.

"Decimal places" means digits to the right of the decimal.

Example 4.8a: *Multiplying Decimal Fractions*

Multiply 3.25 by 0.125

Solution:

$$\begin{array}{r c l}
3.25 & \rightarrow & \text{2 decimal places} \\
\times\,\underline{0.125} & \rightarrow & \text{3 decimal places} \\
1625 & & \\
6500 & & \\
\underline{32500} & & \\
0.40625 & \rightarrow & 2+3=5 \text{ decimal places}
\end{array}$$

The zeros shown in blue are placeholders used to help line up the columns of numbers

Example 4.8b: *Multiplying Decimal Fractions*

Multiply 8.33 by 0.0025

Solution:

$$\begin{array}{r c l}
8.33 & \rightarrow & \text{2 decimal places} \\
\times\underline{0.0025} & \rightarrow & \text{4 decimal places} \\
4165 & & \\
\underline{16660} & & \\
0.020825 & \rightarrow & 2+4=6 \text{ decimal places}
\end{array}$$

EXERCISES

4.17 Perform the indicated multiplication:

a) 59375.0 × 0.00482

b) 49.579 × 3.1416

c) 0.4967 × 5.478

d) 0.587 × 0.472 × 0.068

e) 0.4792 × 0.0468

f) 23.56 × 5.73 × 23

g) 0.00367 × 4973.22 × 0.473

h) 0.000056 × 0.0059

i) 0.0045 × 0.00067

j) 46.36 × 0.10405

k) 9.637 × 0.00073

l) 19.10 × 3.1

m) 2.35 × 1.25

n) 9.12 × 6.3 × 10.2

4.6 Division of Decimal Fractions

The division of decimal fractions is done in the same way as the division of whole numbers, but with special preparation of the position of the decimal points in the denominator and the numerator.

Multiplying both the numerator and denominator by the same number is equivalent to multiplying by 1.

Consider the problem $\dfrac{92.3647}{8.654}$.

To carry out this division, the denominator must be changed to a whole number. This can be accomplished by multiplying both the numerator and denominator by 1000 since the last decimal place in the denominator is *thousandths*. In effect, we simply move the decimal point in both the denominator and numerator three places to the right.

Therefore, the problem becomes $\dfrac{92364.7}{8694}$ and is solved in Example 4.9A.

EXAMPLE 4.9A: ***Division of Decimal Fractions***

Perform the following division: $\dfrac{92.3647}{8.694}$

STEP 1: Rewrite in long division format: $8.654\overline{)92.3647}$

STEP 2: Multiply both numerator and denominator by 1000 so that the denominator becomes a whole number.

STEP 3: Proceed with long division, maintaining the position of the decimal point.

$$
\begin{array}{rrl}
 & 10.673 & \\
8654 & \overline{)92364.700} & \\
 & -\underline{8654} & \\
 & 58247 & \\
 & -\underline{51924} & \\
 & 63230 & \\
 & -\underline{60578} & \\
 & 26520 & \\
 & -\underline{25962} & \\
 & 558 & \text{remainder}
\end{array}
$$

Notice the remainder. Observe that we can continue dividing to as many decimal places as are desired by adding zeros to the remainder and continuing the long division process. Sometimes, if the division is carried out far enough, the remainder eventually becomes 0.

A useful method for learning division of decimals is shown in Example 4.9B.

A practical method for setting up decimal division problems is to count the number of digits to the right of the decimal point in the denominator and then move the decimal point the same number of places to the right. Then count off the same number to the right of the decimal point in the numerator.

EXAMPLE 4.9B: *Division of Decimal Fractions*

Notice the differences among the problems and estimate the answer before looking at the solution.

a) $20 \div 5$

b) $20 \div 0.5$

c) $2 \div 5$

d) $0.2 \div 5$

e) $20 \div 0.05$

f) $0.02 \div 5$

g) $2 \div 0.005$

h) $0.02 \div 0.5$

Solution:

a) $\begin{array}{r} 4 \\ 5\overline{)20} \end{array}$

b) $0.5\overline{)20} \rightarrow \begin{array}{r} 40. \\ 5.\overline{)200.} \end{array}$

c) $\begin{array}{r} 0.4 \\ 5\overline{)2.0} \end{array}$

d) $\begin{array}{r} 0.04 \\ 5\overline{)0.20} \end{array}$

e) $0.05\overline{)20} \rightarrow \begin{array}{r} 400. \\ 5.\overline{)2000.} \end{array}$

f) $\begin{array}{r} 0.004 \\ 5\overline{)0.020} \end{array}$

g) $0.005\overline{)2.0} \rightarrow \begin{array}{r} 400. \\ 5.\overline{)2000.} \end{array}$

h) $0.5\overline{)0.020} \rightarrow \begin{array}{r} 0.04 \\ 5.\overline{)0.20} \end{array}$

EXAMPLE 4.10: *Long Division of Decimal Fractions*

Perform the following division: $0.42\overline{)\ 24.136}$

Solution: To make the denominator a whole number, multiply both divisor and dividend by 100 (that is, move the decimal point two places to the right on both numbers).

$$42\overline{\big|2413.6}$$

Notice that the zero no longer needs to be written. Proceed with long division, maintaining the position of the decimal point:

$$\begin{array}{r}
\overline{57.46} \\
42\big|2413.6 \\
-\underline{210} \\
313 \\
-\underline{294} \\
196 \\
-\underline{168} \\
280 \\
-\underline{252} \\
28 \quad \text{remainder}
\end{array}$$

EXERCISES

4.18 Perform the indicated operation:

a) $5739 \div 0.0946$
b) $4857 \times 0.00369 \div 0.0005$
c) $8.694 \div 4.968 \times 3.475$
d) $472849 \div 0.0000065$
e) $56.0052 \div 0.00047$
f) $38.75 \times 0.0051 \div 0.075$
g) $4.6948 \div 0.0055 \times 5.793$
h) $10.62 \div 2.31$
i) $325.4 \times 45.68 \div 929.017$
j) $25.15 \div 2.6$
k) $87.3495 \div 0.0045$

4.19 Divide after taking product as indicated:
a) $(6.73 \times 2.69) \div 3.66$
b) $(2.93 \times 15.63 \times 19.2) \div (33.1 \times 1.75)$

4.20 Perform the indicated operations. Pay attention to the rules for order of operations.

a) $22.34 \times 3.56 \div 2.63 + 4.75 - 2.38 \times 5.83$
b) $4\frac{3}{8} + 2\frac{5}{16} + 3.587 \times 2\frac{7}{8}$

4.7 Place Value and Rounding

Common Rounding

In many applications where precision is not critical, rounding is accomplished simply by looking at the value of the digit immediately to the right of where the rounding is to occur and applying the following two rules.

- If the value is 0, 1, 2, 3, or 4 then truncate at the point of rounding.
- If the value is 5, 6, 7, 8, or 9 then increase by 1 the value of the digit at the point of rounding.

Rules for Common Rounding

EXAMPLE 4.11A: ***Common Rounding***

Round the number 147.3325689 to the *hundredths* place.

Solution: Since the value in the *thousandths* place is 2, truncate all the digits to the right of the *hundredths* place to get 147.33.

EXAMPLE 4.11B: ***Common Rounding***

Round the number 147.3325689 to the *thousandths* place.

Solution: Since the value in the *ten-thousandths* place is 5, increase the value of the *thousandths* place by 1 and truncate the rest to get 147.333.

EXAMPLE 4.11C: ***Common Rounding***

Round the number 147.3325689 to the *tens* place.

Solution: Since the value in the *ones* place is 7, increase the value of the *tens* place by 1 and truncate the rest to get 150.

Use zeros as placeholders if necessary. When rounding 38.98 to the tenths place, the answer should be written 39.0, not simply 39. The zero in the tenths place indicates that the number was rounded to the tenths place. Simply writing the 39 without the decimal point and zero indicates that the number was rounded to the ones place.

EXAMPLE 4.11D: ***Common Rounding***

Round the number 147.3325689 to the *hundreds* place.

Solution: Since the value in the *tens* place is 4, truncate all the digits to the right of the *hundreds* place to get 100.

EXAMPLE 4.11E: ***Common Rounding***

Round the number 147.3325689 to the *ones* place.

Solution: Since the value in the *tenths* place is 3, truncate all the digits to the right of the *ones* place to get 147.

Specialized Rounding

Some situations in the shop require highly accurate measurements, others less accurate.

Rounding is sometimes referred to as "rounding off."

Whenever a measurement is made, a number is *rounded* to an approximation of the actual value based on the precision of the measuring instrument. The preciseness of a measurement is referred to in terms of *significant digits*.

The rules of rounding are described in terms of most significant digit, least significant digit, and non-significant digits.

The leftmost non-zero digit of a measurement is the *most significant* digit. Leading zeros serve as placeholders and are never considered significant digits.

The *least significant* digit is the last digit in the rounded number. The least significant digit can be a zero. The first *non-significant* digit is the first digit after the point where rounding will occur.

The following rules apply to specialized rounding of numbers.

Rule 1

If the first non-significant digit is less than 5, then the least significant digit remains unchanged.

Rule 2

If the first non-significant digit is greater than 5, the least significant digit is increased by 1 (incremented by 1).

Rule 3

If the first non-significant digit is 5, the least significant digit is either incremented or left unchanged depending on whether the second non-significant digit is odd or even. If odd, increment the least significant digit by 1. If even, leave it unchanged.

Rule 4

All non-significant digits are removed.

Rule 3 does not really matter if one is only rounding a single number. However, when rounding a series of numbers that will be used in a calculation, Rule 3 prevents accumulating round-off errors and thereby overstating or understating the value of the rounded number.

Rule 3 is based on the fact that there are equal numbers of even and odd numbers. Incrementing only for the odd digits insures that overstating and understating are equally probable rounding errors..

EXAMPLE 4.12: *Rounding with First Non-Significant Digit Less than 5*

Round 53.5217 to 3 significant figures.

Solution: The first non-significant digit is the 2 in the hundredths place. Since it is less than 5, Rule 1 applies and the 5 in the tenths place remains.

The final result is 53.5.

EXAMPLE 4.13: *Rounding with First Non-Significant Digit Greater than 5*

Round 53.5217 to 5 significant figures.

Solution: The first non-significant digit is the 7 in the ten-thousandths place. Since it is greater than 5, Rule 2 applies and the 1 in the thousandths place is incremented by 1.

The final result is 53.522.

EXAMPLE 4.14A: *Rounding with First Non-Significant Digit Equal to 5*

Round 53.5217 to 2 significant figures.

Solution: The first non-significant digit is the 5 in the tenths place. Since it is equal to 5, Rule 3 applies and the 3 in the ones place is left unchanged because the second non-significant digit, 2, is even.

The final result is 53.

EXAMPLE 4.14B: *Rounding with First Non-Significant Digit Equal to 5*

Round 92.572 to 2 significant figures.

Solution: The first non-significant digit is the 5 in the tenths place. Since it is equal to 5, Rule 3 applies and the 2 in the ones place is incremented by 1 because the second non-significant digit, 7, is odd.

The final result is 93.

EXAMPLE 4.15: *Rounding with Leading Zeros*

Round 0.00347 to 2 significant figures.

Notice that since leading zeros are not considered significant figures, the first significant digit is the 3 in the thousandths place.

Solution: The first non-significant digit is the 7. Since it is greater than 5, Rule 2 applies and the 4 is incremented by 1.

The final result is 0.0035.

Rounding errors occur when a rule is misapplied or ignored. They also occur when a series of operations are done and intermediate answers are rounded before a final answer is found. Approximations of intermediate answers, for example, $4.51 + 2.176 = 6.686$, might cause the final answer to be off by at least one place.

If this result is rounded to three significant figures to give 6.69 and then added to 3.125, the answer is 9.815, which in turn rounds to 9.82. However, if the intermediate answer is not rounded, the final answer would be $6.686 + 3.125 = 9.811$, which rounds to 9.81.

EXERCISES

4.21 Using the rules for common rounding, round each number to the specified place:

a) 32.15 (tenths place)
b) 992.6 (ones place)
c) 0.00245 (thousandths place)
d) 15.259 (hundredths place)
e) 3256.3 (tens place)
f) 1.39732 (ten-thousandths place)
g) 635.2219 (hundredths place)
h) 949.31 (hundreds place)
i) 18.3357 (thousandths place)
j) 0.3217 (ones place)

4.22 Using the rules for specialized rounding, round each number to the specified number of significant digits:

a) 0.467494 to four, three, and two significant figures

b) 57.8947 to two, three, and five significant figures

c) 5.55555 to three, two, and one significant figure

d) 0.00593875 to five significant figures

e) 0.0059532 to four significant figures

f) 4567.535 to four significant figures

g) 4567.835 to two significant figures

h) 45.67352 to three significant figures

i) 82.68499 to two significant figures

j) 0.77773 to three significant figures

4.23 Figure 4.2 shows a typical gauging setup for measuring the center distance of small holes. Gage pins are fitted in the holes and their surfaces measured in relation to the surface plate. Applying the rules for common rounding and giving the answer in thousandths of inches, solve for the center distance, $D_c = M_1 - M_2 + \frac{D_1}{2} - \frac{D_2}{2}$ when

a) $M_1 = 1.125$ inches, $M_2 = 0.2188$ inch, $D_1 = 0.625$ inch and $D_2 = 0.500$ inch.

b) $M_1 = 0.9688$ inch, $M_2 = 0.0781$ inch, $D_1 = 0.4844$ inch, and $D_2 = 0.2813$ inch.

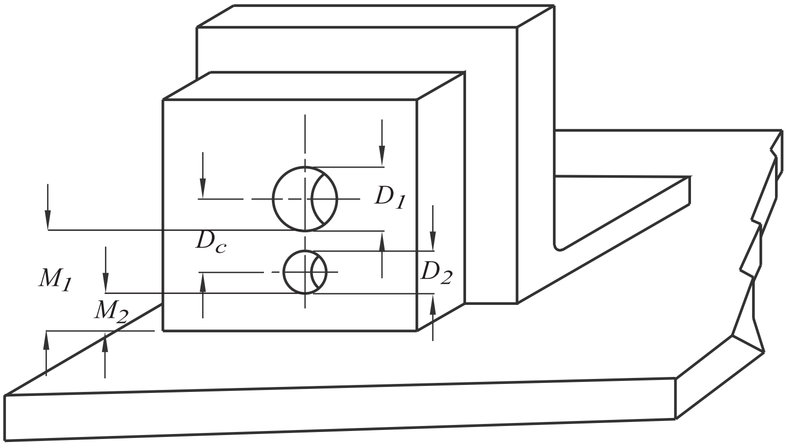


FIGURE 4.2 ***Gauging set-up***

4.8 Measurement Arithmetic

Addition and Subtraction of Measurements

A sum or difference of measurements is only as precise as the *least precise* measurement. Consider Example 4.16. Since the least precise addend is 3.1, the answer must be rounded to the tenths place.

For sum or difference of measurements, express the answer to as many decimal places as the least precise figure being added or subtracted.

Addition of Measurements

EXAMPLE 4.16: ***Addition of Measurements***

Add the measurements and round to the appropriate place.

$$\begin{array}{r} 6.34 \\ 5.192 \\ 3.1 \\ \underline{+\ 8.9658} \\ 23.5978 \approx 23.6 \end{array}$$

Solution: Since the number 3.1 is the least precise of the numbers being added, it determines where the result should be rounded. Accordingly, the answer is rounded to the tenths place.

Notice that if each measurement had been recorded to the nearest ten-thousandth place, the answer would be given as 23.5978:

$$\begin{array}{r} 6.3400 \\ 5.1920 \\ 3.1000 \\ \underline{+\ 8.9658} \\ 23.5978 \end{array}$$

Multiplication and Division of Measurements

A product or quotient of measurements is only as accurate as the *least accurate* measurements given, due to *error compounding*. This means that the result must have only as many significant digits as the least accurate measurement involved. For example, if each side of a square measures 13.2 inches, its area is 13.2 × 13.2 = 174 square inches. The answer is given to three significant digits, like the data.

Multiplication or Division of Measurements

For product or quotient of measurements, express the answer to as many significant digits places as the least accurate figure being added or subtracted.

Example 4.17: ***Multiplication of Measurements***

Find the area of a rectangle 12.31 cm long by 1.05 cm wide.

Solution:

The answer will be rounded to three significant digits, since the least accurate dimension is 1.05 cm and it only has three significant digits. Using area of a rectangle = length × width:

$$\text{Area} = 12.31\text{ cm} \times 1.05\text{ cm} = 12.9255\text{ square cm} \cong 12.9\text{ square cm}$$

Measurement Error

Measurement error is due to human error in reading an instrument or in handling the piece being measured. It also arises due to inaccurate calibration of the measuring device itself.

A *positive error* is one in which a stated figure is higher than the actual measurement. A *negative error* is one in which a stated figure is lower than the actual measurement. Any difference between actual value and measured value, positive or negative, is called *absolute error*.

The numbers 2500 and 2500.0 may mean the same thing to a person who is not working with measurement; but to a worker in one of the technical trades, the two numbers have very different meanings. The first conveys the idea that the accuracy of the number is known only with certainty as far as the hundreds place. The second number implies that the accuracy of the number is known only with certainty as far as the tenths place.

In machine detail dimensioning, for instance, dimensions and tolerances are often given to the thousandths place or finer. A measurement of $\frac{10}{1000}$ inch is written 0.010 inch, rather than 0.01 inch. Including the zero after the hundredths place shows accuracy to the thousandths place.

In the same manner, a measurement of $\frac{10}{10000}$ inch is written as 0.0100 inch, with two zeros after the hundredths place to show its accuracy to the ten-thousandths place.

A Word about Precision and Accuracy

Measurements are only *approximations* of a quantity's true value. Measurements can never be exact, but they may improve with the skill of the person using the device and with the precision of the measuring device.

The reproducibility of a measurement indicates its *precision*. If a quantity is measured several times, the preciseness of the measurement is reflected by how close the individual measurements are to one another. In other words, if each measurement is close to the average of all the measurements, the measurement has good reproducibility and is said to be precise. A wide dispersion among the measurements indicates poor precision. *Accuracy*, on the other hand, reflects how close a measurement is to a known or accepted value.

As an illustration of the meaning of precision and accuracy, consider a series of measurements of the freezing point of pure water at sea level using the same thermometer.

The true freezing point of pure water at sea level is known to be 0° C.

Measured data for the freezing point of pure water at sea level				
0.2°C	0.1°C	0.3°C	0.1°C	0.2°C

The measurements are said to be precise because the data appear to be reproducible with minimal variation from trial to trial. However, the measurement is not very accurate because the measurements are not very close to the known true value of 0° C. In fact, all of the measurements are above 0° C, which may indicate that the thermometer is not calibrated correctly, or that the thermometer is being misread, or that some other factor is influencing the result. Figure 4.3 depicts the meaning of precision and accuracy using a bull's eye analogy.

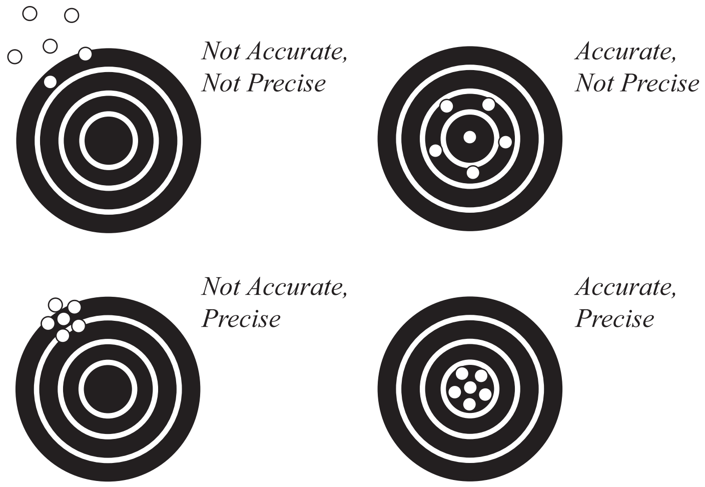


FIGURE 4.3 ***Bull's Eye Analogy for Precision and Accuracy***

EXERCISES

4.24 Add the measurements and round to the appropriate place:

a) $36.213 + 810.2 + 3.13 + 6.0001$ **b)** $102.91 + 336.231 + 429.331698$

4.25 Subtract the measurements and round to the appropriate place:

a) $196.42 - 91.34 - 18.1$ **b)** $2372.1 - 106.41 - 9.3122$

4.26 Multiply the measurements and round to the appropriate place:

a) 100.1×42 **b)** 9×9.1
c) 712.33125×819.31

4.27 Explain the difference of meaning between the words *accuracy* and *precision*.

4.28 Explain some of the causes of measurement errors.

4.9 Decimal Tolerances

Modern mass production depends on parts that may have been produced in different factories to fit together properly at the final assembly stage. Unfortunately, making every part to the exact size specified on the drawing is not possible due to the inherent limitations of measurement precision and accuracy. Parts can be made to very close dimensions, even to a few millionths of an inch, but the cost to achieve this standard for mass production is prohibitive.

The finer the tolerance, the greater the need to make accurate measurements using high precision instruments, resulting in higher production costs.

Fortunately, exact sizes are not needed if accuracy is related to functional requirements. The required accuracy determines the amount of *variation* permitted in the sizes of the parts. This amount of variation is called *tolerance*—the variation tolerated by production and assembly requirements. In many applications, such as the fit of a bolt through a clearance hole, great precision is not required and little attention is given to anything other than nominal dimensions. Where more precision is required, the detail drawing will call for the appropriate tolerances.

Formally, tolerance is the amount by which a given dimension may vary, or the difference between the *limits* of dimensions. In Figure 4.4 the diameter dimension of the hole is written $\begin{smallmatrix}1.251\\1.250\end{smallmatrix}$. The two numbers represent the *limits* of the tolerance dimension. They tell the machinist that the diameter of the hole may not be machined less than 1.250 inches nor greater than 1.251 inches. Likewise, the shaft diameter must not exceed 1.248 inches nor be smaller than 1.247 inches.

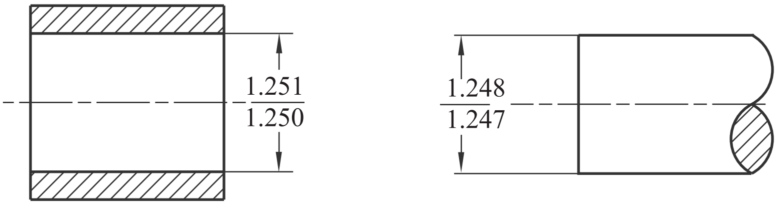


FIGURE 4.4 ***Limit dimensioning***

Tolerances are often expressed with the plus-or-minus symbol (±). For example, the dimension parameters in Figure 4.4 could be shown with tolerances rather than limits, with the hole dimensioned as $1.250^{+0.001}_{-0.000}$ and the shaft dimensioned as $1.248^{+0.000}_{-0.001}$. This gives the same information as in Figure 4.5.

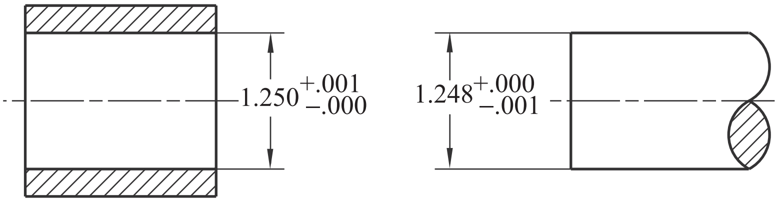


FIGURE 4.5 ***Tolerance dimensioning***

Typical examples of other tolerances are:

$6.122^{+0.002}_{-0.003}$, 0.500 ± .001, 0.8748 ± .0002, and $0.0938^{+0.0001}_{-0.0002}$.

A grinding operation with a tolerance of ±0.0002 inch may be checked by a Vernier micrometer reading to ten-thousandths of an inch. Lapping, which may require accuracies within a few millionths of an inch, would be checked with gage blocks in conjunction with other precise instruments.

A dimension with a tolerance of ± 0.001 *is read: "plus or minus one thousandth."*

Sometimes tolerances are given in fractional inches, such as $\pm\frac{1}{16}$ or $\pm\frac{1}{32}$. Tolerances calling for $\pm\frac{1}{64}$ may be measured by a rule or a micrometer.

Tolerance on a dimension must be properly called out so that there is no verbal misunderstanding in the shop. For example, $4.375^{+0.003}_{-0.000}$ is read *four and three hundred seventy-five thousandths plus three thousandths, minus nothing*.

EXAMPLE 4.18: *Limits and Total Tolerance*

A bored hole in a machine detail is to have a diameter between 2.750 and 2.756 inches. What are the limits? What is the *total* tolerance?

Solution:

The limits are the maximum and minimum permissible dimensions of the hole. Hence, the limits are 2.750 and 2.756 inches. The total tolerance is the total permissible variation in the size of the dimension. Hence, the total tolerance is 0.006 inch.

EXAMPLE 4.19: *Tolerance*

In the previous example, the basic size of the bored hole is that size from which the limits are obtained. It is sometimes considered the *ideal* size of the hole. If the ideal size of the hole is 2.753 inches, what is the tolerance?

Solution: With the basic size of the hole given as 2.753 inches, and the limits 2.750 to 2.756 inches, the tolerance is ±0.003 inch. The dimension is written with tolerances as 2.753 ±0.003 inch.

EXAMPLE 4.20: *Limits*

In Figure 4.6, the dimensions are in inches. Calculate the limits for dimension *A*.

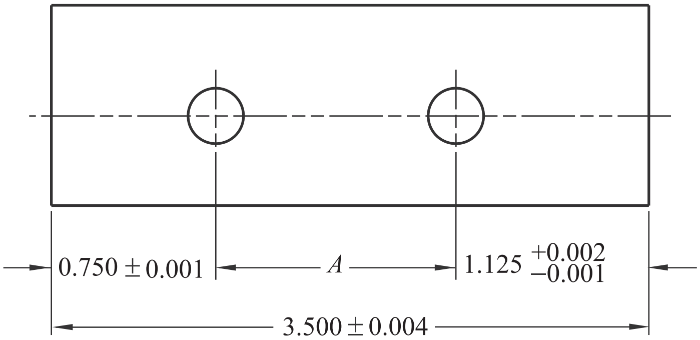


FIGURE 4.6

Solution: The limits are the maximum and minimum distances dimension *A* may have. Notice that distance *A* is equal to the sum of the hole-center dimensions subtracted from the overall dimension of the block.

Add the two hole-center dimensions, as shown:

$$\begin{array}{l} 0.750 + 0.001 - 0.001 \\ \underline{1.125 + 0.002 - 0.001} \\ 1.875 + 0.003 - 0.002 \end{array}$$

Notice that the minimum distance for dimension *A* occurs when the hole-center dimensions are at their maximums on a block cut to its minimum overall length. Likewise, the maximum distance for dimension *A* occurs when the hole-center dimensions are at their minimums on a block cut to its maximum overall length. Stack the tolerances accordingly and subtract the combined hole-center dimensions from the overall dimension as shown:

$$\begin{array}{l} 3.500 - 0.004 + 0.004 \\ \underline{1.875 + 0.003 - 0.002} \\ 1.625 - 0.007 + 0.006 \end{array}$$

Therefore the minimum distance for dimension *A* is:

$$1.625 - 0.007 = 1.618 \text{ inches}$$

and the maximum distance for dimension *A* is:

$$1.625 + 0.006 = 1.631 \text{ inches}$$

Precision Gage Blocks

Gage Measure

Sheet metal thickness, tubing wall thickness, and wire diameter are often specified in terms of *gage*. Several different systems of gage measurement are in use. Generally, larger gage values are associated with a thinner metal sheet or a smaller wire diameter. *Machinery's Handbook* offers several tables showing equivalent gage measurements in several different systems.

The word "gauge" and "gage" are interchangable. Both forms of the word are commonly used.

Various types of *gages* are used to ensure that assembly parts fit into their mating parts as designed. There are many kinds of gages, both hand and automated. Some of the more common hand gages are plug gages for hole sizes, thread gages, ring gages for shaft diameter checking, and taper gages. There are many more specialized types.

Gages must be made to the dimensions specified for the part being checked. Accuracy is achieved in the making of gages by the use of *precision gage blocks*. Precision gage blocks are small, rectangular steel blocks, heat-treated to a high degree of hardness with fine dimensional stability. They are made to have an extremely high dimensional accuracy. The most accurate blocks in a tool room are the *master blocks*, used only to check and calibrate the *working blocks*, which themselves are used in day-to-day checking operations. Gage blocks are assembled in groups. The sum of the dimensions of the individual blocks equals the required checking dimension. A set of gage block dimensions are provided in Sections A.13 and A.14 of the Appendix.

By combining two or more blocks an extremely accurate dimension can be obtained. In selecting blocks to obtain a given dimension, the least number of blocks possible for giving the required dimension should be used. Example 4.21 illustrates the process for setting up blocks.

Another common use of gage blocks is in conjunction with a *sine plate* as shown in Figure 4.7. The stack of blocks gives an accurate vertical measure for setting the sine plate to a precise angle.

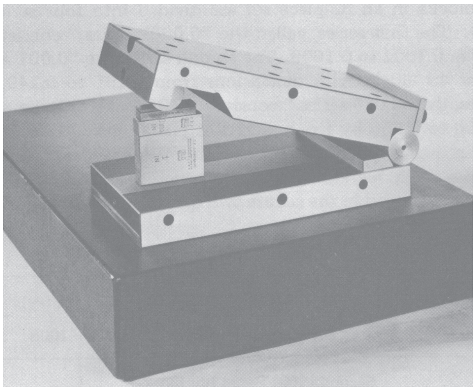


FIGURE 4.7 ***Using gage blocks with sine plate to measure angles***

EXAMPLE 4.21: ***Gage Block Assembly***

Assemble blocks to give 3.6742 inches. Refer to Appendix, A.13 and A.14, for gage block sizes.

Solution:

STEP 1: Eliminate the 0.0002 by selecting a 0.1002 block. Subtract 0.1002 from 3.6742.

$$\begin{array}{r} 3.6742 \\ \underline{-0.1002} \\ 3.5740 \end{array}$$

STEP 2: Eliminate the 0.004 by selecting a 0.124 block. Subtract 0.124 from the difference of Step 1:

$$\begin{array}{r} 3.5740 \\ \underline{-0.124} \\ 3.450 \end{array}$$

STEP 3: Eliminate the 0.450 with a 0.450 block. Subtract 0.450 from the difference of Step 2:

$$\begin{array}{r} 3.450 \\ \underline{-0.450} \\ 3.000 \end{array}$$

STEP 4: Use the 3.000 block for the final step. Assemble and check for required dimension:

$$\begin{array}{l} 0.1002 \\ 0.124 \\ 0.450 \\ \underline{+3.000} \\ 3.6742 \text{ inches} \end{array}$$

EXERCISES

4.29 Determine the gage blocks required to obtain stacks that are equal to 1.125 inches and 0.2188 inch. Sections A.13 and A.14 in the Appendix list available gage block sets.

4.30 Determine the gage blocks required to obtain stacks that are equal to 2.365 inches. Sections A.13 and A.14 in the Appendix list available gage block sets.

4.31 Calculate the depth of a roughing cut to which a cutting tool must be positioned if the initial diameter of a part is 1.000 inch, the final dimension is to be 0.875 inch, and 0.030 inch is to be left on the part for the finish cut.

4.32 Calculate the depth of a roughing cut to which a cutting tool must be positioned if an equal amount is to be machined off opposite sides of a part (see Figure 4.8). The initial size of the part is 1.250 inches, the final size is to be 1.000 inch. Notice that 0.015 inch is to be left on both sides of the part for taking a finish cut.

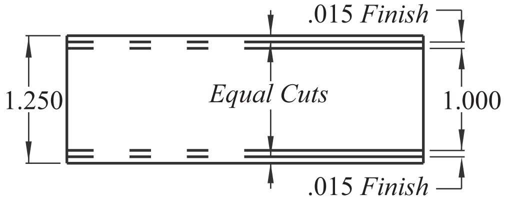


FIGURE 4.8

5

OPERATIONS WITH PERCENTS

Percent means *per one hundred*, hence expressing a fractional or mixed number quantity as a percent is the same as converting that quantity to a fraction with a denominator of 100. Thus, a percent signifies the number of parts per 100 being considered. In general,

$$n\% \equiv \frac{n}{100}$$

For example, $4\% \equiv \frac{4}{100}$.

Notice that $100\% \equiv \frac{100}{100} = 1$. That is, one whole equals 100 percent. Any number less than one whole represents a quantity less than 100 percent and any number greater than one whole represents a quantity greater than 100 percent. Percent can also be expressed as a decimal fraction. Converting a percent to this form is usually easier than converting it to a common fraction.

Becoming proficient with percents involves the ability to readily convert between percents, common fractions, and decimal fractions.

5.1 Working with Percents

Converting a Common Fraction to a Percent

Many times common fractions need to be expressed in percent form. For example, a civil engineer might plan a road elevation change of 1 foot in height for every 10 feet of distance. This amounts to a 10 percent incline, or a 10% *grade*.

The common fraction $\frac{1}{10}$ is converted to 10 percent by recalling that a percent can be thought of as a fraction with 100 as the denominator. Raising $\frac{1}{10}$ to an equivalent fraction with a denominator of 100 by multiplying it, top and bottom, by 10 results in a percent:

$$\frac{1}{10}\times\frac{10}{10}=\frac{10}{100}=10\%$$

Fractions like this—whose denominators are factors of 100—are easy to convert to percent. The conversion of $\frac{7}{25}$, is also straightforward:

$$\frac{7}{25}\times\frac{4}{4}=\frac{28}{100}=28\%$$

Other fractions are not so easily manipulated. For example, $\frac{1}{7}$ cannot be expressed as an equivalent fraction with a denominator of 100. The conversion of this kind of fraction to percent always gives an *approximate* answer. It begins with the division implied by the fraction bar; that is, dividing numerator by denominator.

Three dots, called an elipsis, at the and of a decimal fraction indicate a repeating pattern.

$$\frac{1}{7}=7\overline{)1.00}=0.142857142857\ldots$$

The resulting decimal fraction repeats its first six decimal digits, as indicated by the *ellipsis*. Before converting this decimal to an approximate percent equivalent, we round it off to a specified number of decimal places. To round to the third decimal place, for example, we write $0.142857142 \approx 0.143$

The percent form is also an approximation. We make the final conversion to percent by multiplying the result by 100 and adding the % symbol.

$$(0.143)(100) = 14.3\%$$

Notice that an easy way to mentally multiply by 100 is to simply move the decimal point 2 places to the right.

EXAMPLE 5.1A: *Converting a Common Fraction to Percent*

Convert $\frac{3}{4}$ to percent.

Solution: Since 4 divides into 100 exactly 25 times, we can raise $\frac{3}{4}$ to an equivalent fraction with denominator of 100.

$$\frac{3}{4}=\left(\frac{3}{4}\right)\left(\frac{25}{25}\right)=\frac{75}{100}\equiv 75\%$$

EXAMPLE 5.1B: *Converting a Common Fraction to Percent*

Convert $\frac{1}{6}$ to percent, rounding to the nearest tenth of a percent.

Solution: Since 6 does not divide into 100 exactly, a long division must be performed.

$$\frac{1}{6}=6\overset{\displaystyle 0.1667}{\overline{)1.0000}}$$

Multiply the decimal fraction by 100 and round to the tenths place to get 16.7%.

Converting a Percent to a Decimal Fraction

Sometimes a quantity given as a percent would be expressed better as a decimal fraction. This is accomplished by dividing by 100 and removing the % symbol.

Note that an easy way to mentally divide by 100 is to simply move the decimal point 2 places to the left. Thus, 7% = 0.07.

This technique works because a fraction signifies division; a percent—which is a fraction whose denominator is 100—is simply a division by 100. For example,

$$15\% \equiv \frac{15}{100} = 15 \div 100 = 0.15$$

EXAMPLE 5.2A: ***Converting a Percent to a Decimal Fraction***

Convert 34% to a decimal fraction.

Solution:

Divide 34 by 100 to get $34\% \equiv \frac{34}{100} = 100\overset{\displaystyle 0.34}{\overline{\big)34.00}} = 0.34$

Notice the movement of the decimal point two places left, from 34% to 0.34.

EXAMPLE 5.2B: ***Converting a Percent to a Decimal Fraction***

Simply moving the decimal point two places to the left readily converts a percent to a decimal fraction as shown in the following conversions.

$45\% = 0.45$ $6\% = 0.06$

$8.8\% = 0.088$ $1.4\% = 0.014$

$0.12\% = 0.0012$ $0.003\% = 0.00003$

$127\% = 1.27$ $346\% = 3.46$

The examples of 127% and 346% demonstrate how quantities greater than 100 percent convert to decimal representations greater than 1. Notice the insertion of placeholders in several of these examples.

Converting a Percent to a Common Fraction

Many percents are easily converted to fraction form. As we have seen, we simply write the number before the percent symbol as the numerator over a denominator of 100.

For example, $53\% = \frac{53}{100}$.

The fraction form of a percent should be reduced to lowest terms, if possible.

For example, $24\% = \frac{24}{100} = \frac{6}{25}$.

EXAMPLE 5.3A: ***Converting a Percent to a Common Fraction***

Convert 48% to a common fraction.

Solution:

Divide 48 by 100 and reduce the resulting fraction to lowest terms to get

$$48\% = \frac{48}{100} = \frac{12}{25}$$

EXAMPLE 5.3B: ***Converting a Percent to a Common Fraction***

Convert 125% to a common fraction.

Solution:

Divide 125 by 100 and reduce the resulting fraction to lowest terms to get

$$125\% = \frac{125}{100} = \frac{5}{4} = 1\frac{1}{4}$$

Converting a Mixed Number Percent to a Common Fraction

Conversion of a *mixed number percent*, such as $6\frac{1}{4}\%$, to a common fraction is more complicated. Two methods for doing this conversion are shown.

Method 1 *for converting a mixed number percent to a common fraction.*

The easiest method is to pass from the mixed number through the decimal form and then to the percent form.

EXAMPLE 5.4A: ***Converting a Mixed Number Percent to a Common Fraction***

Convert $6\frac{1}{4}\%$ to fraction form using Method 1.

Solution:

STEP 1: First, convert the fractional part of the mixed number, $\frac{1}{4}$, to a decimal through the division 1 ÷ 4 to get 0.25.

STEP 2: Next, move the decimal point two places to the left to eliminate the percent sign: $6\frac{1}{4}\% = 6.25\% = 0.0625$. Since the last digit is in the ten-thousandths place, the decimal fraction can be written in common fraction form as: $\frac{625}{10,000}$.

EXAMPLE 5.4A: *Converting a Mixed Number Percent to a Common Fraction (Continued)*

STEP 3: Finally, reduce the fraction to lowest terms:

$$\frac{625}{10{,}000}=\frac{(125)(5)}{(125)(80)}=\frac{5}{80}=\frac{1}{16}$$

To check the answer, simply convert this fraction back to decimal form through division:

$$\frac{1}{16}=16\overset{0.0625}{\overline{\big)1.0000}}\text{ or }6.25\% = 6\frac{1}{4}\%$$

Write the mixed number percent as a complex fraction. Perform the fraction division and reduce:

$$6\frac{1}{4}\%=\frac{6\frac{1}{4}}{100}=\frac{\frac{25}{4}}{100}=\frac{25}{4}\div 100=\frac{25}{4}\times\frac{1}{100}=\frac{25}{400}=\frac{1}{16}$$

Method 2 *for converting a mixed number percent to a common fraction.*

EXAMPLE 5.4B: *Converting a Mixed Number Percent to a Common Fraction*

Convert $5\frac{1}{2}\%$ to fraction form.

Solution:

Using Method 1,

$$5\frac{1}{2}\% = 5.5\% = 0.055$$

Since the last digit is in the *thousandths* place, the decimal fraction can be written in common fraction form as:

$$\frac{55}{1000}=\frac{11}{200}$$

Check: $11\div 200=0.055=5.5\%=5\frac{1}{2}\%.$

Alternatively, using Method 2,

$$5\frac{1}{2}\%=\frac{\frac{11}{2}}{100}=\frac{11}{2}\div 100=\frac{11}{2}\times\frac{1}{100}=\frac{11}{200}$$

EXAMPLE 5.4C: ***Converting a Mixed Number Percent to a Common Fraction***

Convert $71\frac{2}{5}\%$ to a fraction in lowest terms.

Solution: Using Method 1,

$$71\frac{2}{5}\% = 71.4\% = 0.714$$

Since the last digit is in the *thousandths* place, the decimal fraction can be written in common fraction form as:

$$\frac{714}{1000} = \frac{357}{500}$$

Check: $357 \div 500 = 0.714 = 71.4\% = 71\frac{4}{10}\% = 71\frac{2}{5}\%$.

Alternatively, using Method 2,

$$71\frac{2}{5}\% = \frac{71\frac{2}{5}}{100} = \frac{\frac{357}{5}}{100} = \frac{357}{5} \div 100 = \frac{357}{5} \times \frac{1}{100} = \frac{357}{500}$$

Summary of Rules for Percent Conversion

Common fraction to percent	If possible, raise the fraction to an equivalent fraction with denominator of 100 and then express with % symbol. If no equivalent fraction exists, divide numerator by denominator, approximate answer to desired number of places, and move decimal point two places to the right, attaching the percent symbol.
Percent to decimal fraction	Move decimal point two places to the left and drop %.
Percent to common fraction	Drop % symbol and write the amount in the numerator and 100 in the denominator. Reduce.
Decimal fraction to percent	Move decimal point two places to the right and attach % symbol.

Table 5.1 shows numerous examples of equivalent percentages, decimal fractions, and common fractions.

TABLE 5.1: **Equivalent Percentages, Decimal Fractions, and Common Fractions**

Percent	Decimal Fraction	Common Fraction	Percent	Decimal Fraction	Common Fraction
$\frac{1}{2}\%$	0.005	$\frac{1}{200}$	40%	0.40	$\frac{40}{100}$
5%	0.05	$\frac{1}{20}$	$43\frac{3}{4}\%$	0.4375	$\frac{7}{16}$
10%	0.10	$\frac{1}{10}$	50%	0.50	$\frac{1}{2}$
$12\frac{1}{2}\%$	0.125	$\frac{1}{8}$	60%	0.60	$\frac{3}{5}$
$15\frac{5}{8}\%$	0.15625	$\frac{5}{32}$	$62\frac{1}{2}\%$	0.625	$\frac{5}{8}$
$16\frac{2}{3}\%$	0.16667	$\frac{1}{6}$	$66\frac{2}{3}\%$	0.66667	$\frac{2}{3}$
$18\frac{3}{4}\%$	0.1875	$\frac{3}{16}$	70%	0.70	$\frac{7}{10}$
20%	0.20	$\frac{1}{5}$	75%	0.75	$\frac{3}{4}$
$21\frac{7}{8}\%$	0.21875	$\frac{7}{32}$	80%	0.80	$\frac{4}{5}$
25%	0.25	$\frac{1}{4}$	$83\frac{1}{3}\%$	0.83333	$\frac{5}{6}$
$28\frac{1}{8}\%$	0.28125	$\frac{9}{32}$	$87\frac{1}{2}\%$	0.875	$\frac{7}{8}$
$31\frac{1}{4}\%$	0.3125	$\frac{5}{16}$	90%	0.90	$\frac{9}{10}$
$33\frac{1}{3}\%$	0.33333	$\frac{1}{3}$	100%	1.00	1
$37\frac{1}{2}\%$	0.375	$\frac{3}{8}$	110%	1.10	$\frac{11}{10}$

EXERCISES

5.1 Convert the indicated common fraction to a percent.

a) $\frac{3}{4}$ **b)** $\frac{4}{31}$ **c)** $\frac{33}{34}$ **d)** $\frac{1}{16}$

e) $\frac{19}{65}$ **f)** $\frac{7}{18}$ **g)** $\frac{9}{52}$ **h)** $\frac{2}{11}$

5.2 Convert the indicated percent to a decimal fraction.

a) 68% **b)** 75% **c)** 125% **d)** 91%

e) $2\frac{2}{3}\%$ **f)** $3\frac{9}{11}\%$ **g)** 43% **h)** 233%

5.3 Convert the indicated percent to a common fraction.

a) 50% **b)** 531.25 **c)** 23.52% **d)** 33%

e) $42\frac{1}{4}\%$ **f)** $6\frac{1}{3}\%$ **g)** 110% **h)** $12\frac{99}{100}\%$

5.4 Change the indicated decimal fraction to a percent:

a) 0.0725 **b)** 1.15 **c)** 0.0025

d) 0.1875 **e)** 10.250 **f)** 0.001352

5.2 Solving Percent Problems

Applied mathematics problems in many fields involve calculations in which two of the three parts of a percent problem are given and the third is unknown. These three parts of a percent problem relate mathematically as

$$\mathit{Percentage} = \mathit{Rate} \times \mathit{Base}$$

This basic formula can be written equivalently in two other versions:

$$\mathit{Rate} = \frac{\mathit{Percentage}}{\mathit{Base}} \qquad \mathit{Base} = \frac{\mathit{Percentage}}{\mathit{Rate}}$$

These other versions are simply *algebraic rearrangements* of the variables in the first formula.

Figure 5.1 provides an easy way to remember the relationships between the three parts of a percent problem. The relative position of each quantity in the triangle suggests which operation needs to be done between the known values to find the unknown quantity. We will use the letters P, B, and R to indicate *percentage*, *base*, and *rate*, respectively. Simply cover the letter of the unknown quantity and the formula involving the other two known values becomes apparent. For example, if percentage is desired, cover the P and the triangle shows that the base (B) must be multiplied by the rate (R). Likewise, if the rate is desired, cover the R and the triangle shows that the percentage (P) must be divided by the base (B). Finally, if the base is desired, cover the B and the triangle shows that the percentage (P) must be divided by the rate (R).

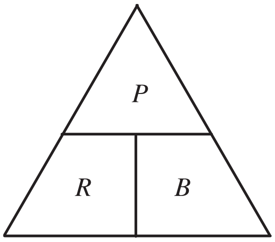


FIGURE 5.1 ***Variable Relationships in Percent Problems***

The three percent formulas are written compactly as

$$P = R \times B \qquad R = \frac{P}{B} \qquad B = \frac{P}{R}$$

Percentage When Base and Rate Are Known

The most familiar percent problem is the kind in which a rate of percent and a base are given and a percentage is sought. Notice that percentage is an amount that is a portion of a base amount. For example, determine the amount of sales tax in dollars and cents to be paid on an item costing \$90 in a state with a 5% sales tax rate. The answer is calculated from the basic formula, *Percentage* = *Rate* × *Base* or $P = R \times B$, in which sales tax is percentage P.

$$\text{sales tax} = (5\%)(\$90) = (0.05)(\$90) = \$4.50$$

- The operation for taking a percent of a quantity is multiplication.
- Percent must be expressed in decimal form before the multiplication is done.

Two important points concerning the basic percent problem.

EXAMPLE 5.5A: ***Percentage When Base and Rate Are Known***

What is 35% of 358 tons of scrap iron?

Solution: Using $P = R \times B$ with $R = 35\%$ and $B = 358$ tons,

$$P = 0.35 \times 358 = 125.3 \text{ tons}$$

EXAMPLE 5.5B: ***Percentage When Base and Rate Are Known***

A person earning \$32,500 per year pays an income tax of 15%. How much does he pay the government?

Solution: Using $P = R \times B$ with $R = 15\%$ and $B = \$32{,}500$,

$$P = 0.15 \times 32500 = \$4875$$

Rate When Base and Percentage Are Known

Another common application is finding a percent, or rate, increase of a base quantity given the resulting percentage. This type of problem is a division problem in which the base is the denominator and the percentage is the numerator.

$$Rate = \frac{Percentage}{Base} \qquad \text{or} \qquad R = \frac{P}{B}.$$

EXAMPLE 5.6A: ***Rate When Base and Percentage Are Known***

The average age in Emerald City in 2000 was recorded to be 2 years above the 1990 average of 35 years. What was the percent increase in the average age?

EXAMPLE 5.6A: ***Rate When Base and Percentage Are Known (Continued)***

Solution: Using $R = \frac{P}{B}$ again, noting in this example that "rate" is a *rate increase*, we have

$$\text{rate increase} = \frac{\text{age difference}}{\text{base age}} = \frac{2 \text{ years}}{35 \text{ years}} \times 100 \approx 0.06 = 6\%$$

EXAMPLE 5.6B: ***Rate When Base and Percentage Are Known***

75 out of every 1000 hexnuts coming off a production line are defective. What percent of the parts on this line are defective?

Solution: The base is 1000 and the percentage is 75. The rate we seek is the rate of defective parts to all parts in the production process.

From $R = \frac{P}{B}$, rate of defective part production $= \frac{75 \text{ hex nuts}}{1000 \text{ hex nuts}} = 0.075$

To express the defective parts rate as a percent, simply convert from decimal to percent by moving the decimal point two places to the right. Thus, 0.075 = 7.5% is the rate of defective part production.

EXAMPLE 5.6C: ***Rate When Base and Percentage Are Known***

The original price tag on a set of wrenches reads $80. During a recent sale, the price was reduced to $60. What was the percent price reduction?

Solution: In this type of problem, the base is the original price, $80. The percentage, however, is the change in price, $20.

From $R = \frac{P}{B}$, $R = \frac{\$20}{\$80} = 0.25$

To express the price reduction as a percent, simply convert from decimal to percent by moving the decimal point two places to the right. Thus, 0.25 = 25% is the price reduction.

Base When Percentage and Rate Are Known

A third variation on percent problems involves finding an unknown base given a rate and percentage.

$$Base = \frac{Percentage}{Rate} \text{ or } B = \frac{P}{R}$$

EXAMPLE 5.7A: ***Base When Percentage and Rate Are Known***

A builder estimates that 34,033 bricks are needed for a building. Experience shows that only 91% of the bricks purchased will be remain after bricks are lost through scrap or pilferage. How many bricks should the builder order to ensure 34,033 usable bricks?

EXAMPLE 5.7A: *Base When Percentage and Rate Are Known (Continued)*

Solution:

The base in this problem is the number of bricks to start the job, and this is the quantity sought. Using the formula $B = \frac{P}{R}$ gives

$$\text{bricks to start} = \frac{34{,}033 \text{ bricks remaining}}{0.91} = 37{,}399 \text{ bricks}$$

EXAMPLE 5.7B: *Base When Percentage and Rate Are Known*

Elena's calibration laboratory wants to have a 25% profit margin on each transducer it calibrates. If Elena's cost to calibrate a transducer is \$75, how much should she charge her customer?

Solution:

The base, B, is the unknown in this problem. It is the price Elena needs to charge her customer to achieve her desired profit margin. The percentage, P, is her cost before profit. The value to use for R is not readily apparent. At first, we might think that R should be 25%, but upon inspection we can see that R needs to reflect all of Elena's costs less the profit margin. In other words, R needs to be 100% – 25% or 75%.

Since the unknown quantity is the base, using the formula $B = \frac{P}{R}$ we find that

$$B = \frac{\$75}{0.75} = \$100 \cdot$$

EXERCISES

5.5 18 is what percent of 200?

5.6 What is 30% of 760 board feet?

5.7 What percent of 40 inches is 35 inches?

5.8 883.20 is 32% of what number?

5.9 A person earning \$22,500 per year pays an income tax of 15%. How much tax is paid?

5.10 The walls of a house are to be covered with insulation board. The actual area is 2604 square feet. How many square feet of insulation board must be ordered to allow for 7% waste?

5.11 A contractor estimates that 10,032 cement blocks are needed for a job. How many additional blocks must be ordered to provide for 12% scrap and damage?

5.12 Import duties are listed as percents of the value of goods imported.

a) What would be the total cost of a Spanish ironwork sold in Spain for \$3250 and subject to a duty of 8 percent?

b) What would be the sales price in the United States after a 20% retail mark up?

5.13 A manufacturer's agent averages a 30% profit on annual sales of machine tools. If the agent's annual gross income was \$68,000, what was the total price of the machines sold during that year?

5.14 \$30.60 is what percent of \$255?

5.15 \$21.91 is 7% of what number?

5.16 $\frac{1}{2}$ is 25% of what number? $\frac{6}{7}$ is $12\frac{1}{2}\%$ of what number?

5.17 If a person works 40 hours a week, what percent of the hours in the week does he work?

5.18 A machine detail has a $\pm 0.12\frac{1}{2}\%$ tolerance on length. If the base length is 14.625 inches, what are the minimum and maximum allowable lengths?

5.19 The clearance between a punch and a die for blanking an aluminum part is specified to be 8% of the stock thickness. Calculate the clearance required for blanking a sheet of $\frac{1}{16}$-inch thick aluminum. The answer should be given in terms of thousandths of an inch.

5.20 Babbit metal is used in low-speed, high-pressure bearings. It is a mixture of the following percents of metals: tin, 85%; antimony, 10%; copper, 4%; and lead, 1%. How many pounds of each metal will be required to rebabbit a bearing using 8.5 pounds of the alloy?

5.21 Solder used by tinsmiths usually contains 59% tin and 41% lead. Solder used in plumbing has less tin and more lead (35% tin and 65% lead). A tinsmith has 60 pounds of plumber's solder to convert to tinsmith's solder. What metal must be added and how much of it?

5.22 An iron meteorite weighing 4450 lb was found to be composed of 85.35% iron, 8.62% nickel, and traces of other elements. Calculate the weight of the iron and the nickel in the meteorite.

5.23 A sample lot from a ton of coal was found to contain 9.95% moisture. How many pounds of water was in that ton of coal?

5.24 When water freezes, it expands to 8.5% of its original volume. How many cubic feet of water are required to make 335 cubic feet of ice?

5.3 Simple Interest

Interest problems belong a category of percent problem that appear in all matters of commerce and personal finance. The definitions of the quantities in an interest problem and how each correlates to the parts of a percent problem are:

Amount of money earned or paid *in one year* on money saved, invested, lent, or borrowed. (This is the *percentage*.)	***Simple Interest***
Rate at which interest accrues per year. (This is the *rate*.)	***Interest Rate***
Amount of money originally saved, invested, lent, or borrowed. (This is the *base*.)	***Principal***

A quick calculation of any of the quantities can be done using one of the formulas that follow. Notice that these formulas mirror the percent problem formulas presented earlier. The only difference is that it is customary to refer to the percent as the *rate* itself, so it is necessary to include the conversion factor of 100 in the formulas.

$$Interest(\$) = \frac{Principal(\$) \times Rate(\%)}{100}$$

$$Rate(\%) = \frac{Interest(\$)}{Principal(\$)} \times 100 \qquad Principal(\$) = \frac{Interest(\$)}{Rate(\%)} \times 100$$

EXAMPLE 5.8A: ***Simple Interest***

What is the amount of interest earned in one year in dollars when an annual rate of 6% is applied to a principal of $300?

Solution:

$$Interest = \frac{Principal \times Rate}{100} = \frac{\$300 \times 6}{100} = \$18$$

Simple interest is interest calculated once in the compounding period.

EXAMPLE 5.8B: ***Simple Interest***

What is the annual rate in percent applied to a principal of $600 that earns an interest amount of $30 per year?

Solution:

$$Rate = \frac{Interest}{Principal} = \frac{\$30}{\$600} \times 100 = 5\%$$

EXAMPLE 5.8C: ***Simple Interest***

How much principal earns \$60 per year when an annual interest rate of 5% is applied to it?

Solution: $$Principal = \frac{Interest}{Rate} = \frac{\$60}{5} \times 100 = \$1200$$

EXERCISES

5.25 Find the amount of interest paid per year on a the indicated loan.

a) \$2,500 at 6%
b) \$8,320 at 6.125%
c) \$52,500 at 3.75%
d) \$15,000 at $4\frac{1}{8}\%$
e) \$10,350 at $5\frac{1}{4}\%$
f) \$3,200 at $7\frac{3}{8}\%$
g) \$1,500 at 8.5%
h) \$2,800 at 4.25%

5.26 Determine how much money was borrowed if a 1-year bank loan at 8% annual interest costs the borrower the indicated amount.

a) \$48
b) \$64
c) \$15
d) \$100
e) \$8
f) \$94
g) \$36.22
h) \$19.63

5.27 What annual interest rate was charged if a \$200 loan cost a borrower the indicated amount?

a) \$80 per year
b) \$100 per year
c) \$150 per year
d) \$40 per year
e) \$35 per year
f) \$60 per year
g) \$5 per month
h) \$10 per month

5.28 A bank gives 4% annual interest on savings. What would a depositor be required to have in an account to provide the indicated monthly income? Assume annual interest divided by 12 months.

a) \$300
b) \$352.12
c) \$400
d) \$418.19
e) \$500
f) \$581.27
g) \$600
h) \$630.54

5.4 List Price and Discounts

Wholesale houses and distributors often use a system of list price and discount in their catalogs. *List price* is the fair retail price charged to persons buying only a few items at a time.

Large quantity buyers are favored by lower prices which are obtained by applying a percent discount to the list price. In some cases only one discount is given, while in others multiple discounts are granted. The resulting price is known as the wholesale price.

$$\text{Wholesale price} = \text{List price} - (\text{Discount rate} \times \text{List price})$$

Wholesale price formula

.

EXAMPLE 5.9A: *List Price and Discounts*

The list price on a single box of roofing shingles is \$45. A contractor is given a discount of 15%. What is the wholesale price?

Solution: Applying the formula directly,

$$\$45 - (0.15 \times \$45) = \$45 - 6.75 = \$38.25$$

EXAMPLE 5.9B: *List Price and Discounts*

Large machine tools are sold at a discount to an agent. The amount of discount from the list or market price is the agent's gross profit on the sale. Some perishable tools such as drills and reamers have list prices in the manufacturer's catalog followed by a series of percents, such as

$\frac{1}{2}$ – 13 UNC tap. List: \$1.65, less 15%, 10%, 6%.

These percents are multiple discounts and will be deducted from the list price, depending on the quantity ordered. The catalog has a *discount schedule* indicating quantities required to get the discounts. Suppose the schedule indicates 1–5: List; 6–20: 15%; 21–50: 15% and 10%; 51 and over: 15%, 10%, and 6%.

Orders for amounts in the four quantity ranges will then get the discounts indicated. Thus, an order of 12 items will get 15% off the list price; an order of 51 or more will get a multiple discount of 15%, 10%, and 6%.

An order calls for 60 reamers listing at \$10 each. The discounts for this quantity are 15%, 10%, and 6%. How much will the 60 reamers cost the customer?

Solution: In multiple discounts the percentages must be applied in the order they are stated, one at a time. *Each succeeding rate is applied to the new cost after the discount is taken.*

This is done as follows:

60 reamers @ \$10:	List price totals \$600
First discount of 15%:	$\$600 \times 0.15 = \$90$
Net after first discount:	$\$600 - \$90 = \$510$
Second discount of 10%:	$\$510 \times 0.10 = \$51$
Net after second discount:	$\$510 - \$51 = \$459$
Third discount of 6%:	$\$459 \times 0.06 = \$27.54$
Net after third discount:	$\$459 - \$27.54 = \$431.46$

The complicated computation done in Example 5.9B can be shortened to one calculation by multiplying the list price by (100% – discount rate), and doing so as many times as there are discount rates. That is,

$$\begin{aligned} \textit{Wholesale price} = \textit{List price} & \\ &\times (100\% - \textit{Discount rate } 1) \\ &\times (100\% - \textit{Discount rate } 2) \\ &\times \ldots \end{aligned}$$

In other words,

$$\begin{aligned} \$600 \times (1.00 - 0.15) \times (1.00 - 0.10) \times (1.00 - 0.06) &= \$600 \times 0.85 \times 0.90 \times 0.94 \\ &= \$431.46 \end{aligned}$$

EXERCISES

5.29 A mill supply firm offers multiple discounts of 20%, 15%, 10%, and 5%. These discounts are applied successively to orders totaling \$2,000, \$6,000, \$9,000, and \$18,000, respectively. The discounts are compounded also, so that an order of \$18,000 or over gets all four discounts; \$9,000 gets the first three, and so on. How much will the discounted price be on separate orders of \$4,000, \$8,000, \$11,000, and \$22,000?

5.30 A merchant's invoice from a supplier totals \$1550. The supplier offers a $2\frac{1}{2}\%$ discount on invoices paid in 30 days. How much can the merchant save by prompt payment? (This practice is referred to in business as "discounting one's bills.")

5.31 Which is most profitable to a buyer: a single discount of 12%; multiple discounts of 8% and 4%; or multiple discounts of 7%, 3%, and 2%?

5.32 A dozen drills list priced at \$75 are discounted \$15. What percent discount was given?

5.33 A machinery distributor buys a lathe from a manufacturer for \$750 and sells it for the manufacturer's list price of \$1250. What was the discount percentage offered by the manufacturer?

5.34 The price tag on a suit reads \$325 and was reduced for sale to \$225. What was the percent reduction?

5.35 An auto mechanic receives an 8% discount on repair parts from a distributor. How much does the mechanic save when the list price on parts purchased totals \$350?

6

EXPONENTS: POWERS AND ROOTS

We now turn to that aspect of multiplication that is so important to technical mathematics: taking a power or a root of a number. Both of these operations are represented by exponents.

6.1 Powers of Positive and Negative Bases

The terms *exponent*, *base*, *root*, and *power*, have special meanings in mathematics. To help commit these terms to memory, study the diagram shown in Figure 6.1.

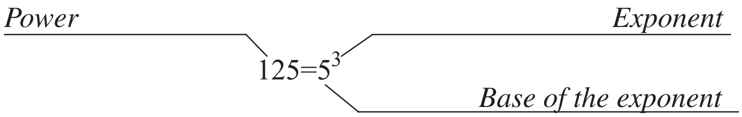


FIGURE 6.1 *Exponent, base, root, and power*

A power of a real number b is a product b^n in which b is the base and the integer n is the exponent. Hence, b is used as a factor n times:

$$b^n = b \cdot b \cdot b \cdot \ldots \cdot b$$

Definition of b^n

EXAMPLE 6.1A: ***Meaning of a Power***

Find the value of 3^2.

Solution: $3^2 = 3 \cdot 3 = 9$ The second power of 3 is $9 \cdot$

We say, “3 squared is 9” or “3 to the second power is 9.”

EXAMPLE 6.1B: ***Meaning of a Power***

Find the value of $(-1)^5$

Solution: $(-1)^5 = (-1)(-1)(-1)(-1)(-1) = -1$ The fifth power of -1 is -1.

b^n is usually read as *b to the nth power* instead of *b to the exponent n.* Power also refers to the result of raising a number to an exponent. For example, the powers of 3 are 3, 9, 27, and so on, because $3^1 = 3$, $3^2 = 9$, $3^3 = 27$.

The powers b^2 and b^3 are read as "b squared" and "b cubed", respectively. The geometric interpretation of finding the area of a square and the volume of a cube gives us these phrases.

Raising a base to increasing exponents quickly produces numbers much greater in size than the original base. The example of 2^n illustrates this:

$$\begin{aligned} 2^1 &= 2 \\ 2^2 &= 4 \\ 2^3 &= 9 \\ 2^4 &= 16 \\ &\ldots \\ 2^{10} &= 1024 \\ &\ldots \end{aligned}$$

In fact, powers of 2 appear commonly in science and technology, particularly in systems using the binary number system. The process of doubling anything produces numbers with values 2^n.

As we have seen, when a base is *positive* all powers of the base are positive, since any positive number multiplied by another positive yields a positive. When the base is *negative* only *even* exponents will produce a positive result, while *odd* exponents produce negative results. For example

A negative base raised to an even exponent gives a positive result. A negative base raised to an odd exponent gives a negative result.

$$(-3)^1 = -3$$
$$(-3)^2 = (-3)(-3) = 9$$
$$(-3)^3 = (-3)(-3)(-3) = -27$$
$$(-3)^4 = (-3)(-3)(-3)(-3) = 81, \text{ and so on.}$$

Parentheses are needed around a negative base when it is raised to an exponent, as the following example shows.

EXAMPLE 6.2: *Using Parentheses for a Negative Base*

Compare -3^2 to $(-3)^2$.

Solution: $-3^2 = -(3)(3) = -9$ $\qquad$ $(-3)^2 = (-3)(-3) = 9$

EXERCISES

6.1 In the equation: $256 = 2^8$

a) Check to see if 256 is the correct answer.
b) Name the exponent, the base, and the power.

6.2 What number is the fourth power of 10?

6.3 Simplify each expression.

a) -2^2

b) $(-2)^2$

c) -2^3

d) $3^2 \times 3^3$

e) $\left(5^2\right)^3$

f) $(-3)^2 + 4^2 - 2^3$

g) $(-2)^3$

h) -1^5

i) $(-1)^5$

j) $4^3 - 8^2$

k) $\frac{2^3}{4^2}$

l) $-3^2 + 4^2 - 2^3$

m) -1^6

n) $(-1)^6$

o) 2^6

p) 10^9

q) $\frac{2^5}{5^2}$

6.4 True or False?

-3^3 is the same as $(-3)^3$
-5^2 is the same as $(-5)^2$
-5^2 is the same as (-5^2)
$-x^4$ is the same as $(-x)(-x)(-x)(-x)$

6.5 Why does a negative number raised to an odd power always produce a negative result?

6.2 Exponent Rules Part 1

Several rules must be understood to work with exponents. The first group of rules deals with exponents that are whole numbers.

Special Exponents: 1 and 0

Any number or variable raised to a power of 1 is itself. Any number or variable (except 0) raised to a power of 0 is 1. These rules are stated as follows.

$$b^1 = b$$

For example, $8^1 = 8$. If an exponent is not shown, the number 1 is understood.

Exponent Rule 1

$$0^n = 0 \text{ for } n > 0$$

0^0 is undefined.

When zero is raised to any exponent greater than zero the result is zero. This is true because $0^n = 0 \cdot 0 \cdot 0 \cdot \ldots \cdot 0$

Exponent Rule 2

$$b^0 = 1 \text{ for } b \neq 0$$

The exponent 0 always results in 1, except for 0^0, which is undefined.

Exponent Rule 3

EXAMPLE 6.3: ***Base Raised to the Zero Power***

$$10^0 = 1 \qquad \left(\frac{1}{7}\right)^0 = 1 \qquad 314{,}902^0 = 1.$$

Negative Exponent

Exponent Rule 4

$$\text{For } b \neq 0,\ b^{-n} = \frac{1}{b^n}$$

$$\text{Likewise, } \frac{1}{b^{-n}} = b^n$$

A negative exponent in the numerator is equivalent to moving the number in the numerator to the denominator and making the exponent positive. Likewise, a negative exponent in the denominator is equivalent to moving the number in the denominator to the numerator and making the exponent positive.

For example, $8^{-2} = \frac{1}{8^2}$.

Also, $\frac{4^3}{2^{-1}} = 4^3 \cdot 2^1 = 64 \cdot 2 = 128$.

In the case of a fraction, where both numerator and denominator have negative exponents, the positions of the numbers are simply switched and the exponents made positive.

$$\frac{3^{-2}}{7^{-1}} = \frac{7^1}{3^2} = \frac{7}{9}$$

Operations with Exponents

Exponent Rule 5

$$b^m \cdot b^n = b^{m+n}$$

The product of exponents with the same base is equal to the base raised to the sum of the exponents.

Notice that 2=2^1

For example, $2^2 \cdot 2 \cdot 2^3 = 2^{3+1+2} = 2^6$ because $(2^2)(2)(2^3) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$.

Exponent Rule 6

$$\frac{b^m}{b^n} = b^{m-n}$$

The quotient of exponents with the same base is equal to the base raised to the difference of the exponents.

Once again, the reasoning behind this rule is obvious if one considers an example:

$$\frac{3^5}{3^2} = \frac{3 \cdot 3 \cdot 3 \cdot \cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3}} = 3^3$$

The exponent of the answer is the difference of the original exponents: $3^3 = 3^{5-2}$.

Another example: $\frac{4^2}{4^5} = 4^{2-5} = 4^{-3} = \frac{1}{4^3}$.

We stated in Rule 3 that $b^0=1$ for $b \neq 0$, and in Rule 4 that $b^{-n} = \frac{1}{b^n}$. Rule 6 gives a rationale for both:

$$1 = \frac{b^n}{b^n} = b^{n-n} = b^0 \qquad \frac{1}{b^n} = \frac{b^0}{b^n} = b^{0-n} = b^{-n}$$

$$\left(\frac{b}{c}\right)^n = \frac{b^n}{c^n}$$

A quotient raised to an exponent is equal to the quotient of the numerator and the denominator, each raised to the exponent.

Exponent Rule 7

For example, $\left(\frac{3}{5}\right)^2 = \frac{3^2}{5^2} = \frac{9}{25}$.

Power of a Power

$$(b^m)^n = b^{mn}$$

A power raised to a power is equal to the base raised to the product of the exponents.

Exponent Rule 8

Again, a simple example explains why this is so:

$$\left(4^2\right)^3 = 4^2 \cdot 4^2 \cdot 4^2 = 4^{2+2+2} = 4^6 = 4^{2\cdot 3} = 4096$$

Notice that a product or a quotient of *like bases* can always be found without applying Rules 5, 6, or 7, simply by expanding the expressions as shown. However, the rules *do not apply to unlike bases*, which must be expanded or entered on a calculator when a solution is sought. For example,

$$4^2 \cdot 10^3 = 16 \cdot 1000 = 16{,}000 \qquad \text{and} \qquad \frac{6^1}{8^2} = \frac{6}{64} = \frac{3}{32}.$$

EXERCISES

6.6 Simplify each expression.

a) 6^1 **b)** 6^0 **c)** 793^1 **d)** 793^0

e) $(-14)^1$ **f)** -14^0 **g)** 0^1 **h)** 0^0

6.7 Simplify each expression.

a) 6^{-1} **b)** 2^{-2} **c)** $(-14)^{-3}$ **d)** 14^{-3}

e) 2^{-3} **f)** -3^{-2} **g)** 0^{-8} **h)** 4^{-5}

6.8 Simplify each expression.

a) $6^{-1}\cdot 6^{1}$ **b)** $2^{2}\cdot 2^{5}$ **c)** $5^{-2}\cdot 5^{3}$

d) $4^{-5}\cdot 4^{1}$ **e)** $11^{10}\cdot 11^{2}$ **f)** $7^{-1}\cdot 7^{-3}$

g) $14^{2}\cdot 14^{3}$ **h)** $-8^{-2}\cdot 8^{-3}$ **i)** 14^{-3}

6.9 Rewrite each expression using only positive exponents.

a) $\left(2^{2}\right)\left(2^{-3}\right)$ **b)** $\left(1^{2}\right)\left(2^{-2}\right)\left(3^{2}\right)$ **c)** $\left(\frac{2}{3}\right)^{-3}$ **d)** $\left(\frac{1}{2}\right)^{2}(2)^{-2}$

e) $2^{2}\div 2^{-2}$ **f)** $1\div 4^{-3}$ **g)** $\left(3^{2}\right)\left(2^{0}\right)\left(4^{-2}\right)$ **h)** $5^{2}\div 5^{-3}$

i) 3^{-4} **j)** $\left(4^{-1}\right)\left(4^{2}\right)$ **k)** $(-14)(4)^{-1}$ **l)** $(5)^{2}(-4)^{-1}$

6.10 Simplify each expression.

a) $\left(2^{2}\right)\left(2^{-3}\right)$ **b)** $\left(1^{2}\right)\left(2^{-2}\right)\left(3^{2}\right)$ **c)** $\left(\frac{2}{3}\right)^{-3}$ **d)** $\left(\frac{1}{2}\right)^{2}(2)^{-2}$

e) 3^{-4} **f)** $\left(4^{2}\right)\left(5^{0}\right)$ **g)** $2^{2}\div 2^{-2}$ **h)** $1\div 4^{-3}$

i) $5^{2}\div 5^{-3}$ **j)** $\left(3^{2}\right)\left(2^{0}\right)\left(4^{-2}\right)$ **k)** $\left(4^{-1}\right)\left(4^{2}\right)$ **l)** $\left(5^{0}\right)\left(5^{1}\right)\left(5^{2}\right)$

6.3 Exponent Rules Part 2

The second group of exponent rules deals with fractional exponents.

Square Roots

The symbol used to denote square root is the *radical*: $\sqrt{\ }$. The number inside the radical is called the *radicand*. The *square root of b* is symbolized by $\sqrt{b}$. It represents a number that multiplied by itself gives b. For example, $\sqrt{16}=4$ because $4^2 = 16$. Likewise, the square root of 100 is 10 since $10 \cdot 10 = 100$.

The positive square root of a number is called the principle root.

While –4 is also a square root of 16 since $(-4)(-4) = 16$, the notation $\sqrt{b}$ *in this book* is intended to signify the *positive* square root of b. We would need to write $-\sqrt{b}$ if we want to indicate the negative root of b. For example, $-\sqrt{16}=-4$.

The operation of taking a number's square root is the inverse operation of squaring a number. Square roots are sometimes represented using a fractional exponent instead of the radicand symbol, as in

$$\sqrt{25} = 25^{\frac{1}{2}} = 5$$

5 is the square root of 25 because 5 × 5 = 25.

$$\sqrt{4} = 4^{\frac{1}{2}} = 2$$

2 is the square root of 4 because 2 × 2 = 4.

Higher Roots

We can extract roots of numbers higher than square roots, just as we can raise a number to a power higher than two. The n^{th} root of b is a number that when used as a factor n times gives b. It is represented by $\sqrt[n]{b}$ or $b^{\frac{1}{n}}$. For example,

$$\sqrt[3]{8} = 2$$

$$27^{\frac{1}{3}} = 3$$

2 is the third root of 8 because 2 · 2 · 2 = 8.

3 is the third root of 27 because 3 · 3 · 3 = 27.

When $n = 2$, the n^{th} root is called the *square root* and when $n = 3$, the n^{th} root is called the *cube root*.

In general, the n^{th} root of b is shown in exponential notation by $\sqrt[n]{b}$ or $b^{\frac{1}{n}}$. n is the *index* of the radical.

$$\sqrt{b} = b^{\frac{1}{2}} \qquad \text{and} \qquad \sqrt[n]{b} = b^{\frac{1}{n}}$$

Exponent Rule 9

Example 6.4: ***Roots as Fractional Exponents***

$$125^{\frac{1}{3}} = \sqrt[3]{125} = 5$$

$$40^{\frac{1}{4}} = \sqrt[4]{40} = 2.5148668...$$

$$32^{\frac{1}{5}} = \sqrt[5]{32} = 2$$

Roots of Negative Numbers

The radicand of an even root such as square root or fourth root must be positive if the solution is to be a real number. Consider the following examples:

$$\sqrt{4} \qquad \sqrt{-4} \qquad \sqrt[3]{8} \qquad \sqrt[3]{-8}$$

*Taking a root of a number is also called **extracting a root**.*

Only three of these have real number solutions:

$$\sqrt{4} = 2, \quad \text{since } 2^2 = 4$$

$$\sqrt[3]{8} = 2, \quad \text{since } 2^3 = 8$$

$$\sqrt[3]{-8} = -2, \quad \text{since } (-2)^3 = -8$$

$\sqrt{-4}$ does not have a real number solution. We see that $(2)(2) = 4$ and $(-2)(-2) = 4$, but no *real number* multiplied by itself produces -4.

EXAMPLE 6.5: ***Root***

$$\sqrt{100} = 100^{\frac{1}{2}} = 10 \qquad \sqrt{-100} \text{ has no real number solution}$$
$$\sqrt[3]{27} = 27^{\frac{1}{3}} = 3 \qquad \sqrt[3]{-27} = (-27)^{\frac{1}{3}} = -3$$
$$\sqrt[5]{1} = 1^{\frac{1}{5}} = 1 \qquad \sqrt[5]{-1} = (-1)^{\frac{1}{5}} = -1$$

The rules of integer exponents also apply to exponents that indicate roots. Because the exponents in the case of roots are fractions, operations of fractions are followed. For example,

$$8^{\frac{1}{3}}8^{\frac{1}{2}} = 8^{\frac{1}{3}+\frac{1}{2}} = 8^{\frac{5}{6}} \qquad \frac{8^{\frac{1}{3}}}{8^{\frac{1}{2}}} = 8^{\frac{1}{3}-\frac{1}{2}} = 8^{-\frac{1}{6}} \qquad (8^4)^{\frac{1}{3}} = 8^{\frac{4}{3}}$$

Solving for these roots is usually difficult without the help of a calculator. Using a calculator we find that $8^{\frac{5}{6}} = 5.66$ (rounded to the nearest hundredth) and that $8^{-\frac{1}{6}} = 0.71$ (rounded to the nearest hundredth). The third example is solved easily by noting that $(8^4)^{\frac{1}{3}} = \left(8^{\frac{1}{3}}\right)^4 = 2^4 = 16$. We extract the cube root of 8 first, to get 2, then raise 2 to the fourth power, to get 16.

Real number even roots can only be extracted from positive numbers.

Real number odd roots can be extracted from either positive or negative numbers.

EXAMPLE 6.6: ***Power Raised to a Fractional Power***

$$\left(32^7\right)^{\frac{1}{5}} = 32^{\frac{7}{5}} = \left(32^{\frac{1}{5}}\right)^7 = 2^7 = 128$$

Simplifying Radicals

Often a rounded decimal estimate of a root is not sought; the simplified radical is sufficient. In these cases the next rule is very helpful. It shows that we can often extract the root of certain *factors* of the radical.

Exponent Rule 10

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

Consider $\sqrt{75}$. The radicand 75 can be broken into factors of 3 and 25. Noticing that 25 is a perfect square, the radical is thus simplified as:

$$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$$

EXAMPLE 6.7: *Simplifying Radicals*

Observe how the following radicals are reduced.

$$\sqrt{1000} = \sqrt{100 \cdot 10} = \sqrt{100}\sqrt{10} = 10\sqrt{10}$$

$$\sqrt[4]{80} = \sqrt[4]{16 \cdot 5} = \sqrt[4]{16}\sqrt[4]{5} = 2\sqrt[4]{5}$$

Next, the extraction of the *root of a fraction* is treated in Rule 11.

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Exponent Rule 11

EXAMPLE 6.8: *Root of a Fraction*

Observe how the following radicals are reduced.

$$\sqrt{\frac{49}{25}} = \frac{\sqrt{49}}{\sqrt{25}} = \frac{7}{5} \qquad \sqrt[3]{\frac{1000}{64}} = \frac{\sqrt[3]{1000}}{\sqrt[3]{64}} = \frac{10}{4} = \frac{5}{2}$$

Exponent Rule 11 is very useful when numerator and denominator are perfect powers of the root indicated (for example, perfect squares or cubes). If the numerator and denominator are not both perfect squares, the fraction is converted to a decimal, and the approximate root taken using a calculator. Sometimes, however, a partial root can be extracted as just seen.

Finally, to prepare for algebraic equations containing radicals, we need the last rule of exponents.

$$b^n = x \Rightarrow b = \sqrt[n]{x}$$

Exponent Rule 12

EXAMPLE 6.9: *Exponent Rule 12*

Observe how Exponent Rule 12 is applied.

$$2^3 = 8 \Rightarrow 2 = \sqrt[3]{8} = 8^{\frac{1}{3}}$$

$$3^2 = 9 \Rightarrow 3 = \sqrt{9} = 9^{\frac{1}{2}}$$

EXERCISES

6.11 Find the indicated square root.

a) $\sqrt{64}$ **b)** $\sqrt{25}$ **c)** $\sqrt{39}$ **d)** $\sqrt{345.25}$

e) $\sqrt{0.638}$ **f)** $\sqrt{49.1}$ **g)** $\sqrt{45}$ **h)** $\sqrt{81}$

i) $\sqrt{620}$ **j)** $\sqrt{7.60}$ **k)** $\sqrt{\frac{9}{64}}$ **l)** $\sqrt{\frac{25}{100}}$

m) $\sqrt{235}$ **n)** $\sqrt{1444}$ **o)** $\sqrt{3.25}$ **p)** $\sqrt{36.3}$

q) $\sqrt{\frac{4}{25}}$ **r)** $\sqrt{\frac{16}{49}}$

6.12 Find the indicated root.

a) $\sqrt[3]{27}$ **b)** $\sqrt[4]{128}$ **c)** $\sqrt[4]{256}$ **d)** $\sqrt[5]{125}$

e) $\sqrt[3]{-64}$ **f)** $\sqrt[5]{-10000}$ **g)** $\sqrt[5]{505.36}$ **h)** $\sqrt[3]{32.45}$

i) $\sqrt[3]{-216}$ **j)** $\sqrt[8]{4325}$ **k)** $\sqrt[3]{-68.9}$ **l)** $\sqrt[6]{64}$

6.13 Rewrite the indicated root using fractional exponents.

a) $\sqrt{64}$ **b)** $\sqrt{25}$ **c)** $\sqrt{39}$ **d)** $\sqrt[3]{27}$

e) $\sqrt[4]{256}$ **f)** $\sqrt[4]{128}$ **g)** $\sqrt[6]{64}$ **h)** $\sqrt[5]{125}$

i) $\sqrt[3]{-216}$ **j)** $\sqrt[3]{-64}$ **k)** $\sqrt[5]{-10000}$ **l)** $\sqrt[5]{505.36}$

m) $\sqrt[3]{32.45}$ **n)** $\sqrt[3]{-68.9}$ **o)** $\sqrt[8]{4325}$

6.14 Simplify each expression.

a) $8^{\frac{2}{3}}$ **b)** $625^{\frac{3}{4}}$ **c)** $-8^{\frac{2}{3}}$ **d)** $8^{-\frac{2}{3}}$

e) $10000^{\frac{3}{2}}$ **f)** $-27^{\frac{5}{3}}$ **g)** $-27^{-\frac{5}{3}}$ **h)** $64^{-\frac{1}{4}}$

6.4 Scientific Notation

Scientific notation is a method of writing or displaying numbers in terms of a decimal number between 1 and 10 multiplied by a power of 10. The number 10,492, for example, can be expressed in scientific notation as 1.0492×10^4.

Scientific notation is used to simplify calculations and insure accuracy, particularly with respect to the position of decimal points. It also simplifies the expression of numbers that are so large or so small as to be unwieldy. For example, the metric unit for pressure, the pascal, is equivalent to 0.00000986923 atmospheres or 0.0001450377 pounds per square-inch. In scientific notation these figures are 9.86923×10^{-6} atmospheres and 1.450377×10^{-4} pounds per square inch. Scientific notation also facilitates adaptation of numbers for electronic data processing and computer readout.

Expressing Numbers in Scientific Notation

In scientific notation every number is expressed by two factors, one of which is some integer from 1 to 9 often followed by a decimal and the other is some power of 10. Thus, 10,000 is expressed as 1×10^4 and 10,463 as 1.0463×10^4.

The number 43 is expressed as 4.3×10 and 568 is expressed 5.68×10^2. In the case of numbers less than 1, the power of 10 is negative, as we saw above. We now explain why.

Consider the decimal 0.0001, which as a fraction is $\frac{1}{10,000}$. This number could be thought of as $\frac{1}{10^4}$, which we know from the rules of exponents can be written as 10^{-4}, or equivalently, 1×10^{-4}.

Accordingly, the decimal 0.0001463 is expressed as 1.463×10^{-4}, the decimal 0.498 is expressed as 4.98×10^{-1} and the decimal 0.03146 is expressed as 3.146×10^{-2}.

Rules for Converting any Number to Scientific Notation

Any number can be converted to scientific notation by means of one of two rules.

If the number is a whole number or a whole number and a decimal, the decimal point is moved a sufficient number of places to the left to place it immediately right of the first non-zero digit.

Rule 1
Scientific Notation

With the decimal point shifted to this position, the number so written is the first factor in the scientific notation form of the original number. The number of places that the decimal point is moved to the left to bring it immediately to the right of the first non-zero digit is the *positive* power of 10 that is the second factor of the scientific notation. Thus, to write 4639 in scientific notation, the decimal point is moved three places to the left giving the two factors: 4.639×10^3.

Similarly, $431.412 = 4.31412 \times 10^2$ and $986{,}388 = 9.86388 \times 10^5$.

If the number is a decimal, with digits only to the right of the decimal point, then the decimal point is moved a sufficient number of places to the right to place it immediately right of the first non-zero digit.

Rule 2
Scientific Notation

With the decimal point shifted to this position, the number so written is the first factor in the scientific notation form of the original number. The number of places that the decimal point is moved to the right to bring it immediately to the right of the first digit is the *negative* power of 10 that is the second factor of the scientific notation. Thus, to bring the decimal point in 0.005721 to the immediate right of the first non-zero digit, it must be moved three places to the right, giving the two factors: 5.721×10^{-3}.

Similarly, $0.469 = 4.69 \times 10^{-1}$ and $0.0000516 = 5.16 \times 10^{-5}$.

Multiplying Numbers Written in Scientific Notation

When multiplying two numbers written in scientific notation, the procedure is to multiply the first factor of one number by the first factor of the other number to obtain the first factor of the product. Then multiply the second factors — the powers of 10 — and write the answer in the two new factors, noting that the position of the decimal point and the power of 10 may need to be adjusted so that the decimal point only has one non-zero digit to its left. This process is illustrated in the next two examples.

EXAMPLE 6.10A: ***Multiplying Numbers Written in Scientific Notation***

Observe the following multiplication of numbers written in scientific notation.

$$(4.31 \times 10^{-4}) \times (9.0125 \times 10^{2}) = (4.31 \times 9.0125) \times 10^{-4+2} = 38.8 \times 10^{-2}$$

The decimal must now be moved one place to the left and the exponent adjusted to get the final answer in scientific notation: 3.88×10^{-1}.

EXAMPLE 6.10B: ***Multiplying Numbers Written in Scientific Notation***

Observe the following multiplication of numbers written in scientific notation.

$$(5.986 \times 10^{4}) \times (4.375 \times 10^{3}) = (5.986 \times 4.375) \times 10^{4+3} = 26.189 \times 10^{7}$$

The decimal must now be moved one place to the left and the exponent adjusted to get the final answer in scientific notation: 2.6189×10^{8}.

Dividing Numbers Written in Scientific Notation

When dividing one number by another when both are written in scientific notation, the procedure is to divide the first factor of one number by the first factor of the other to obtain the first factor of the quotient. Then divide the second factors — the powers of 10 — and write the answer in the two new factors, noting that the position of the decimal point and the power of 10 may need to be adjusted so that the decimal point only has one non-zero digit to its left. This process is illustrated in the next example.

EXAMPLE 6.11: ***Dividing Numbers Written in Scientific Notation***

Observe the following division of numbers written in scientific notation.

$$\frac{4.31 \times 10^{-2}}{9.0125 \times 10^{1}} = (4.31 \div 9.0125) \times 10^{-2-1} = 0.4782 \times 10^{-3}$$

The decimal must now be moved one place to the right and the exponent adjusted to get a final answer in scientific notation: $4.78 \times 10^{-4.}$

EXERCISES

6.15 Rewrite each number using scientific notation.

a) 10,000
b) 0.0001
c) 173,221
d) –235
e) 0.15583
f) 1,031,023
g) 0.001445
h) – 0.223589

6.16 Evaluate the expression and write the result using scientific notation.

a) $\left(6.02\times10^{23}\right)\left(5.6\times10^{4}\right)$
b) $\left(-235\times10^{8}\right)\left(12\times10^{-2}\right)$
c) $\dfrac{-235\times10^{8}}{12\times10^{-2}}$
d) $\dfrac{6.02\times10^{23}}{5.6\times10^{4}}$
e) $\left(3.09\times10^{9}\right)\left(19\times10^{5}\right)$
f) $\left(1.77896\times10^{-8}\right)\left(3.4\times10^{12}\right)$
g) $\dfrac{1.77896\times10^{-8}}{3.4\times10^{12}}$
h) $\dfrac{3.09\times10^{9}}{19\times10^{5}}$

6.5 Logarithms

The power to which a base, such as 10, must be raised to produce a given number is called a *logarithm*.

$$\log_n a = x \qquad \text{means} \qquad a^x = n$$

Many natural events, such as population growth and radioactive decay, are modeled using logarithmic expressions. Data are often graphed on logarithmic scales to allow for compact presentation or to illustrate the behavior of a particular phenomenon.

Symbolically, we write $\log_n a = x$ and say the logarithm of a, with n as the base, is x. The number a is the *antilog* of the equation.

Base 10 Logarithms

Although any number can be used as a base for logarithms, the best place to start studying is with the base 10 because we are already accustomed to thinking in terms of powers of 10. The base 10 logarithm is called the *common logarithm*.

To find the base 10 logarithm of a number, ask, "To what exponent is 10 raised to get the number?"

For example, $10^3 = 1{,}000$; therefore, $\log_{10} 1{,}000 = 3$. In other words, 10 has to be raised to the third power to get 1000.

EXAMPLE 6.12: ***Logarithm with Base 10***

Observe the following base 10 logarithms.

$\log 10{,}000 = 4$ $\qquad$ $\log 100 = 2$ $\qquad$ $\log 1{,}000{,}000 = 6$

Notice that we do not need to write the base for common logarithm problems.

Logarithms of numbers that are powers of 10 are easy to do mentally. To find more difficult logarithms, most scientific calculators have a *log* key. For example the log of 56 is 1.75 rounded to the hundredths place. In other words, 10 has to be raised to the 1.75 power to get 56.

Inverse or Antilogarithms

The number for which a given logarithm stands is called an *inverse logarithm* or *antilogarithm*. This function is commonly found on scientific calculators as a key designated as 10^x or INV LOG for base 10 logarithms.

EXAMPLE 6.13: ***Inverse or Antilogarithm***

What is the inverse base 10 logarithm of 1.75?

Solution: $10^{1.75} = 56$

Natural Logarithms

Natural logarithms use 2.71828… as a base. This special number is designated by the letter *e*. Although this base appears awkward to work with, it simplifies many calculations that need to be made in the laboratory or shop. Fortunately, most scientific calculators have natural logarithm keys in addition to the common base 10 logarithm and inverse logarithm keys discussed earlier. These calculator keys usually appear as *ln* for the natural logarithm and e^x or INV LN for the inverse natural logarithm.

EXAMPLE 6.14A: ***Natural Logarithm***

Find the natural logarithm of 100.

Solution: ln 100 = 4.61 rounded to the hundredths place.

In other words, 2.71828... has to be raised to an exponent of 4.61 to get 100.

EXAMPLE 6.14B: ***Natural Logarithm***

Find the natural logarithm of 78.

Solution: ln 78 = 4.36 rounded to the hundredths place.

In other words, 2.71828... has to be raised to an exponent of 4.36 to get 78.

Inverse Natural Logarithm

The inverse natural logarithm is the number that results when the base *e*, or 2.71828…, is raised to a power.

EXAMPLE 6.15: ***Inverse Natural Logarithm***

Find the inverse natural logarithm of 4.61.

Solution: $e^{4.61} = 100$.

A Few General Observations about Logarithms

A few general observations about logarithms are:

Natural logarithms are sometimes referred to as Napierian logarithms.

- The logarithm of 0 is undefined. This is true because there is no number to which 10 or any other base can be raised to give 0 as a result.
- The logarithm of 1 is 0 since 10^0 is 1.
- Logarithms of numbers less than 1 are negative.

EXERCISES

6.17 Find the indicated base 10 logarithm.

a) $\log_{10}100{,}000$ **b)** $\log_{10}456$ **c)** $\log_{10}7889$ **d)** $\log_{10}1$

e) $\log_{10}1000$ **f)** $\log_{10}2$ **g)** $\log_{10}0.625$ **h)** $\log_{10}0.33$

6.18 Find the inverse base 10 logarithm, 10^x.

a) 6 **b)** –0.225 **c)** 0.10 **d)** – 0.5

e) 0.625 **f)** 1 **g)** –1 **h)** –5

6.19 Find the indicated natural logarithm.

a) ln 0.33 **b)** ln 100,000 **c)** ln 456 **d)** ln 7889

e) ln 1 **f)** ln 1000 **g)** ln 2 **h)** ln 0.625

6.20 Find the inverse natural logarithm, e^x, where values of x are given as indicated.

a) 6 **b)** – 0.225 **c)** 0.10 **d)** – 0.5

e) 0.625 **f)** 1 **g)** –1 **h)** –5

7

MEASUREMENT

Measurement is the basis of all science and engineering work, and hence, all industrial applications. Every measurement has two components: a *quantity* and a *unit*. The quantity gives a magnitude; the unit gives the type of measurement and a reference for comparison to other quantities of the same type. Together they describe *how many*, *how much, how long, how heavy, how hard, how fast*, and so on. Common units of measure include meter (m), second (sec), pound (lb), gallon (gal), calorie (cal), and combinations of these or other units.

Unit conversion tables are given in Appendix A.

7.1 Systems of Measurement

The main dimensional systems of measurement are the *English system* and the *SI* or *metric system*.

English System

The English system of measurement evolved over many centuries and reflects the patterns of commerce and local customs of earlier times. In the English system, units of length were based on the human body because it was easily referred to and understood. The standard unit of linear measure, the *foot*, was originally the length of the king's foot. The foot, in turn, was divided into 3 hands, 4 palms, 16 digits, or 12 *inches*. A farm field measuring 4 rods by 40 rods was an *acre*. Similarly, the weight of a barleycorn was called a *grain*. With so many units in use, and no unifying factor, the English system became a complicated measurement scheme. Eventually, fundamental units in the English system were defined. They are *inch* for length, *second* for time, *pound* for weight, and *gallon* for volume. The United States is one of the few countries in which the English system of measure is still in use.

The word "inch" was a synonym for "thumb."

International System of Units (SI)

The abbreviation SI stands for Système International d'Unites—French for international system of units.

Most other countries use the International System of Units, or *SI*, which is adapted from the metric system. The metric system was developed in France in the years following the French Revolution. While the metric system is not necessarily more precise than the English system, it is easier to use once learned. Metric prefixes indicate relative sizes of a particular kind measurement, such as mass or length. Each prefix refers to a power of 10. For example,

1 kilogram = 1000 grams, since *kilo* means thousand;

1 millimeter = $\frac{1}{1000}$ meter, since *milli* means thousandth.

Important prefixes used in the metric/SI system are given in Table 7.1.

TABLE 7.1: Common Metric/SI Prefixes

Increasing Magnitude			Decreasing Magnitude		
Prefix	**Power of 10**	**Symbol**	**Prefix**	**Power of 10**	**Symbol**
kilo	10^3	k	milli	10^{-3}	m
mega	10^6	M	micro	10^{-6}	μ
giga	10^9	G	nano	10^{-9}	n
tera	10^{12}	T	pico	10^{-12}	p

The prefixes given in Table 7.1 are the most common ones. In each of these, the exponents of 10 are in multiples of 3. Measurements expressed in other powers of 10 are avoided in technical work. For example, quantities expressed in *deci* (10^{-1}), *centi* (10^{-2}), *deka* (10^1), and *hecto* (10^2) are not commonly used. These less common prefixes are shown in Table 7.2.

TABLE 7.2: Less Common Metric/SI Units

10 meters = 1 dekameter	1 meter = 10 decimeters
100 meters = 1 hectometer	1 meter = 100 centimeters

Fundamental and Derived Units in SI

The SI measurement system has seven precisely defined fundamental units. Each is related to a real, unchanging physical quantity — a *standard of measurement*. For example, 1 second is the amount of time elapsed during the radioactive emission of 9,192,631,770 wavelengths by an atom of the isotope Cesium-133. Table 7.3 shows the seven fundamental SI units and their symbols.

TABLE 7.3: Fundamental Units in SI

Quantity	Unit	Symbol
length	meter	m
mass	kilogram	kg
time	second	s or sec
electricity (current)	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

All other units in SI are derived from these seven fundamental units. Table 7.4 contains some common derived SI units.

TABLE 7.4: Common Derived SI Units

Quantity	Unit	Symbol
area	square meter	m^2
volume	cubic meter	m^3
mass density	kilogram per cubic meter	$\frac{kg}{m^3}$
force	Newton $\left(\frac{kg \cdot m}{sec^2}\right)$	N
acceleration	meter per second per second	$\frac{m}{sec^2}$
angular velocity	radian per second $\left(\frac{rad}{sec}\right)$	ω
pressure	Pascal $\left(\frac{N}{m^2}\right)$	Pa
work, energy, heat	Joule (N · m)	J
power	Watt $\left(\frac{J}{sec}\right)$	W
electrical charge	Coulomb	C
electrical current	Ampere $\left(\frac{C}{sec}\right)$	A
electromotive force	Volt	V
electrical resistance	Ohm	Ω

Unit Conversion

We frequently need to convert data units from one kind to another. This can be a simple process, as when feet are converted to inches through multiplication by 12, or it can be a laborious and not so straightforward process, as when several conversion factors are needed to arrive at the desired unit. Fortunately, the *factor-label method* of unit conversion simplifies the more complicated situations.

Many reference books such as Machinery's Handbook *and* CRC Handbook of Chemistry and Physics *contain extensive lists of conversion factors.*

Factor-label refers to the conversion factors that are used in a series of multiplications, and their unit labels or names, which are canceled or retained. These conversion factors can be used to make a variety of conversions within and between the SI and English systems.

The best way to learn how to use the factor-label method is to study examples and then begin to apply the technique to a variety of situations. Numerous illustrations of the factor-label method are given in the examples found in the next several sections of this chapter.

EXERCISES

7.1 Describe the English and SI measurement systems.

7.2 Compare the advantages and disadvantages of the English and SI measurement systems.

7.3 Name the SI prefix for:

a) 1 million **b)** 1 thousandth **c)** 1 millionth **d)** 1 billion

7.4 What is the difference between a fundamental unit and a derived unit?

7.5 Name the seven fundamental units used in the SI system. Research and state the physical basis for each.

7.6 Name two derived units and state their fundamental units.

7.2 Measures of Length, Area, and Volume

Length, area and volume are measures of extension.

Linear Measure

Linear measure describes an object's length or a distance traveled by a body. It is a one-dimensional measure, given in meters, kilometers, feet, inches, and so on.

The secret to mastering the factor-label method is to recognize how to arrange conversion factors so that units cancel.

EXAMPLE 7.1: ***Distance Unit Conversion***

Convert 1700 meters to kilometers using the factor-label method.

Solution:
This is a simple problem requiring multiplication by only one conversion factor. The factor-label method is shown in 3 steps.

STEP 1: Restate the problem as an equation using the # ("number of") symbol and appropriate unit symbol to indicate the unknown quantity on the left side of the equal sign and the known quantity on the right side of the equal sign.

$$\# \text{ km} = 1700 \text{ m}$$

EXAMPLE 7.1: *Distance Unit Conversion (Continued)*

The given quantity, 1700 m becomes the first factor in the conversion process.

STEP 2: Find the appropriate conversion factor and write it as fraction so that the desired unit, km, is in the numerator and the given unit, m, is in the denominator.

$$\frac{1\text{ km}}{1000\text{ m}}$$

STEP 3: Do the multiplication so that m cancels, leaving km.

$$\#\text{ km} = 1700\ \cancel{\text{m}} \cdot \frac{1\text{ km}}{1000\ \cancel{\text{m}}} = \frac{1700\text{ km}}{1000} = 1.7\text{ km}$$

Since 1 km is the same as 1000 m, multiplying by the fraction 1 km/1000 m is the same as multiplying by 1.

EXAMPLE 7.2: *Factor-Label Method of Unit Conversion*

Convert 291 meters to yards using the factor-label method.

Solution:
This requires a conversion from SI to the English system. The conversion factors needed are 1 inch = 2.54 cm, and 1 yard = 36 inches.

$$\#\text{ yd} = 291\ \cancel{\text{m}} \cdot \frac{100\ \cancel{\text{cm}}}{1\ \cancel{\text{m}}} \cdot \frac{1\ \cancel{\text{in}}}{2.54\ \cancel{\text{cm}}} \cdot \frac{1\text{ yd}}{36\ \cancel{\text{in}}} = 318\text{ yd}$$

Notice that the answer has the same number of significant digits as the given measurement, 291 m.

EXAMPLE 7.3: *Converting Kilometers to Feet*

Use the factor-label method to determine the number of feet in 1 km.

Solution:
The conversion factor for meters to feet can be found in Appendix A.

$$\#\text{ ft} = 1\ \cancel{\text{km}} \cdot \frac{1000\ \cancel{\text{m}}}{1\ \cancel{\text{km}}} \cdot \frac{1\text{ ft}}{0.3048\ \cancel{\text{m}}} = 3281\text{ ft}$$

If a conversion table is not available, the calculation can still be done by remembering that 2.54 cm = 1 inch and 12 inches = 1 foot. Thus,

$$\#\text{ ft} = 1\ \cancel{\text{km}} \cdot \frac{1000\ \cancel{\text{m}}}{1\ \cancel{\text{km}}} \cdot \frac{100\ \cancel{\text{cm}}}{1\ \cancel{\text{m}}} \cdot \frac{1\ \cancel{\text{in}}}{2.54\ \cancel{\text{cm}}} \cdot \frac{1\ \cancel{\text{f}}\text{t}}{12\ \cancel{\text{in}}} = 3281\text{ ft}$$

EXAMPLE 7.4: *Converting Millimeters to Inches*

A dimension for a machine part is 56.3 mm. Use the factor-label method to convert the dimension to inches.

Solution:

$$\#\text{ in} = 56.3\ \cancel{\text{mm}} \cdot \frac{1\ \cancel{\text{cm}}}{10\text{ mm}} \cdot \frac{1\text{ inch}}{2.54\ \cancel{\text{cm}}} = 2.22\text{ in}$$

Area Measure

Any two-dimensional object can be thought to contain small squares whose areas, when added together, give the total area of the object. Hence, area measure is also referred to as *square measure*.

Exponents may be applied to units.

Familiar area equivalencies are $1\ \text{ft}^2 = (12\ \text{inches})(12\ \text{inches}) = 144\ \text{in}^2$, and $1\ \text{yd}^2 = (3\ \text{feet})(3\ \text{feet}) = 9\ \text{ft}^2$.

Some less familiar area equivalences are:

1 rod = 5.5 yd

$1\ \text{rod}^2 = 30.25\ \text{yd}^2$

$1\ \text{km}^2 = 1{,}000{,}000\ \text{m}^2 = 100$ hectares (ha)

$1\ \text{acre} = 160\ \text{rod}^2 = 43{,}560\ \text{ft}^2 = 4840\ \text{yd}^2 = 0.4046856\ \text{ha}$

$1\ \text{mi}^2 = 1$ section = 640 acres

1 township = 36 sections = $36\ \text{mi}^2$

EXAMPLE 7.5: ***Hectares***

Use the factor-label method to determine how many hectares fit into a township.

Solution:

Convert from township to square miles to acres to hectares as follows:

$$\#\,\text{ha} = 1\,\cancel{\text{township}} \cdot \frac{36\,\cancel{\text{mi}^2}}{1\,\cancel{\text{township}}} \cdot \frac{640\ \cancel{\text{acre}}}{1\,\cancel{\text{mi}^2}} \cdot \frac{0.4046856\ \text{ha}}{1\,\cancel{\text{acre}}} = 9324\ \text{ha}$$

Notice that the words "acre" and "hectare" are defined to be area units, and so are not referred to as "square acres" or "square hectares."

EXAMPLE 7.6: ***Converting Square Feet to Square Yards***

A house has $2000\ \text{ft}^2$ of floor that needs to be carpeted. Carpeting is purchased in square yards. How many square yards of carpeting must the homeowner purchase?

Solution:

$$\#\,\text{yd}^2 = 2000\,\cancel{\text{ft}^2} \cdot \frac{1\ \text{yd}^2}{9\,\cancel{\text{ft}^2}} = 222.2\ \text{yd}^2$$

Purchase $225\ \text{yd}^2$ to allow for waste.

Volume Measure

Any three-dimensional object can be thought to contain small cubes whose volumes, when added together, give the total volume of the object. Hence, volume measure is also referred to as *cubic measure*.

Capacity refers to the volume of a container. If a container's dimensions are stated in feet, it is likely that volume will be expressed in cubic feet (ft^3); a container with dimensions in meters will likely have its volume expressed as cubic meters (m^3). Volume measure may also be given in units other than cubic units; for example, gallons, liters, and barrels. In the English system, units of dry volume are sometimes different from units of liquid volume. Table 7.5 gives some measures of liquid and dry volume in both SI and the English system.

The abbreviation for liter is often a script letter ℓ, since the plain letter l *looks so much like the number* 1. *The others are:*

gal = gallon,
qt = quart,
pt = pint,
c = cup,
oz = ounce.

TABLE 7.5: Some SI and English Volume Units and their Equivalents

Measures of liquid volume (SI and English system)	**Measures of dry volume (English system)**
1 ml = 1 cc (cubic centimeter)	1 peck = 8 qt = 16 pt = 32 c
1 liter = 1000 ml = 1000 cc	1 bushel = 4 pecks
1ft^3 = 7.48 U.S. gal	1 bushel = 1.2445 ft^3
1 British Imperial gal = 1.2009 U.S. gal	1 register ton = 100 ft^3 (used for shipping)
1 gal = 4 qt = 8 pt = 16 c = 128 oz	1 cord of wood = 4 feet × 4 feet × 8 feet = 128 ft^3
	1 perch of masonry = $16\frac{1}{2}$ feet × $1\frac{1}{2}$ feet × 1 feet = $24\frac{3}{4}$ ft^3

EXAMPLE 7.7: ***Converting Liters to Cubic Inches***

Use the factor-label method to determine how many cubic inches of displacement a 1.5-liter engine has (round to a whole number).

Solution:

The problem is to convert liters to cubic inches. Since the conversion factor for liters-to-in^3 is not readily available we will use what we already know.

In linear measurement, 1 inch = 2.54 cm. To go from linear to volume measure, raise the conversion unit to the third power. Thus, 1 in^3 = (2.54 cm)^3. Raise both the number and the unit.

$$\#\ \text{in}^3 = 1.5\ \cancel{\ell} \cdot \frac{10^3\ \cancel{\text{cm}^3}}{1\ \cancel{\ell}} \cdot \frac{1\ \text{in}^3}{2.54^3\ \cancel{\text{cm}^3}} = 92\ \text{in}^3$$

In linear measurement, 1 in = 2.54 cm. To go from linear to volume measure, raise the conversion unit to an exponent of 3.

Thus, 1 in^3 = (2.54 cm)^3. Raise both the number and the unit.

EXAMPLE 7.8: ***Bushels***

How many bushels of wheat can a 40 ft by 12 ft by 10 ft truck carry?

Solution:

The volume of the truck is 40 ft × 12 ft × 10 ft = 4800 ft^3.

$$\#\text{bushels} = 4800\,\cancel{\text{ft}^3} \cdot \frac{1.2445\ \text{bushels}}{1\,\cancel{\text{ft}^3}} = 5973.6\ \text{bushels}$$

Board Measure

Although linear units are used in measuring the length of a piece of lumber, lumber quantity is given in the volume measure known as *board feet*. A standard board foot is a piece of lumber that measures 1 foot in length by 1 foot in width by 1 inch in thickness as shown in Figure 7.1 Hence,

1 board foot = 1 $\text{ft}^2 \times 1$ in

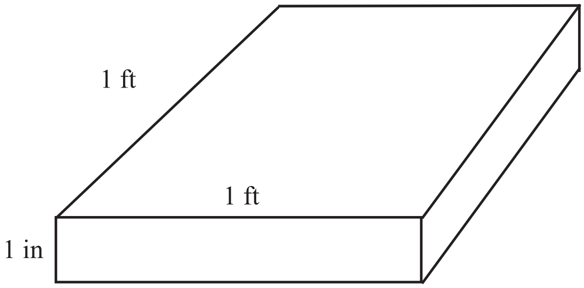


FIGURE 7.1 1 *board foot*

Stock that is less than 1 inch thick is sometimes (but not always) figured as if it were 1 inch. If it is thicker than 1 inch, the number of board feet is figured by its actual thickness.

When calculating quantity of lumber in board feet, care must be taken since the definition gives two dimensions in feet and one dimension in inches. Typically, lumber quantity is calculated from the thickness of wood required for a project. Orders are made for a number of board feet of $\frac{1}{2}$-inch stock, number of board feet of $\frac{3}{4}$-inch stock, and so on.

EXAMPLE 7.9: *Board Feet*

A construction project requires 150 2-by-6-inch boards. Each board is 16 feet long. How many board feet should be ordered?

Solution:

To calculate board feet, two of the dimensions need to be in feet and one dimension needs to be in inches. According, the board's thickness, 2 inches, is left in inches, and the width, 6 inches, is converted to feet by dividing by 12 (because 1 foot has 12 inches). The result is the following expression that converts the dimensional units to board feet.

$$\begin{aligned}\#\text{board feet} &= 150 \cdot 2\,\text{inches} \cdot \frac{6}{12}\,\text{feet} \cdot 16\,\text{feet}\\ &= 2400\ \text{feet}^2 \cdot \text{inches}\\ &= 2400\ \text{board feet}\end{aligned}$$

EXAMPLE 7.10A: *Board Feet*

A construction project requires 22 sheets of $\frac{3}{4}$-inch plywood and 10 sheets of $\frac{1}{2}$-inch plywood. If each sheet of plywood is 4 feet by 8 feet, how many board feet are required? Use actual thickness rather than rounding to 1 inch.

Solution:

For the $\frac{3}{4}$-inch plywood:

$$22 \cdot \frac{3}{4} \text{ inch} \cdot 4 \text{ feet} \cdot 8 \text{ feet} = 528 \text{ feet}^2 \cdot \text{inch} = 528 \text{ board feet}$$

For the $\frac{1}{2}$-inch plywood:

$$10 \cdot \frac{1}{2} \text{inch} \cdot 4 \text{feet} \cdot 8 \text{feet} = 160 \text{board feet}$$

Total board feet = 528 board feet + 160 board feet = 688 board feet

EXERCISES

7.7 Convert 220 yards to:

a) meters
b) centimeters
c) millimeters

7.8 Convert 2.6875 inches to:

a) millimeters
b) miles
c) yards

7.9 Convert from inches to millimeters:

a) $3\frac{17}{32}$ inches
b) $4\frac{1}{32}$ inches
c) $2\frac{11}{16}$ inches
d) $5\frac{31}{32}$ inches
e) $5\frac{3}{8}$ inches
f) 1.53125 inches
g) 10.3125 inches
h) 0.9375 inch
i) 6.4375 inches
j) 2.250 inches
k) 2.90625 inches
l) 0.09375 inch
m) 0.6875 inch

7.10 Convert from millimeters to decimal inches:

a) 15.875 mm
b) $4,710,055$ mm
c) 1234 mm
d) 8.459 mm
e) 24.6062 mm
f) 0.00456 mm
g) 45.836 mm
h) 6.8467 mm

7.11 How many feet are in 50 rods?

7.12 A 5-meter board is cut into 60-cm lengths. If the saw cut is 2 mm wide, how many pieces can be cut to length and what is the length of the scrap piece? How many usable board feet are available?

7.13 How many inches are in $3\frac{1}{2}$ yards?

7.14 Change 5 yards $2\frac{1}{2}$ feet to inches.

7.15 A ball of chalk cord contains 500 feet of cord. How many yards does it contain?

7.16 How many $4\frac{1}{2}$ inch long pieces can be cut from a piece of molding that is 6 feet 8 inches long? Allow $\frac{1}{8}$-inch waste per piece for cutting.

7.17 Convert 15,675 cm to kilometers.

7.18 Change 6,115 m to millimeters.

7.19 Convert 532 cm to meters.

7.20 Change 452 km to meters.

7.21 A timber is 5.5 m long. How many 150-cm pieces can be cut from it, allowing 3 millimeters waste per cut?

7.22 How many yards are in 1 mile?

7.23 Change 4 yard 2 feet 3 inches to centimeters.

7.24 Convert 456 inches to meters.

7.25 Convert 52.5 feet to centimeters.

7.26 How many pieces, each measuring 8.3 cm, can be cut from a bar 245.6 feet in length? Allow 3 mm waste per cut.

7.27 A Texas ranch contains 7.5 sections. How many acres does it have?

7.28 How many square meters are in 1 acre?

7.29 A sailboat has a sail with 15 m^2 of sailcloth. How many square feet is this?

7.30 The foundation of a building required 346 cubic yards of concrete. The concrete was hauled in buggies holding 9 cubic feet each. How many loads were required to pour the foundation?

7.31 An oil drum holds 1.57 cubic feet of oil. How many cubic centimeters is this?

7.32 A 2-by-6-inch plank that is 12 feet long contains how many board feet?

7.33 A lumber yard has 100 hardwood boards, $1\frac{1}{4}$ inches thick by 8 inches wide and 8 feet long in stock. How many board feet is this?

7.34 How many board feet are there in each linear foot of a piece of lumber 2 inches thick by:

a) 4 inches wide?
b) 6 inches wide?
c) 8 inches wide?
d) 12 inches wide?

7.35 High quality fir sells for $153 per 1000 board feet. How much will 36 pieces of 2-by-12-inch fir boards that are each 18 feet long cost?

7.36 A contractor buys 500 Douglas fir planks, 2 inches by 12 inches by 20 feet each, for scaffolding. Find the number of board feet of lumber in one plank and the total number of board feet in the total order.

7.37 How many board feet are there in 10,000 railroad ties that are 8 inches by 8 inches by 8 feet long?

7.38 Wooden posts were purchased for a fence. The dimensions were 4 inches by 4 inches by 12 feet. How many board feet are in one post? How many are in 358 posts?

7.39 Perform the following conversions:

a) 6 liters to quarts
b) 45 gallons to liters
c) 35 gallons to kiloliters
d) 2 barrels to liters
e) 14,065 milliliters to pints
f) 500 deciliters to pints

7.40 A car is driven 25 mph for $4\frac{1}{2}$ hours. How many miles did it travel? How many kilometers did it travel?

7.41 A security fence is 12 feet 6 inches high. What is its height in meters?

7.42 What is the difference in area between 30 square inches and 100 square centimeters? Give answer in square millimeters.

7.43 A mountain is 11,367 feet high. What is its height in meters?

7.44 A mountain peak is 1,358 meters high. How many feet is this?

7.45 A mountain is 32,756 feet high. How many meters is this?

7.46 A French farm covers 75 hectares. How many acres are there on this farm?

7.47 A ball of string line contains 385 feet of line. How many meters is this? How many inches?

7.48 A cubic inch of aluminum alloy is equivalent to how many cubic millimeters?

7.49 The road sign shows 45 km to Madrid. How many miles is this? The sign also gives the speed limit at 55 kilometers per hour. How fast is this speed in miles per hour?

7.50 How many liters are in a 752-cubic inch tank?

7.51 A 500-cubic foot tank contains how many cubic meters?

7.52 A V-8 engine has a 352-cubic inch displacement (cid). What is its metric displacement?

7.53 How many cubic inches will an 8-liter racing car engine displace?

7.54 The density of gasoline is 0.67 kg per liter. What is this density in pounds per gallon?

7.55 A ranch in Texas is said to contain 200 square miles. How many acres would this be? How many sections?

7.56 An Italian machine detail is 345 mm long and 25 mm in diameter. What are its English system dimensions?

7.57 A steel bar is 5.5 inches wide, 0.45 inch thick, and 6 feet 7 inches long. What are its metric dimensions?

7.58 An American tourist in France purchases 16 liters of wine. Since the duty is figured in gallons by the United States customs, how many gallons does he have to declare?

7.3 Angle Measure

Angle measurement is very important in virtually every technical occupation. The most common system of angle measurement has units of *degrees*, in which a circle is divided into 360 equal units called degrees. In some fields, degrees are subdivided into decimal fractions, while in other fields degrees are subdivided into *minutes* and *seconds*. An alternative system of angle measurement, used in electrical work and for many engineering calculations, is the *radian*. The radian unit is related to the radius of a circle and its relationship to circumference, and therefore, unlike the degree, is not arbitrarily defined. A third unit for angle measure, not commonly used except in land surveying, is the *gradian*, in which a circle is arbitrarily divided into 400 equal units called gradians, or simply *grads*. This chapter discusses degrees and radians. The geometry of angles is discussed in Chapter 15.

Many devices are used to measure angles. A ordinary carpenter's square and plumb bob are adequate for most construction work. Laser systems, sine bars, dividing heads, angle gauge blocks, and clinometers are used where greater accuracy is required.

Degrees Decimal and the Degree-Minute-Second System

The most familiar system for angle measure is the *degree decimal system* in which the circle is divided into 360 equal parts, each with a measure of 1 degree (1°). In the degree decimal system, parts of a degree are represented by a decimal fraction, as in 35.375°. Some industries, however, particularly those where older machines are in use, express angle measure in degrees, minutes, and seconds, rather than in degrees decimal. In this system, degrees are subdivided into 60 equal parts called *minutes*. Each minute is further subdivided into 60 equal parts called *seconds*. For example, 43.392° in degree decimal notation is approximately equal to 43 degrees 23 minutes and 31 seconds in degree-minute-second notation. This is written as 43° 23′ 31″.

Many scientific calculators have a key that converts degrees decimal (sometimes identified on calculators as DD) to degree-minute-second (DMS). The conversion can also be done manually as illustrated in the next examples.

EXAMPLE 7.11A: ***Converting from Degrees Decimal to Degree-Minute-Second System***

Convert 52° 27′ 33″ to degrees decimal.

Solution:

STEP 1: 52° is unchanged because the degree portion needs no conversion.

STEP 2: 27′ is converted to a degree decimal through division by 60:

$$27\,\cancel{\text{min}} \cdot \frac{1\,\text{deg}}{60\,\cancel{\text{min}}} = \frac{27\,\text{deg}}{60} = 0.45^\circ.$$

STEP 3: 33″ is converted to degrees decimal through division by 3600, since 1° has 60 min and 1 min has 60 sec. Thus, (60)(60) = 3600″. Hence,

$$33\,\cancel{\text{sec}} \cdot \frac{1\,\cancel{\text{min}}}{60\,\cancel{\text{sec}}} \cdot \frac{1\,\text{deg}}{60\,\cancel{\text{min}}} = \frac{33\,\text{deg}}{3600} = 0.0092^\circ$$

Now the decimal numbers that resulted from the minutes and seconds conversions are added: 0.45° + 0.0092° = 0.4592°

Finally, this sum is appended to the degrees:

$$52^\circ\ 27'\ 33'' = 52.4592^\circ$$

This result can be confirmed with a calculator that has a DD → DMS button. Note that the "inverse" button may be needed to perform this calculation.

EXAMPLE 7.11B: ***Converting from Degrees Decimal to Degree-Minute-Second System***

Convert 36.3112° to degrees-minutes-seconds.

Solution:

STEP 1: 36° remains unchanged because the degree portion needs no conversion.

EXAMPLE 7.11B: ***Converting from Degrees Decimal to Degree-Minute-Second System (Continued)***

STEP 2: The decimal part converts to minutes and seconds. Using the factor-label method, change the decimal part of the degree to minutes first, using the fact that 1 degree contains 60 minutes:

$$0.3112\,\cancel{\text{deg}} \cdot \frac{60\text{ min}}{1\,\cancel{\text{deg}}} = 18.67\text{ min}$$

STEP 3: Now convert the decimal part of the minutes to seconds, using the conversion of 1 minute equals 60 seconds:

$$0.67\,\cancel{\text{min}} \cdot \frac{60\,\text{sec}}{1\,\cancel{\text{min}}} = 40\,\text{sec}$$

Hence, 36.3112° = 36° 18′40″.

Notice that if the conversion process were to result in a figure of more than 60 minutes, then 1 degree is added for every 60 minutes in the answer. The same applies when the answer has more than 60 seconds: 1 minute is added for every 60 seconds.

Working with Angle Measure Greater than 360°

An angle measure that is greater than 360° indicates more than one revolution or rotation of an object.

Sometimes angles greater than 360° are measured. Example 7.12 shows how to handle these angles.

EXAMPLE 7.12: ***Working with Angles Measuring Greater than 360 °***

How many turns of a crank are indicated by an angle of 470°?

Solution:

One turn of a crank corresponds to a full circle, or 360°. Hence,

$$470° = 360° + 90° = 1\tfrac{1}{4}\text{ turns.}$$

Positive angles indicate a counterclockwise direction. Negative angles indicate a clockwise direction.

Radians

A *radian* is defined by the angle formed when the arc of a circle has the same length as its radius. The definition of a radian is illustrated in Figure 7.2, where both the radius and the highlighted arc have the same length and are designated by the letter *r*. Notice that 1 radian is a fairly large angle. In fact, 1 radian is approximately 57.3 degrees. Generalizing the definition of radian to any arc length divided by the circle's corresponding radius leads to the measure of an angle in units of radians.

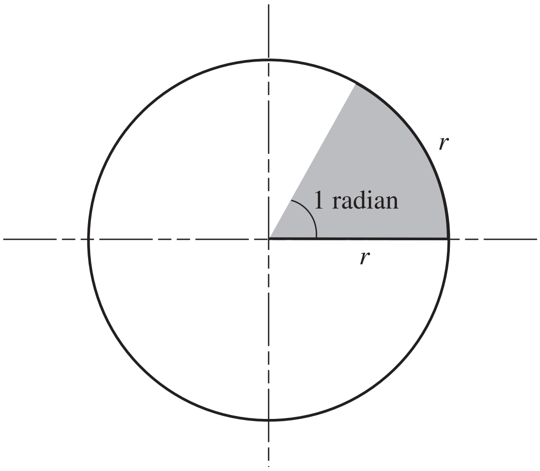


FIGURE 7.2 ***Definition of a radian***

Related to the definition of radian, is the definition of the irrational number pi, represented by the Greek letter π. In fact, the number pi is defined by how many times a circle's diameter will fit along its circumference as shown in Figure 2 for a circle with diameter d. For any circle, the resulting value for pi rounded to the fifth decimal place is approximately 3.14159. The exact number cannot be expressed as a decimal fraction because the number of digits continues indefinitely, so we use the Greek letter pi to designate the number exactly.

An irrational number is a number whose decimal form does not repeat or terminate.

Circumference is discussed in more detail in Chapter 16.

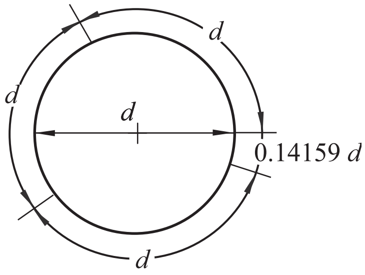


FIGURE 7.3 ***The diameter of a circle fits along its circumference approximately 3.14159 times***

Using the illustration in Figure 7.3 and recognizing that a circle's diameter is twice its radius, we can see that a full circle has an angle of 2π radians.

To equate degrees to radian measure, we note that a full circle is an arc of 360° and has a circumference of $2\pi r$. Replacing r (for "radius") with "radian", we have $360° = 2\pi$ radians, or equivalently, π radians = 180°. Either of these relationships can be used to convert degree measure to radian measure or vice versa.

A handy rule to remember when working with arc lengths is that

$$\text{radian measure} = \frac{\text{arc length}}{\text{radius}}$$

EXAMPLE 7.13: ***Converting Degrees to Radians***

Convert 82° to radians. Round answer to the hundredths place.

Solution: Using the relationship π radians = 180° in the factor-label method:

$$\text{\# radians} = 82° \cdot \frac{\pi \text{ radians}}{180°} = 1.43 \text{ radians}$$

EXAMPLE 7.14: ***Converting Radians to Degrees***

Convert 2.34 radians degrees. Round answer to the hundredths place.

Solution:

Using the relationship π radians = 180° in the factor-label method:

$$\#\text{ radians} = 2.34\text{ radians}\cdot\frac{180^\circ}{\pi\text{ radians}} = 134.07\text{ degrees}$$

When working in radians, common angles such as 30°, 45°, 60°, 90°, 180°, and 360° are usually expressed as fractions involving π. These fractions can be easily determined from the relationship that π radians = 180°. Table 1 shows some common angles in both degree measurement and radian measurement

"Radian" is abbreviated as "rad."

TABLE 7.6: Equivalent Angles in Degree and Radian Measure

Degrees	Radians
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$

EXERCISES

7.59 Convert 48.8573° to degrees, minutes, and seconds.

7.60 Convert 35° 25′ 32″ to a decimal degrees.

7.61 Add: 28° 45′ 27″ and 18° 25′ 63″.

7.62 Subtract: 95° 46′ 53″ from 145° 23′ 42″.

7.63 Convert: 72.5834° to degrees, minutes, and seconds.

7.64 Convert: 65° 22′ 47″ to decimal degrees.

7.65 Add: 38° 22′ with 45′ 38″ and 25° 56′ 33″.

7.66 Subtract: 25° 58′ 49″ from 29° 42′ 38″.

7.67 Add: 45° 0′ 22″ and 59′ 58′.

7.68 Add: 39.5745° + 3° 39′ 22″ + 59.75°.

7.69 Indicate the direction and number of turns of a crank passing through the indicated angle:

a) –554° **b)** 1040.55° **c)** 367° 0′ 59″
d) –784° 12′ 45″ **e)** – 484°

7.70 Convert the following angles to radian measure:

a) 23° **b)** 133° **c)** 65.22°
d) 1° **e)** 75° **f)** 102.1°
g) 19° 28′ 45″ **h)** 372° 12′ 15″

7.71 Convert the following angles to degree measure:

a) 1 radian **b)** 2 radian **c)** 3 radian
d) 2.114 radian **e)** 5.349 radian **f)** 2π radian
g) $\frac{3\pi}{2}$ radians **h)** $\frac{5\pi}{4}$ radians

7.4 Weight and Mass Measure

The fundamental unit of *mass* in the SI system is the kilogram. *Weight*, rather than mass, is the more commonly measured quantity in the English system. The English unit for weight is the pound (lb). The difference between mass and weight is often not clear to students starting out in technical and scientific studies. Simply put, mass is the *amount* of matter an object possesses, while weight is the *force* felt by an object due to the pull of *gravity*.

In the English system, the unit for mass is the slug.

The mass of a given object on the moon is the same as its mass on Earth, but it weighs only 0.15 as much on the moon. On Earth, the weight of a 1 kg mass is approximately 2.2 lb. The same 1 kg mass on the moon weighs 0.33 lb.

EXAMPLE 7.15: ***Weight and Mass Conversion***

How much does 1 gram of water weigh in pounds at sea level?

Solution: The conversion is simple:

$$1\,\cancel{g}\cdot\frac{1\text{ lb}}{454\,\cancel{g}}=0.00221\text{ lb}$$

For a relatively small weight such as this, it may be better to use the smaller unit of the ounce. Thus,

$$0.00221\,\cancel{\text{lb}}\cdot\frac{16\text{ oz}}{\cancel{\text{lb}}}=0.035\text{ oz}$$

EXAMPLE 7.16: ***Mass of Water***

The mass of 1 milliliter of water is 1 gram. What is the mass of 1 liter of water?

EXAMPLE 7.16: ***Mass of Water (Continued)***

Solution:

$$\frac{1\text{g}}{1\,\cancel{\text{ml}}} \cdot \frac{10^3\,\cancel{\text{ml}}}{1\ell} = \frac{10^3\text{ g}}{\ell} = \frac{1\text{kg}}{\ell}$$

That is, a liter of water has the mass of 1 kg. It weighs 2.2 lb.

EXERCISES

7.72 Perform the following conversions:

- **a)** 90,857 g to kg
- **b)** 2.3 kg to oz
- **c)** 478 kg to lb
- **d)** 56.7 oz to lb
- **e)** 4586 lb to tons
- **f)** 14 tons to kg

7.73 A man weighs 198 lb. What is his mass in kg?

7.74 The water in a reservoir weighs 1,000,000 tons. How many gallons of water are in the reservoir?

7.75 An Italian machine tool weighing 500 kg is to be shipped into the United States. The shipping rate quoted was 23 cents per pound. What is the total shipping cost in dollars?

7.5 Measures of Temperature and Heat

Various temperature scales are used in technical applications. Some are *relative* temperature scales and some are *absolute* temperature scales. While temperature gives us a notion of how hot or cold something is, heat is a measure of energy.

Relative Temperature: Fahrenheit and Celsius

Gabriel Daniel Fahrenheit *(1686–1736) was a German-Dutch physicist who manufactured the first mercury-based thermometer in 1715.*

Anders Celsius *(1701–1744) was a Swedish astronomer.*

The familiar temperature scale in the English system is the Fahrenheit (°F) scale. On it, 32°F is arbitrarily defined as the freezing point of water at sea level, and 212°F is arbitrarily defined as the boiling point of water at sea level.

Similarly, the temperature scale used in the metric system is the Celsius (°C) scale in which 0°C is arbitrarily defined as the freezing point of water at sea level and 100°C is arbitrarily defined as the boiling point of water at sea level. For this reason, temperatures given in Celsius are referred to as *degrees centigrade*, for the 100 gradations of the Celsius scale that mark the range of temperature between freezing and boiling for water.

Both the Fahrenheit and Celsius temperature scales are *relative* scales because they were arbitrarily defined. Temperatures below zero are possible on both scales.

Absolute Temperature: Rankine and Kelvin

Absolute temperature is used in many formulas involving calculations for heating, air conditioning, power generation, and chemical processes.

The unit of temperature in the SI system is the *Kelvin* (K). The Kelvin scale is an *absolute scale*; 0 K is called *absolute zero*. Absolute zero is the temperature at which all molecular motion ceases and is therefore the coldest temperature possible. Thus, on the Kelvin scale temperatures are never below zero.

British physicist and mathematician, William Thomson, the first Baron of Kelvin, devised the Kelvin scale.

The relationship between Kelvin absolute zero and zero degree Celsius is

$$0\text{ K} = -273.15°\text{C}$$

Kelvin is an absolute scale of temperature. Celsius is a relative scale.

The interval of temperature measured by 1 K is the same as 1°C. Hence, to convert temperature from Celsius to Kelvin scales, we simply add 273.15 to the Celsius temperature:

$$\text{K} = °\text{C} + 273.15$$

Likewise, to convert from Kelvin to Celsius, we simply subtract 273.15 from the Kelvin temperature:

$$°\text{C} = \text{K} - 273.15$$

In the English system of measurement, the *Rankine* is the fundamental unit of temperature. Absolute zero on the Rankine scale is 0 R and, like the Kelvin, is the temperature at which all molecular motion ceases. The interval of temperature measured by 1 R is the same as 1°F. Since absolute zero is – 459.67°F, the conversion from Fahrenheit to Rankine is

The degree symbol (°) is not used for reporting temperature on the Kelvin or Rankine scales.

$$\text{R} = °\text{F} + 459.67$$

To convert from Rankine to Fahrenheit, the formula is reversed:

$$°\text{F} = \text{R} - 459.67$$

Conversion between Temperature Scales

The freezing point of water is 0°C, which is equivalent to 32°F. However, to convert from °C to °F, simply adding 32 to a Celsius reading is not sufficient. This is because a Fahrenheit degree is almost twice the size of a Celsius degree ($\frac{9}{5}$ its size, to be exact). The conversion from °C to °F is given by the formula:

$$°\text{F} = \frac{9}{5}°\text{C} + 32$$

and the conversion from °F to °C is given by the formula

$$°\text{C} = \frac{5}{9}(°\text{F} - 32)$$

EXAMPLE 7.17: ***Converting Fahrenheit to Celsius, Kelvin, and Rankine***

Change 72.0°F to the equivalent temperature on the Celsius, Kelvin, and Rankine scales. Round to tenths.

Solution:

$$°\mathrm{C} = \frac{5}{9}(°\mathrm{F} - 32) = \frac{5}{9}(72.0 - 32.0) = 22.2°\mathrm{C}$$

$$\mathrm{K} = °\mathrm{C} + 273.15 = 22.2 + 273.15 = 295.4\ \mathrm{K}$$

$$°\mathrm{R} = °\mathrm{F} + 459.67 = 72.0 + 459.67 = 531.7°\mathrm{R}$$

Heat

Closely associated with temperature is *heat*, a form of energy. In the English system, the unit of heat is the *British thermal unit* (Btu). One Btu is the amount of energy required to raise the temperature of 1 lb of water 1°F at sea level. In the metric system, the unit of heat is the calorie (cal). One calorie is the amount of energy required to raise 1 g of water 1°C at sea level.

The conversion between Btu and calorie is:

$$1\ \mathrm{Btu} = 251.996\ \mathrm{cal}$$

Food calories are actually kilocalories, but they are commonly referred to as simply "calories." 1 kilocalorie = 1000 calories. Scientists may refer to a kilocalorie as a "big calorie" and capitalize it: Calorie.

In SI, the *joule* (J) is the unit of energy. The joule is a derived unit:

$$\mathrm{J} = \frac{\mathrm{newton{\cdot}meter}}{\mathrm{second}} = \frac{\mathrm{N{\cdot}m}}{\mathrm{sec}}$$

Joule equates to calorie and Btu in the following way:

$$1\ \mathrm{J} = 0.238846\ \mathrm{cal} = 9.478 \times 10^{-4}\ \mathrm{Btu}$$

Often (and misleadingly) furnaces are rated as having so many Btu. The actual unit is Btu per hour.

Furnaces and other heating equipment are often rated in terms of *power*. Power is simply a rate of energy consumption. One way to express power in the English system is in units of Btu per hour. The *watt* (W) is the unit of power in the SI system. 1 *watt* = 1 *Joule* per second. An important and handy equivalence to remember for power is:

$$1\frac{\mathrm{Btu}}{\mathrm{hr}} = 2.931 \times 10^{-4}\ \mathrm{kW}$$

This is sometimes expressed in terms of energy as

$$1\ \mathrm{Btu} = 2.931 \times 10^{-4}\ \mathrm{kW \cdot hr}$$

EXAMPLE 7.18: ***Btu Rating***

An electrical heating element for a small control room in a wastewater treatment plant is rated at 15 kW. What is the equivalent rating in Btu per hour?

EXAMPLE 7.18: ***Btu Rating (Continued)***

Solution:

Using the factor-label method with the conversion factors just given,

$$\#\frac{\text{Btu}}{\text{hr}} = 15\ \cancel{\text{kW}} \cdot \frac{1\ \text{Btu}}{2.931\times10^{-4}\ \cancel{\text{kW}}\cdot\text{hr}} = 51{,}180\frac{\text{Btu}}{\text{hr}}$$

EXERCISES

7.76 Explain the concept of absolute temperature.

7.77 Explain the difference between temperature and heat.

7.78 Perform the following temperature conversions:

a) 50°F into Celsius degrees
b) 350°F into Celsius degrees
c) – 28°C into Fahrenheit degrees
d) 412°C into Kelvin
e) 219 K into Celsius degrees
f) – 40°F into Celsius degrees
g) 110°C into Fahrenheit degrees
h) – 40°C into Fahrenheit degrees
i) 125°C into Rankine
j) 89 R into Kelvin

7.79 What is the normal clinical temperature of a healthy resident of Paris, France? (In the United States it is 98.6°F.)

7.80 Perform the following heat-related conversions:

a) 24,000 Btu to cal
b) 1.5×10^6 J to kW·hr
c) 100,000 J to cal
d) 100 kW to $\frac{\text{Btu}}{\text{hr}}$
e) 250,000 J to $\frac{\text{Btu}}{\text{hr}}$
f) 96,000 $\frac{\text{Btu}}{\text{hr}}$ to kW

7.6 Measures of Pressure

Pressure may be defined as the force exerted by a fluid over the area of some surface. Hence, pressure is measured in units of force per unit area. The English system pressure units are *pound force per square foot* $\left(\frac{\text{lb}}{\text{ft}^2}\right)$ or *pound force per square inch* (psi). In the metric system the units for pressure are $\frac{\text{N}}{\text{m}^2}$ for *Newton per square meter*. In SI, this unit is called the *pascal* (Pa).

psi = pounds per square inch
ksi = thousand pounds per square inch

$$1\ \text{Pa} = 1\frac{\text{N}}{\text{m}^2}$$

Air Pressure

Air pressure is commonly expressed in terms of the amount of force a column of fluid—mercury (Hg) or water (H_2O), for example—exerts on a surface supporting it. This is the principle of a barometer. Air pressure is also called atmospheric pressure or barometric pressure. It changes with temperature and weather. High pressure fronts are generally associated with clear skies and warmer weather; low pressure is associated with precipitation.

At 5000 feet, atmospheric pressure is 83% of its value at sea level; at 10,000 feet, 69%; at 15,000 feet, 56%; at 20,000 feet, 46%; 25,000 feet, 37%; 30,000 feet, 30% and so on. Consult a reference book for elevation correction factors as needed.

A standard atmosphere (atm) is the average atmospheric pressure observed at sea level. One standard atmosphere is equal to 14.7 psi. Other common pressure equivalences at sea level are shown in Table 7.7. At elevations significantly different from sea level, atmospheric pressure changes because the weight of the air varies inversely with elevation. Atmospheric pressure is less at elevations above sea level because the air exerts a smaller force on a surface supporting it. Below sea level, the opposite is true and atmospheric pressure is greater.

TABLE 7.7: Common Air Pressure Equivalencies at Sea Level

1 atm = 14.7 psi
= 760 mm Hg (or 760 Torr)
= 29.92 inch Hg
= 33.93 feet H_2O
= 1.01325 bar
= 101,325 Pa

EXAMPLE 7.19: ***Atmospheric Pressure Corrected for Elevation***

Denver is often called the Mile High City because of its elevation.

The elevation of Denver, Colorado, is given as 5280 feet. What is the approximate equivalent of 1 standard atmosphere in Denver measured in psi?

Solution:

Since the elevation of Denver is close to 5000 feet, the air pressure should be close to 83% of a standard atmosphere measured at sea level. Or,

$$83\% \text{ of } 14.7 \text{ psi} = (0.83)(14.7) = 12.2 \text{ psi.}$$

Civil engineers often use the term *head* to describe pressure in water towers and water mains. Mechanical engineers use the same term in reference to the suction and discharge sides of a pump. *Feet of head* is the amount of pressure that a water column of the specified height would exert on the area that supports it.

Gases and vapors are considered *compressible fluids*. They exert pressure on the walls of their containers, and when subjected to pressure, they decrease in volume, that is, they are compressed. When pressure is removed from gases and vapors, they expand. Liquids are generally considered incompressible, although compression and expansion of liquids does occur under changing pressure conditions.

EXAMPLE 7.20: ***Water Tower Outlet Pressure***

Use the factor-label method to calculate the pressure in psi at the outlet of a full water tower measuring 200 feet above the outlet.

EXAMPLE 7.20: ***Water Tower Outlet Pressure (Continued)***

Solution:

$$\# \text{psi} = 200 \cancel{\text{ft H}_2\text{O}} \cdot \frac{1 \cancel{\text{atmosphere}}}{33.93 \cancel{\text{ft H}_2\text{O}}} \cdot \frac{14.7 \text{ psi}}{1 \cancel{\text{atmosphere}}} = 86.6 \text{ psi}$$

Conversion factors are provided in Appendix A.

Relative (Gauge) and Absolute Pressure

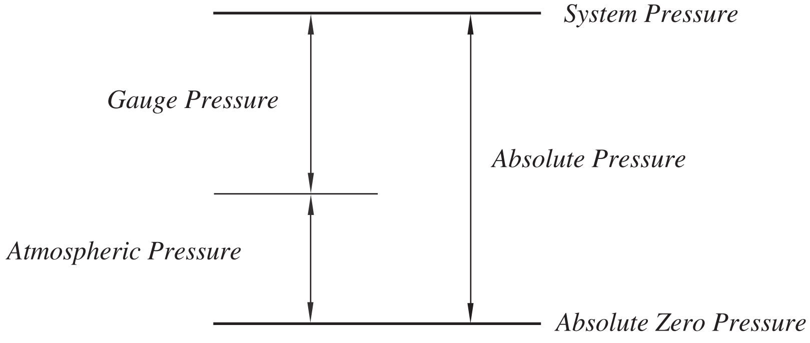
Notice that Example 7.20 does not mention the effect of atmospheric pressure. The reason for this omission is that atmospheric pressure is the same inside and outside a water tower. Therefore, it can be ignored. In certain circumstances, however, the effect of atmospheric pressure must be taken into account. For example, it cannot be ignored in power plant work or in dealing with the low pressures generated by fans and blowers. For this reason, it is important to distinguish between relative – or gauge – and absolute pressure measurements.

A pressure gauge sitting on a table is calibrated to read 0 psi, even though atmospheric pressure is present. Like the gauge thus calibrated, our bodies do not "register" air pressure. Since we are accustomed to the air pressure acting upon us, we sense only that pressure placed on us *in addition* to normal air pressure.

Gauge pressure is a pressure measurement that does not include atmospheric pressure. It is often designated *psig*. Absolute pressure includes atmospheric pressure. It is often designated *psia*. The two pressures are related by the formula:

absolute pressure = gauge pressure + atmospheric pressure

Figure 7.4 illustrates the relationships between absolute, gauge, and atmospheric pressures.



Absolute Pressure = Gauge Pressure + Atmospheric Pressure

FIGURE 7.4 ***Absolute and gauge pressure***

Pressures given as "psi" could mean either "psig" or "psia." Before doing calculations, be sure find out which type of pressure is given.

At sea level the formula becomes:

absolute pressure = gauge pressure + 14.7 psi

psia = psig +14.7 psi at sea level

A gauge reading of 0 psig at sea level indicates that absolute pressure is equal to atmospheric pressure, 14.7 psia.

In very high pressure systems, such as are found in hydraulic machinery, the effect of atmospheric pressure is negligible and can usually be ignored.

EXAMPLE 7.21: ***Absolute Pressure in a Steam Boiler***

A pressure gauge attached to a steam boiler reads 305 psig. What is the absolute pressure of the steam in the boiler?

Solution:

$$\begin{aligned}\text{absolute pressure} &= \text{gauge pressure} + \text{atmospheric pressure}\\ &= 305\,\text{psig} + 14.7\,\text{psi}\\ &= 319.7\,\text{psia}\end{aligned}$$

Vacuums

A vacuum is sometimes thought of as the absence of pressure. Actually, a vacuum exists at any pressure that is less than the prevailing atmospheric pressure. Absolute pressure in a *perfect* vacuum is zero—virtually unattainable, even in the depths of outer space.

Vacuums have many industrial applications. Vacuum chucks hold workpieces on machine tool tables for machining operations in factories. Vacuum furnaces are used to melt metals. In power plants a vacuum is generated in a condenser to improve the efficiency of a steam turbine.

Because a vacuum is almost always measured as pressure below barometric pressure, as shown in Figure 7.5, vacuum pressure gauges report negative pressure.

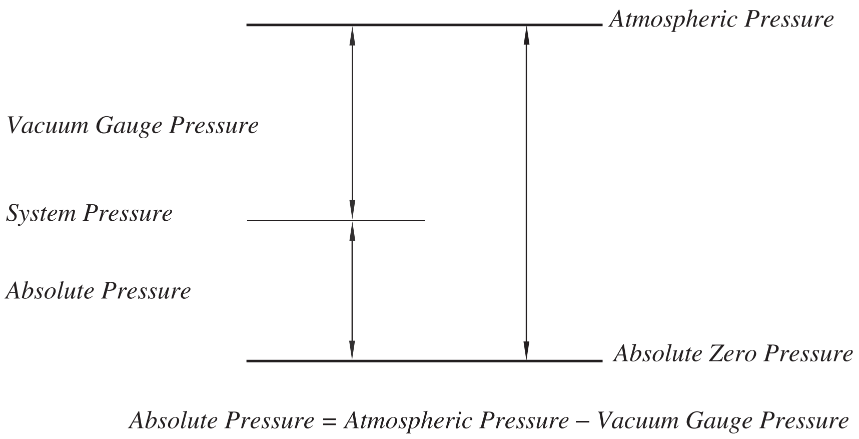


FIGURE 7.5 ***Vacuum measurement***

EXAMPLE 7.22: *Vacuum*

A vacuum gauge attached to a steam condenser reads 15.20 inches Hg. The barometric pressure (corrected for elevation) reads 29.95 inches Hg. What is the absolute pressure in the condenser in terms of inches Hg and psia?

Hg is the symbol for the chemical element mercury.

Solution:

Substitute data into the formula with all units in inches Hg:

absolute pressure = atmospheric pressure – vacuum gauge pressure

= 29.95 inch Hg – 15.20 inch Hg

= 14.75 inch Hg

Convert to psi using the factor-label method:

$$\#\,\text{psi} = 14.75\ \cancel{\text{inches Hg}} \cdot \frac{14.7\ \text{psi}}{29.92\ \cancel{\text{inches Hg}}} = 7.25\ \text{psi}$$

EXERCISES

7.81 A pressure of 50 psi equals how many $\frac{\text{kg}}{\text{cm}^2}$?

7.82 A 3-inch diameter hydraulic cylinder is to be used on a truck tailgate lifting mechanism. If the tailgate is required to lift 1400 lbs, including the weight of the tailgate, what should the pressure in the hydraulic system be?

7.83 The pressure on a helium gas tank reads 240 psig. What is the absolute pressure of the helium?

7.84 A factory requires 150 psig pressure at ground level for its fire lines. How high should the stand-pipe be that is used to obtain this pressure?

7.85 The discharge pressure of a blower is 102 cm H_2O. The actual barometric pressure is 29.90 inches Hg. Calculate the blower discharge pressure in terms of psig, kPa gauge, psia, and kPa absolute.

7.86 A high pressure steam power plant is designed to operate at 50 atmospheres. What is this pressure in terms of psig and kPa gauge?

7.87 A vacuum gauge attached to a vacuum furnace reads 28.70 inches Hg, and the corrected atmospheric pressure is 30.11 inches Hg. What is the absolute pressure in terms of psia?

7.88 A very low vacuum of 0.5 psig is required in an industrial process. The corrected standard atmospheric pressure at the site of this process is 29.90 inches Hg. What should the reading of the vacuum measuring gauge be in terms of inches H_2O and in terms of inches Hg?

7.7 Strain

Stress and strain are often used in the same way in everyday conversation, but are very different physical quantities.

Strain is a very useful quantity in structural engineering. It is closely related to the behavior of materials under various loading conditions. When a material such as steel or aluminum is subjected to a force, it changes shape. Sometimes the change is clearly visible and sometimes the change is only microscopic, but in either case, the change is important to engineers and is quantified as a strain.

Strain is the change in length of a piece of material subjected to a force divided by the original length of the piece of material without force applied to it. Since strain is defined as a length divided by a length, its length units can be any kind, as long as they are identical, for example, $\frac{\text{m}}{\text{m}}$ or $\frac{\text{in}}{\text{in}}$. The lowercase Greek letter epsilon (ε) is a symbol sometimes used to signify strain. Quite often strain is expressed in terms of *microstrain*, that is, $10^{-6}\,\frac{\text{m}}{\text{m}}$ or $10^{-6}\,\frac{\text{in}}{\text{in}}$.

The words "stress" and "strain" are given distinct meanings by engineers and scientists. Strain is the quotient of changes in an object's physical dimensions; stress is the quotient of an applied force by the cross-sectional area over which the force is applied.

Strain is commonly measured with small pieces of wire called *strain gauges*. As the strain gauge becomes deformed under a load, its electrical resistance changes and can be correlated to strain. Strain gauges are often connected in Wheatstone bridges and used in various transducers to measure force, torque, pressure, and other quantities.

EXAMPLE 7.23: *Strain*

The 15-meter cable on a crane stretches 936×10^{-6} m when it lifts a 1500-kg beam. How much strain exists on the cable?

$$\varepsilon = \frac{\text{length change}}{\text{original length}} = \frac{936 \times 10^{-6}\ \text{m}}{15\ \text{m}} = 62\ \text{microstrain}$$

EXERCISES

7.89 A 45-foot steel elevator cable stretches 52×10^{-6} inch when it lifts a 3800-lb elevator car with passengers. How much strain exists on the cable?

7.90 A 25-meter aluminum guy wire supporting a weather tower stretches 123×10^{-6} m during peak wind conditions. How much strain exists on the cable?

7.91 While lifting a 225-lb worker, a 10-foot safety harness stretches to 10 feet, 2 inches. How much strain does the safety harness experience?

7.92 A 10-meter steel post supporting a bridge compresses 456×10^{-6} m when a heavy truck rolls over it. How much strain does the post experience?

7.93 A 25-cm control cable on an airplane lever applies a force of 50 lb to actuate a latch. Upon applying this force, the cable stretches to 25.1 cm. How much strain does the cable experience?

8

ALGEBRAIC EXPRESSIONS

With properties and operations of real numbers in place, we proceed to the work of algebra: the study of the *relationships* among quantities, as represented by *variables* and *constants* in mathematical *expressions* and *equations*. Algebra provides *algorithms*—that is, systematic methods—for solving equations that model physical processes occurring in the “shop,” be it a factory, a garage, a welding site, a construction site, or a laboratory.

8.1 Working with Algebraic Expressions

Technical shop mathematics largely entails applying a formula directly or rearranging a formula to solve for a particular unknown variable. However, before we can solve an equation we need to know how to perform the basic operations on algebraic expressions within the equation: the four arithmetic operations, powers and roots, and the processes of expanding and factoring an expression. For example, flow of fluid through a pipe is given by the formula

$$Q = 2.44d^2V$$

where the variables Q, d, and V represent flow in gallons per minute, diameter of pipe in inches, and velocity of flow in feet per second, respectively. The number 2.44 is a constant in the relationship.

Keep in mind that the properties of real numbers, such as the commutative property of multiplication, also apply to variable expressions.

Definitions

Variable

A variable is a value that is unknown or subject to change. Usually a variable is represented by a letter, such as x, y, z, s or t. The point of most algebra problems is to discover the value of an *unknown* variable. Hence, the definition of *variable* is a letter assigned to a quantity that is unknown or that may change.

Constant

A constant is a value in a mathematical expression that does not change. Usually a constant is an actual number, but sometimes it is represented by a letter.

Term

A term is a constant, a variable, or a product of constants and variables.

A single term may have many factors and the variables in a term may be raised to a positive exponent as in the term $124x^4y^7z^3$. Any single terms such as y, 0.17, $4x$, $-1.6abc^2$, are called *monomials.*

Coefficient

A coefficient is a constant by which a variable is multiplied.

Coefficients in the following terms appear in blue:

$$4x \qquad -1.6abc^2 \qquad \frac{1}{2}m - 3n$$

In the expression $\frac{a}{3} + 6b$ the coefficient of a is $\frac{1}{3}$, and the coefficient of b is 6.

The expression $x^2 - x + 0.004$ has three coefficients: 1, –1, and 0.004.

Several points about coefficients must be emphasized:

- A constant that stands alone is called a *constant coefficient*; for example, the number 3 in the expression $x^2 - x + 3$ is a constant coefficient.
- A variable that stands alone is understood to have a coefficient of 1; for example, 1 is the coefficient of x^2 in $x^2 - x + 3$.
- The value of a coefficient includes the sign of the operation that immediately precedes it; for example, −1 is the coefficient of x in $x^2 - x + 3$.
- A fractional coefficient may assume either of two forms: either as a fraction in front of the variable, as in $\frac{1}{2}m$, or as part of the variable, as in $\frac{a}{3}$, in which the coefficient is $\frac{1}{3}$ since $\frac{a}{3} = \frac{1}{3}a$.

Expression

An expression is a group of terms joined by addition and/or subtraction.

An expression like $\frac{12}{x}$ *is called a* rational expression.

For example,

$$\frac{a}{3} + 6b \qquad \frac{1}{2}m - 3n \qquad x^2 - 0.1x + 0.004$$

Polynomial

In general, a sum of monomials is a *polynomial*.

A *binomial* contains two terms, a *trinomial* contains three. The word “sum” in this sense encompasses both addition and subtraction since subtraction of a term can always be written as *addition of the negative* of that term. “Polynomial” in its broadest sense means anything from a constant on its own to an infinite sum of monomials.

Degree of a Polynomial

The degree of a polynomial is determined by the highest exponent present on a variable in an expression.

Thus, $3x + 7$ is a *first degree polynomial* since the highest exponent is 1; $x^2 + 2x - 5$ is a *second degree polynomial* since the highest exponent is 2. A constant is a *zero degree polynomial*, because a constant c can be regarded as cx^0 since $cx^0 = c \cdot 1 = c$.

An expression in which a variable appears in the denominator is called a *rational expression*.

Rational Expression

For example, $\frac{4}{x}$, $\frac{m_1 m_2}{d^2}$, and $\frac{a+5}{a-5}$ are all rational expressions.

Terms and/or expressions set equal to one another form an *equation*.

Equation

For example, $x + y = 9$ is an equation.

Terms and/or expressions set unequal to one another form an *inequality*. There are several kinds of inequalities.

Inequality

For example, $x \neq 5$, $x > 5$, $x \geq 5$, $x < 5$, and $x \leq 5$ are all inequalities.

Evaluating Expressions

As previously defined, a variable represents an unknown numerical value in a term or expression. For example, in the term $3x^2$ the variable x is unknown. A term or expression can be evaluated by substituting a numerical value for the variable. Thus, $3x^2$ evaluated at $x = -2$ is $3(-2)^2 = 3(4) = 12$. Notice that -2 is squared before it is multiplied by 3 as required by the rules for the order of operations: PEMDAS.

EXAMPLE 8.1A: ***Evaluating an Expression***

Evaluate the expression $ab^3 + 4a^2b - \frac{a}{b}$ for $a = 3$ and $b = -1$.

Solution:

$$\begin{aligned} ab^3 + 4a^2b - \frac{a}{b} &= (3)(-1)^3 + 4(3)^2(-1) - \left(\frac{3}{-1}\right) \\ &= (3)(-1) + 4(9)(-1) - (-3) \\ &= -\cancel{3} - 36 + \cancel{3} = -36 \end{aligned}$$

Notice how the signs and operations are combined using the rules of signed number operations to give the intermediate step, $-3 - 36 + 3$.

EXAMPLE 8.1B: *Evaluating an Expression*

Evaluate the expression $\sqrt{x}-8x^2+\frac{3x}{2}$ for $x = 4$.

Solution:

$$\sqrt{x}-8x^2+\frac{3x}{2}=\sqrt{4}-8(4^2)+\frac{3(4)}{2}=2-8(16)+\frac{12}{2}=2-128+6=-120$$

According to the rules for the order of operations (PEMDAS), operations inside the radical are done first, as if the radical were a set of parentheses (P). Then the radical itself is evaluated, since radicals are like exponents (E). Then multiplication and division (M and D) are done in the order they appear, followed by addition and subtraction in the order they appear (A and S).

EXAMPLE 8.2A: *Evaluating a Formula*

Find the effective outside length L of a V-belt from the following formula:

$$L = 2C + 1.57(D + d) + \frac{(D - d)^2}{4C}$$

where center distance C between sheaves is 750 mm, the effective outside diameter D of the large sheave is 175 mm, and the effective diameter d of the small sheave is 125 mm.

Solution:

Solution of this problem amounts to evaluating the right hand side of the formula using the values given for each variable. This will yield the value of L.

$$\begin{aligned}L&=2C+1.57(D+d)+\frac{(D-d)^2}{4C}\\&=2(750)\text{ mm}+1.57(175\text{mm}+125\text{ mm})+\frac{(175\text{ mm}-125\text{ mm})^2}{4(750\text{ mm})}\\&=1500\text{ mm}+1.57(300\text{ mm})+\frac{50^2\text{ mm}^{\cancel{2}}}{3000\text{ }\cancel{\text{mm}}}\\&=1972\text{ mm}\end{aligned}$$

The final answer is rounded to the ones place.

Notice that the units $\frac{\text{mm}^2}{\text{mm}}$ cancel to give mm.

EXAMPLE 8.2B: *Evaluating a Formula*

G.S. Ohm (1757-1854) discovered more than one law of electricity, but V=IR is what is commonly referred to as "Ohm's Law."

Ohm's law gives the relationship between direct current voltage V, resistance R, and current I flowing through an electrical circuit. The formula for Ohm's law where V has units of volts (V), I has units of amperes (A), and R has units of ohms (Ω), is $V = IR$.

If a 52-milliamp (mA) current is to be maintained through a device with 25 kilo-ohm (kΩ) resistance, what is the applied voltage (V) across the circuit?

EXAMPLE 8.2B: ***Evaluating a Formula (Continued)***

Solution:

Since the formula requires units of amperes (A) and ohms (Ω), we need to convert milliamps and kilo-ohms to the proper units using the factor-label technique learned in Chapter 7.

$$\#\text{A} = 52\ \cancel{\text{m}}\text{A} \cdot \frac{1\ \text{A}}{1000\ \cancel{\text{m}}\text{A}} = 0.052\ \text{A} \qquad \#\Omega = 25\ \cancel{\text{k}}\Omega \cdot \frac{1000\ \Omega}{1\ \cancel{\text{k}}\Omega} = 25{,}000\ \Omega$$

Substituting values in for I and R to find V gives:

$$V = (0.052\ \text{A})(25{,}000\ \Omega) = 1300\ \text{V}$$

V *is the symbol for volts; an italic letter, V, is the variable for voltage.*

EXERCISES

8.1 Identify the coefficients in each expression.

a) $4x^2 - 3x + 2$ **b)** $-2x^3 + x^2 - 4x + 7$ **c)** $-0.1x^2 - 2.3x + 4$

d) $x^2 + \frac{x}{3} + \frac{1}{2}$ **e)** $\frac{1}{2}x^2 - 2x + 3$ **f)** $\frac{x^2}{4} - \frac{2x}{3} + \frac{1}{5}$

g) $x^2 + x - 1$ **h)** $-3.7x^3 + 2x^2 + 7.2x - 3$

8.2 Evaluate $3x^2 - 2x + 1$ for the indicated value of x.

a) $x = 0$ **b)** $x = 2$ **c)** $x = -2$

8.3 Evaluate $-\frac{x^2}{2} - \frac{x}{3} + 2$ for the indicated value of x.

a) $x = -1$ **b)** $x = 3$ **c)** $x = 0$

8.4 Evaluate $5a^2 + 2b - c$ for the indicated values of a, b, and c.

a) $a = 0$, $b = 2$, $c = 3$ **b)** $a = -1$, $b = -2$, $c = 1$ **c)** $a = 2$, $b = 0$, $c = -3$

8.5 Evaluate $\frac{-a^3}{3} + \frac{b^2}{2} + c - d$ for the indicated values of a, b, c, and d.

a) $a = 1 \quad b = 2 \quad c = 4 \quad d = -1$

b) $a = -1 \quad b = -2 \quad c = -2 \quad d = 0$

c) $a = 0 \quad b = 1 \quad c = -1 \quad d = 3$

8.2 Operations on Expressions — Exponents

The process of solving an equation often requires simplification of expressions within the equation, using the four arithmetic operations on algebraic terms, as well as the operations of taking a root and finding an absolute value.

Exponents—Powers of x

Operations on algebraic terms are based on the exponent rules, as covered in Chapter 6. There the discussion was limited to real number *constants*; here we discuss how exponents on *variables* are handled.

The rules for applying exponents to variables are identical to the rules for applying exponents to constants. To become accustomed to x and y notation, the exponent rules are illustrated in Examples 8.3A through 8.3E, using x and y as the variables.

Summary of Rules for Applying Exponents to Variables

$$x^0 = 1 \text{ for } x \neq 0 \qquad x^1 = x \qquad x^m \cdot x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn} \qquad x^{-m} = \frac{1}{x^m} \qquad (xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

EXAMPLE 8.3A: ***Applying Exponents to Variables***

Observe how exponents are applied to variables in each of the following equations.

a) $(4x)^0 = 1 \text{ for } x \neq 0$ since $(4x)^0 = 4^0 \cdot x^0 = 1 \cdot 1 = 1$

b) $\frac{2}{11}x^0 = \frac{2}{11}$ for $x \neq 0$ since $\frac{2}{11}x^0 = \frac{2}{11} \cdot 1 = \frac{2}{11}$

Notice how parentheses indicate that both coefficient and variable are raised to the exponent; the absence of parentheses indicates that only x is raised to the exponent, with the restriction that $x \neq 0$, since 0^0 is undefined.

EXAMPLE 8.3B: ***Applying Exponents to Variables***

$(x + y)^n \neq x^n + y^n$

Observe how exponents are applied to variables in each of the following equations.

a) $x^5 = x \cdot x \cdot x \cdot x \cdot x$

b) $7x^4 \cdot 3x = 7 \cdot 3 \cdot x^4 \cdot x^1 = 21x^{4+1} = 21x^5$

c) $\frac{1}{4}x \cdot \frac{5}{7}x = \frac{1}{4} \cdot \frac{5}{7} \; x^{1+1} = \frac{5}{28}x^2$ or $\frac{5x^2}{28}$

d) $(x + y)^2(x + y)^4 = (x + y)^{2+4} = (x + y)^6$

EXAMPLE 8.3C: *Applying Exponents to Variables*

Observe how exponents are applied to variables in each of the following equations.

a) $\frac{10x^6}{2x^3} = (10 \div 2)x^{6-3} = 5x^3$

b) $\frac{6x}{14x^4} = \frac{3 \cdot \cancel{2}}{7 \cdot \cancel{2} x^{4-1}} = \frac{3}{7x^3}$

c) $\frac{(a+b)^5}{(a+b)^2} = (a+b)^{5-2} = (a+b)^3$

For multiplication of terms, coefficients are multiplied, and exponents are added. When dividing terms, coefficients are divided and exponents are subtracted.

Subtraction of exponents on like bases amounts to *canceling* the greatest common factor (GCF), thus reducing the algebraic fraction to its lowest terms. We can thus write

$$\frac{15x^5}{3x^2} = \frac{\cancel{15} x \cdot x \cdot x \cdot \cancel{x} \cdot \cancel{x}}{\cancel{3} \cancel{x} \cdot \cancel{x}} = 5x^3$$

In division of *unlike* terms, only the coefficients may be reduced. For example,

$$\frac{22\,x^2}{4\,y^2} = \frac{11\,x^2}{2\,y^2}$$

EXAMPLE 8.3D: *Applying Exponents to Variables*

Observe how exponents are applied to variables in each of the following equations.

a) $(x^3)^4 = x^{3\cdot 4} = x^{12}$

b) $(3ab^2)^4 = (3)^4(a)^4(b^2)^4 = 81a^4b^{2\cdot 4} = 81a^4b^8$

c) $\left(\frac{x^4}{7}\right)^2 = \frac{x^{4\cdot 2}}{7^2} = \frac{x^8}{49}$

d) $\left(\frac{4}{3x^2}\right)^2 = \frac{4^2}{3^2 x^{2\cdot 2}} = \frac{16}{9x^4}$

Notice that all terms inside parentheses are raised to the power.

EXAMPLE 8.3E: *Applying Exponents to Variables*

Observe how exponents are applied to variables in each of the following equations.

a) $8x^{-2} = \frac{8}{x^2}$

b) $(3x)^{-1} = \frac{1}{3x}$

c) $\frac{6}{x^{-1}} = 6x$

EXAMPLE 8.3E: *Applying Exponents to Variables (Continued)*

d) $\dfrac{5^{-1}}{x^{-3}} = \dfrac{x^3}{5}$

In the first expression, $8x^{-2}$, only the variable changes position; the coefficient is not affected because it is not raised to the exponent. However, in the second expression, $(3x)^{-1}$, both the coefficient and the variable are raised to a power of -1 so both move to the denominator.

The variable in $\dfrac{6}{x^{-1}}$ moves to the numerator because its power is negative.

Finally, in the last expression, $\dfrac{5^{-1}}{x^{-3}}$, the numerator and denominator change position to eliminate the negative sign of the exponent on each.

EXERCISES

8.6 Evaluate each expression.

a) $3a \cdot 2b$
b) $a \cdot 6b$
c) $x \cdot z \cdot x \cdot z$
d) $x \cdot xy \cdot xyz \cdot x$
e) $3bc(2e - 3g + 2)$
f) $5(3ab - 6bd)$
g) $z \cdot z \cdot z$
h) $(x^2y^3)^2$
i) $\dfrac{6ab}{2a}$
j) $6xy \cdot 3z$
k) $5a \cdot b \cdot 3c$
l) $a \cdot b \cdot c \cdot b$
m) $4x(3y - 2z)$
n) $2(3x)$
o) $3(a + b - c)$
p) $ab \cdot ab$
q) $-4(x - y + z)$
r) $\dfrac{12xy}{6y}$
s) $4b \cdot 3$
t) $a \cdot b \cdot 2d$
u) $3ab \cdot 3ab \cdot 3ab$
v) $12z(xy - 3y + 2x)$
w) $x(x - 2)$
x) $5(b + c) - a$
y) $(-w)^2$
z) $(-w)^3$

8.7 Evaluate each expression.

a) $(d^5)^2$
b) $-(c^6)^2$
c) $w^{0.5} \div w^{0.25}$
d) $35cd \div 7cd$
e) $(x^2y)(x^3)$
f) $\dfrac{-32rst^2}{4rs}$
g) $\left(\dfrac{c^2}{d^3}\right)^{\frac{5}{6}}$
h) $x^2 \cdot x^4$
i) $(-a)(a)(-a)$
j) $20a \div (-4)$
k) $(w^{\frac{1}{2}})^0$
l) $x^{-\frac{1}{2}} \cdot x^{\frac{1}{2}}$

m) $\dfrac{2a(-3ab^2)}{ab}$ **n)** $\dfrac{(pd^2q\cdot p^3dq^2)}{p^{\frac{2}{3}}d^{\frac{5}{2}}q^2}$ **o)** $\left(\dfrac{k^{\frac{1}{4}}}{k^{\frac{1}{2}}}\right)^4$

p) $(5a^2x^3)^2 \div (3b^2xy^3)^3$ **q)** $-56x^3y^4 \div (-7xy^2)$ **r)** $21y^4 \div y$

s) $\dfrac{6(-4a^3b^2)}{-12a}$ **t)** $\dfrac{128a^3bc^2}{-8abc}$

8.3 Operations on Expressions — Radicals

A radical (or root) is a fractional exponent. The *n*th root of *x* is defined as follows: $\sqrt[n]{x} \equiv x^{\frac{1}{n}}$ where *x* itself is the *radicand*.

Thus, $\sqrt{x} \equiv x^{\frac{1}{2}}$, $\sqrt[3]{x} \equiv x^{\frac{1}{3}}$, and so on.

Radical x is written as $\sqrt{x}$. It is the same as $x^{\frac{1}{2}}$.

A more general rule is this:

$$\sqrt[n]{x^m} \equiv x^{\frac{m}{n}}$$

Thus, $\sqrt[2]{x^3} \equiv x^{\frac{3}{2}}$, for example.

The symbol ≡ means "is defined as."

All the rules of integer exponents hold for roots as well. In addition, radicals follow these rules:

$$\sqrt{xy} = \sqrt{x}\sqrt{y} \qquad \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$a\sqrt{x} + b\sqrt{x} = (a+b)\sqrt{x} \qquad \sqrt{x} + \sqrt{y} \neq \sqrt{x+y}$$

The *square root* is the most often encountered radical. Perfect squares, such as 1, 4, 9, …, are easily simplified when they appear in a radical. For example, $\sqrt{9} = 3$, $\sqrt{25} = 5$, and so on. Often, however, a situation arises in which a number or an algebraic term appearing under a radical is not a power of 2 or some other easy-to-work-with root. In such a case, the radicand cannot be completely eliminated, but it can sometimes be simplified. Consider the different cases of the square root.

A perfect square is a number that can be represented by x^2, with x an integer. Thus, x is the square root of x^2.

CASE 1. A constant alone: $\sqrt{20} = \sqrt{4\cdot 5} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

The constant 20 is factored so that a *partial root* can be extracted.

CASE 2. A variable alone with an even exponent: If the exponent on the variable is even (that is, x^{2n}) and a square root is desired, we simply divide the exponent by 2 to get the resulting power on the variable. That is,

$$\sqrt{x^{2n}} = \left(x^{2n}\right)^{\frac{1}{2}} = x^{\frac{2n}{2}} = x^n$$

Thus, $\sqrt{x^4} = \left(x^4\right)^{\frac{1}{2}} = x^{\frac{4}{2}} = x^2$.

Working it through in an alternate way, we could write

$$\sqrt{x^2 \cdot x^2} = \sqrt{x^2}\sqrt{x^2} = x \cdot x = x^2$$

Notice that 2n + 1 is always an odd number.

Case 3. A variable alone with an odd exponent: If the exponent on the variable is odd, the goal is to take out the highest possible root. To do this, the radicand is separated into two factors: one with a base to the highest possible *even* power, the other with a base to a power of 1. We then extract the root on the even power as before and leave the remaining variable in the radical:

$$\sqrt{y^{2n+1}} = \sqrt{y^{2n} \cdot y^1} = \sqrt{y^{2n}}\sqrt{y^1} = y^n\sqrt{y}$$

Thus, $\sqrt{y^7} = \sqrt{y^6 \cdot y} = y^3\sqrt{y}$, and $\sqrt{x^{13}} = \sqrt{x^{12}x^1} = x^6\sqrt{x}$.

Case 4. A constant and a variable:

$$\sqrt{50x} = \sqrt{25 \cdot 2 \cdot x} = \sqrt{25}\sqrt{2x} = 5\sqrt{2x}$$

$$\sqrt{48x^2} = \sqrt{16 \cdot 3 \cdot x^2} = \sqrt{16}\sqrt{3} \cdot x = 4x\sqrt{3}$$

$$\sqrt{\frac{4}{x^2}} = \frac{\sqrt{4}}{\sqrt{x^2}} = \frac{2}{x} \qquad \sqrt{\frac{3x^5}{100}} = \frac{\sqrt{3x^5}}{\sqrt{100}} = \frac{x^2\sqrt{3x}}{10} \qquad \sqrt{\frac{16}{3}x^{11}} = 4x^5\sqrt{\frac{x}{3}}$$

The radical does not need to be broken up to work with the separate parts.

Example 8.4: ***Simplifying Radicals***

Notice that x is always placed to the left of the final radical so that an extracted x is not seen as part of the radicand.

The example $\sqrt{x} + 4\sqrt{x} - 12\sqrt{x} = -7\sqrt{x}$ *illustrates how like radical terms are combined through addition.*

Observe the simplification of the radical expressions.

$$\sqrt{2x} \cdot 3\sqrt{x} = 3\sqrt{2x^2} = 3x\sqrt{2} \qquad x\sqrt[3]{x} = x^1x^{\frac{1}{3}} = x^{1\frac{1}{3}} = x^{\frac{4}{3}}$$

$$\left(\sqrt{2x}\right)^3 = \left(2^{\frac{1}{2}}x^{\frac{1}{2}}\right)^3 = 2^{\frac{3}{2}}x^{\frac{3}{2}} \qquad \frac{\sqrt{20x}}{2x} = \frac{\sqrt{4 \cdot 5x}}{2x} = \frac{\cancel{2}\sqrt{5x}}{\cancel{2}x} = \frac{\sqrt{5x}}{x}$$

$$\frac{\sqrt{x}}{6\sqrt{x^3}} = \frac{\sqrt{x}}{6\sqrt{x^2}\sqrt{x}} = \frac{\sqrt{x}}{6x\sqrt{x}} = \frac{1}{6x} \qquad \sqrt{\frac{16x^3}{9y^4}} = \frac{\sqrt{16x^3}}{\sqrt{9y^4}} = \frac{4x\sqrt{x}}{3y^2}$$

$$11x^{-\frac{1}{2}} = \frac{11}{x^{\frac{1}{2}}} = \frac{11}{\sqrt{x}} \qquad (81x)^{-\frac{1}{2}} = \frac{1}{(81x)^{\frac{1}{2}}} = \frac{1}{81^{\frac{1}{2}}x^{\frac{1}{2}}} = \frac{1}{9\sqrt{x}}$$

In $11x^{-\frac{1}{2}}$ only the variable is moved to the denominator, since the coefficient is not raised to the $-\frac{1}{2}$ power. In the last two examples the denominator should be rationalized, a process discussed in the next section.

EXERCISES

8.8 Express each item in equivalent radical form.

a) $w^{\frac{1}{2}}, z^{\frac{2}{3}}, a^{\frac{4}{3}}, q^{0.4}, x^{-\frac{1}{2}}, p^{2.5}, r^{-\frac{2}{3}}$ **b)** $(x-y)^{\frac{1}{2}}, (x+y)^{\frac{1}{4}}, z^{\frac{6}{5}}, y^{\frac{2}{3}}, x^{\frac{3}{2}}$

c) $4p^{\frac{2}{3}}, (4p)^{\frac{2}{3}}, 5(2x-3y)^{\frac{1}{2}}, 5w^{1.5}, 2z^{3.5}$ **d)** $-3x^{\frac{1}{2}}, (-4z)^{\frac{1}{2}}, -8a^{\frac{1}{3}}, 8b^{\frac{2}{3}}, (8n)^{\frac{2}{3}}$

e) $3b^{\frac{1}{2}}, (3b)^{\frac{1}{2}}, (x^2-y^2)^{\frac{1}{2}}, (-8y)^{\frac{1}{3}}, 2w^{\frac{3}{4}}$

8.9 Multiply and express result in equivalent radical form.

a) $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$ **b)** $w^2 \cdot w^{\frac{1}{4}}$ **c)** $z^{\frac{1}{2}} \cdot z^3 \cdot z^{\frac{2}{3}}$

8.10 Divide and express in equivalent radical form.

a) $z^{\frac{2}{3}} \div z^{\frac{1}{4}}$ **b)** $x^{\frac{3}{4}} \div x^{\frac{1}{2}}$ **c)** $\dfrac{q^{\frac{1}{2}}}{q^{\frac{1}{4}}}$

8.4 Operations on Expressions — Rationalizing the Denominator

Consider a fraction like $\sqrt{\frac{x}{3}}$. This may be written as $\frac{\sqrt{x}}{\sqrt{3}}$.

However, a radical in the denominator of an expression is considered to be in a nonstandard form.

The square root of a number that is not a perfect square is an irrational number.

This is because a number like $\sqrt{3}$ is an *irrational* number, which is often not desirable in a denominator. Such an expression is converted into an equivalent fraction by *rationalizing* its denominator. This is accomplished by multiplying numerator and denominator by the radical found in the denominator. Since a number multiplied by itself is 1, the process of rationalizing the denominator does not change its numerical value – only its form.

EXAMPLE 8.5A: ***Rationalizing the Denominator***

Rationalize $\dfrac{\sqrt{x}}{\sqrt{3}}$

Solution:

$$\frac{\sqrt{x}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3x}}{\sqrt{9}} = \frac{\sqrt{3x}}{3}$$

The factors of 3 in the final answer do not cancel because the 3 under the radical is not equal to 3 standing alone.

EXAMPLE 8.5B: ***Rationalizing the Denominator***

Rationalize the denominator in $\frac{11}{\sqrt{x}}$.

Solution:

$$\frac{11}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{11\sqrt{x}}{x}$$

Since multiplying a radical by itself produces the radicand, we have eliminated the radical in the denominator and left a radical in the numerator only. This is considered correct form for a fraction or rational expression.

EXAMPLE 8.5C: ***Rationalizing the Denominator***

Rationalize the denominator in $\frac{1}{9\sqrt{x}}$.

Solution:

$$\frac{1}{9\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{9x}$$

Notice that we cannot further simplify the expression by canceling *x*. The *x* in the radical is not identical to *x* standing on its own.

EXERCISES

8.11 Rationalize the denominator in each expression.

a) $\frac{3x}{\sqrt{11}}$ **b)** $\frac{10x^4}{\sqrt{8x}}$ **c)** $\frac{9x^3}{2\sqrt{45}}$ **d)** $\sqrt{\frac{x^2}{y}}$ **e)** $\frac{-2x}{\sqrt{12}}$

f) $\frac{-1}{\sqrt{10x}}$ **g)** $\frac{x^2}{\sqrt{32x}}$ **h)** $\frac{3}{4\sqrt{2x}}$ **i)** $\frac{x}{2\sqrt{x}}$ **j)** $\sqrt{\frac{9}{16x}}$

8.5 Operations on Expressions — Combining Like Terms

Two or more terms are called *like terms* if they are composed of the same variable or variables raised to identical exponents.

For example, $11x$, $2x$, $\frac{x}{3}$ are like terms, as are ab^2, $-3ab^2$, $\frac{ab^2}{10}$. Terms that differ in variable or have different exponents on like variables are *unlike terms*. For example, a^2b and ab^2 are unlike terms.

Expressions are simplified by *combining like terms* through addition and subtraction on the coefficients of the like terms.

EXAMPLE 8.6: ***Identifying Like Terms***

Consider the following groups of like terms and identify what each group has in common.

GROUP 1: $x, -5x, \frac{x}{2} \qquad x$

GROUP 2: $m^3n,\ -7.5m^3n,\ \frac{1}{5}m^3n \Rightarrow m^3n$

GROUP 3: $x\sqrt{y},\ 8xy^{\frac{1}{2}},\ 2x\sqrt{y} \Rightarrow x\sqrt{y}$

The coefficients of like terms, in most cases, are different. Only the variable portion of a term determines if it is "like" another term. For example, to simplify the expression $7x + 2y - x - 11y + 8$, we group like terms together: $(7x - x) + (2y - 11y) + (8)$.

In this example the coefficient of $-x$ is understood to be -1; also, $-11y$ is moved next to $+ 2y$; while $+ 8$ stands alone, since it is the only constant in the expression. Simplifying by combining coefficients of like terms gives $6x - 9y + 8$.

If we were to combine the terms within each group from the last example, the process would look like this:

GROUP 1: $x + (-5x) + \frac{x}{2} = x - 5x + \frac{1}{2}x = -3\frac{1}{2}x$

GROUP 2: $m^3n + (-7.5m^3n) + \frac{1}{5}m^3n = m^3n - 7.5m^3n + 0.2m^3n = -6.3m^3n$

GROUP 3: $x\sqrt{y} + 8xy^{\frac{1}{2}} + 2x\sqrt{y} = x\sqrt{y} + 8x\sqrt{y} + 2x\sqrt{y} = 11x\sqrt{y}$

*Like Bases that differ only by the magnitude of an exponent are **not** like terms. For example, rs^2t^3 and rs^2t^2 are unlike terms.*

EXAMPLE 8.7A: ***Combining Like Terms***

Combine like terms in each expression.

a) $2a + 3b + 4a$

b) $6x - 2w - 4x$

c) $3a + 2b - 2c + a$

d) $7ab + 2z - 7a - 2z$

Solution:

a) $2a + 3b + 4a = 6a + 3b$

b) $6x - 2w - 4x = 2x - 2w$

c) $3a + 2b - 2c + a = 4a + 2b - 2c$

d) $7ab + 2z - 7a - 2z = 7ab - 7a$ since $2z - 2z = 0$

A term includes the operation and/or sign immediately before it. When rearranging an expression, failure to move the sign might cause an operation to be "dropped."

EXAMPLE 8.7B: *Combining Like Terms*

Simplify the expression $-4x^2y^2+12x^2y-x^2y^2-5xy^2+3x^2y$.

Solution:

Grouping and combining like terms:

$$\left(-4x^2y^2-x^2y^2\right)+\left(12x^2y+3x^2y\right)-\left(5xy^2\right)=-5x^2y^2+15x^2y-5xy^2$$

The term $-5xy^2$ does not combine with any other term because its exponents are not identical to any others.

EXERCISES

8.12 Combine like terms in each expression.

a) $3b-2a+b-3a$

b) $5x+3z-y+2z$

c) $4h+2k-f+3h-k$

d) $6a-4b-6a+4b$

e) $2a+3b+a+c+4b+2c$

f) $5x+3xy+2y+xy+2x$

g) $a+bc+c+3a+5c$

h) $xy+yz+2x+3xy$

i) $5p+3q+4pq+5p+pq$

j) $7w-3b-2w+5b-w$

k) $3a+b-5a+2b$

l) $5x+4y-2x+3y$

m) $4B+8F+4C+B+BF+3B+2B$

n) $3xyz+2xy+4x+5y+xyz+2x$

o) $5A+8B-3A+AB-5B+2AB$

p) $8yw+y+3w+2y+yw+3y$

q) $2xy-3y+6x+4y+5xy-6y+10x$

r) $9r-2t+3w+5t-5r+2w-2t-r$

s) $(3x+y)-(4x-2y)+(9x-2y)$

t) $(3x+y-z)-(3x-2z)-(2x-6y)$

u) $3p-2q+5q-p$

v) $3x+xy-y+3xy+2y$

w) $(3x-2)-(5x+2)$

x) $(a-6c)-(3a+2b)$

y) $3x^2-2\sqrt{y}+9x^2+4\sqrt{y}$

z) $w^3+w^2-4w^2+2w^3+w$

9

SOLVING EQUATIONS AND INEQUALITIES IN *X*

Solving equations is vital to success in all areas of practical mathematics.We cannot transform shop formulas, analyze geometric relationships, or address the mathematics of the circle and the triangle in trigonometry without this most essential skill.

Operations on algebraic expressions and factorization of polynomials are the building blocks of equation solving. Having studied these building blocks in the previous chapters, we now turn to the topic of solving several kinds of equations and inequalities.

The word "algebra" derives from the Arabic words al, meaning "the" and jabara, meaning "reuniting." The parts of an equation are, in a sense, "reunited" when the equation is solved.

9.1 Solving Linear Equations in One Variable

The first kind of equation we consider is that of a *single variable* raised to an exponent of one. These equations may be very simple or less simple. For example:

$$x = 7 \qquad y = 0.14 \qquad t + 2 = 11 \qquad \frac{z}{4} = 1$$

$$x - 4 = 3x + 2 \qquad \frac{2}{7}(y-1) = 10 + y \qquad \frac{x+6}{3} = \frac{2x-1}{4}$$

These are all *linear equations* in one variable. That is, they represent lines when graphed, as we will later see. These examples by no means represent all types of linear equations. Other linear equations have two variables each raised to an exponent of one. To begin, we will look at linear equations in a single variable.

In general, a linear equation can be written in the form $y = ax + b$ in which a and b represent constants, while x and y are variables.

For most of this chapter we use the letter x to denote the unknown variable. Consider the simplest such equation, that in which x is set equal to a constant. For example, in $x = 7$, the variable stands alone on the left hand side of the equation. A constant stands alone on the right. Indeed, "solving an equation" is defined as "isolating the variable on one side of the equation and a constant on the other side." We will be concerned with getting equations into such a form when presented with equations in which the variable is *not* isolated.

Inverse Operations

Addition and subtraction are mutual inverse operations; multiplication and division are mutual inverse operations.

In each of the next four examples, the variable is on the left, but constants appear on both sides of the equation. Isolating the variable terms on the left side of the equation requires moving non-variable terms to the other side of the equation. This movement is accomplished by using the appropriate *inverse operation*. An inverse operation "undoes" the operation immediately preceding the term to be moved. The basic rule of solving any algebraic equation is that:

Eqation Solving Rule

Any operation applied to one side of the equation must be applied to the other side of the equation.

In this way a term from one side of the equation is "moved" to the other side of the equation.

EXAMPLE 9.1: ***Inverse of Addition Is Subtraction***

Solve for x in the equation $x + 1 = 5$.

Solution:

$$x + 1 = 5$$

Move the 1 from the left side of the equation by subtracting it from *both* sides of the equation. Thus,

$$\begin{aligned} x + 1 - 1 &= 5 - 1 && \text{The inverse of adding 1 is subtracting 1.} \\ x &= 4 \end{aligned}$$

To check the solution, simply substitute the answer into the original problem and verify that the expression is true.

Letting $x = 4 \Rightarrow 4 + 1 = 5$ and so the answer checks.

EXAMPLE 9.2: ***Inverse of Subtraction Is Addition***

Solve for y in the equation $y - 9.2 = 7.1$.

Solution:

Move the – 9.2 from the left side of the equation by *adding it to both sides* of the equation.

$$\begin{aligned} y - 9.2 + 9.2 &= 7.1 + 9.2 && \text{The inverse of subtracting 9.2 is adding 9.2.} \\ y &= 16.3 \end{aligned}$$

To check the solution, simply substitute the answer into the original problem and verify that the expression is true.

Letting $y = 16.3 \Rightarrow 16.3 - 9.2 = 7.1$ and so the answer checks.

Example 9.3: ***Inverse of Multiplication Is Division***

Recall that a coefficient is a number by which a variable is multiplied.

Solve for t in the equation $-2t = 6$.

Solution:

Move the –2 coefficient from the left side of the equation by dividing *both* sides of the equation by –2.

$$\frac{-2t}{-2} = \frac{6}{-2} \qquad \text{The inverse of multiplying by } -2 \text{ is dividing by } -2.$$

$$t = -3$$

To check the solution, simply substitute the answer into the original problem and verify that the expression is true.

Letting $t = -3 \Rightarrow (-2)(-3) = 6$ and so the answer checks.

Example 9.4: ***Inverse of Division Is Multiplication***

Solve for x in the equation $\frac{x}{3} = 12$.

Solution:

Move the 3 from the denominator of the left side of the equation by multiplying *both* sides of the equation by 3.

$$\frac{x}{3} = 12$$

$$\frac{x}{3} \cdot 3 = 12 \cdot 3 \qquad \text{The inverse of dividing by 3 is multiplying by 3.}$$

$$x = 36$$

To check the solution, simply substitute the answer into the original problem and verify that the expression is true.

Letting $x = 36 \Rightarrow \frac{36}{3} = 12$ and so the answer checks.

We can simplify the discussion of inverse operations for the purpose of isolating a variable with the concepts of *additive inverse* and *multiplicative inverse*.

The *additive inverse* of a number a is that number which, when added to a, gives zero. Thus, the additive inverse of a is $-a$ because $a + -a = 0$. For example the additive inverse of –9.2 is 9.2.

Definition of Additive Inverse

The *multiplicative inverse* of a number a is that number which, when multiplied by a, gives one. Thus, $a \cdot \frac{1}{a} = 1$, for $a \neq 0$. For example, the multiplicative inverse of 5 is $\frac{1}{5}$. Zero has no multiplicative inverse since $\frac{1}{0}$ is *undefined*.

Definition of Multiplicative Inverse

Sometimes $\frac{n}{0}$ indicates a number whose value is undefined or is infinity.

The multiplicative inverse of a number is also called its *reciprocal*. In general, the *reciprocal* of $\frac{a}{b}$ is $\frac{b}{a}$ for $a, b \neq 0$. Although the reciprocal of zero is undefined, $\frac{0}{b} = 0$ for any real number $b \neq 0$.

The previous examples can now be solved with the idea of additive and multiplicative inverse used in place of inverse operation:

$x+1+(-1)=5+(-1)$	The additive inverse of $+1$ is -1
$y-9.2+9.2=7.1+9.2$	The additive inverse of -9.2 is $+9.2$
$\frac{-1}{2}\cdot -2t=\frac{-1}{2}\cdot 6$	The multiplicative inverse of -2 is $-\frac{1}{2}$
$3\cdot\frac{1}{3}x=3\cdot 12$	The multiplicative inverse of $\frac{1}{3}$ is 3

In the first two examples the idea is clear. In the third example we multiply by $-\frac{1}{2}$, which is the multiplicative inverse of the coefficient –2. In the last example we use the fact that $\frac{x}{3} = 12$ is the same as $\frac{1}{3}x = 12$. Multiplication by the coefficient's reciprocal in both cases changes the coefficient to $1x$, or simply, x.

We summarize the process of isolating the variable in this way:

- Moving an *entire* term requires addition or subtraction. Hence, we apply the *additive inverse* to a term to move it across the equal sign.
- Removing a coefficient *from* a term requires division or multiplication. Hence, we apply the *multiplicative inverse* to a coefficient to remove it from a variable.

The multiplicative inverse of n, an integer ≠0, is $\frac{1}{n}$.

Solving for a Variable

The next three examples illustrate how to solve for a variable in simple one-variable linear equations. Although these examples can be solved by inspection, we give the details of the *process* to illustrate how to solve equations of any complexity.

Example 9.5: ***Solving for a Variable***

Solve $x - 11 = 7 + 3x$ for x.

Solution:

The variable appears on both sides of the equation. To solve for x, we need to move the variable terms to one side of the equation and the constants to the other side. First, move –11 to the right by adding 11 to both sides of the equation:

$$x - 11 + 11 = 7 + 3x + 11$$

$$x = 18 + 3x \qquad \text{Combine like terms.}$$

Recall that "like terms" are terms in which variable and exponent are identical. Constants are also like terms.

EXAMPLE 9.5: ***Solving for a Variable (Continued)***

Now, move the variable term 3*x* by applying its additive inverse to both sides of the equation:

$$\begin{aligned} x &= 18 + 3x \\ x + (-3x) &= 18 + 3x + (-3x) \\ -2x &= 18 \qquad \text{Combine like terms.} \end{aligned}$$

The additive inverse of 3*x* is –3*x*. When this is added to both sides of the equation the result on the left is –2*x*. Apply $-\frac{1}{2}$, the multiplicative inverse of the coefficient –2, to both sides:

$$\begin{aligned} -2x &= 18 \\ -\frac{1}{2} \cdot -2x &= -\frac{1}{2} \cdot 18 \\ x &= -9 \qquad \text{Cancel.} \end{aligned}$$

Check the solution: Letting $x = -9$,

$$\begin{aligned} x - 11 &= 7 + 3x \\ -9 - 11 &\overset{?}{=} 7 + (3)(-9) \\ -20 &\overset{?}{=} 7 + (-27) \\ -20 &= -20 \end{aligned}$$

Since the final statement is true, the answer checks.

The multiplicative inverse of –1 is –1, since (–1)(–1) = 1.

EXAMPLE 9.6: ***Solving for a Variable***

Solve $4x + \frac{1}{8} = x - \frac{3}{4}$ for *x*.

Solution:

$$4x + \frac{1}{8} = x - \frac{3}{4}$$

$$4x - x + \frac{1}{8} - \frac{1}{8} = x - x - \frac{3}{4} - \frac{1}{8}$$ Move variable terms to the left, constants to the right.

$$3x = -\frac{7}{8}$$ Combine like terms.

$$\frac{1}{3} \cdot 3x = \frac{1}{3} \cdot \left(-\frac{7}{8}\right)$$ Multiply by reciprocal of coefficient to solve for *x*.

$$x = -\frac{7}{24}$$

EXAMPLE 9.6: *Solving for a Variable (Continued)*

Check the solution: Letting $x = \frac{-7}{24}$,

$$4x + \frac{1}{8} = x - \frac{3}{4}$$

$$4 \cdot \left(-\frac{7}{24}\right) + \frac{1}{8} \overset{?}{=} -\frac{7}{24} - \frac{3}{4}$$

$$-\frac{28}{24} + \frac{1}{8} \overset{?}{=} -\frac{7}{24} - \frac{3}{4}$$

$$-\frac{28}{24} + \frac{3}{24} \overset{?}{=} -\frac{7}{24} - \frac{18}{24}$$

$$-\frac{25}{24} = -\frac{25}{24}$$

The answer checks because the final statement is true.

EXAMPLE 9.7: *Solving for a Variable*

Solve $12x - 14 + 7(x - 1) = -8 + x - 9\left(\frac{x}{3} + 1\right)$ for *x*.

Solution:

Expand expressions on either side of the equation before moving terms.

$12x - 14 + 7(x - 1) = -8 + x - 9\left(\frac{x}{3} + 1\right)$	Apply distributive property.
$12x - 14 + 7x - 7 = -8 + x - 3x - 9$	Combine like terms.
$19x - 21 = -17 - 2x$	Move variable terms to the left, constants to the right.
$19x + 2x - 21 + 21 = -17 + 21 - 2x + 2x$	Combine like terms.
$21x = 4$	Multiply by reciprocal of coefficient to solve for *x*.
$\frac{1}{21} \cdot 21x = \frac{1}{21} \cdot 4$	
$x = \frac{4}{21}$	(Perform the check.)

- Combine like terms on each side of the equation.
- Move variable terms to one side of the equation and constants to the other using additive inverses.
- Combine like terms that result from previous step.
- Eliminate coefficient of resulting variable term using the coefficient's multiplicative inverse (reciprocal).

Summary of Steps for Solving a Single-Variable Linear Equation

Solving for a Variable: An Application

Recall that all rational numbers can be written as fractions or repeating or terminating decimal fractions. As we learned in Chapter 4, converting a fraction to decimal form is easy: simply divide the numerator by the denominator. Converting from decimal to fraction form is more difficult, but as long as the decimal form terminates, this can be done with arithmetic. For example, 0.122 is read as "one-hundred twenty-two thousandths." In fraction form, this is $\frac{122}{1000}$, reducing to $\frac{61}{500}$. The problem arises about what to do with a repeating decimal such as 0.111…? We cannot write it as a fraction whose denominator is some power of 10. However, since it repeats it must also have a fractional form. Finding the fraction for this or any repeating decimal number is a simple algebra problem with a clever solution.

Let $x = 0.111\ldots$.

Then $10x = (10)(0.111\ldots) = 1.111\ldots$.

Subtract these equations as follows:

$$\begin{array}{r} 10x = 1.111... \\ \underline{-x = 0.111...} \\ 9x = 1.000 \end{array}$$

Solve for x: $\frac{1}{9} \cdot 9x = \frac{1}{9} \cdot 1 \rightarrow x = \frac{1}{9}$

Checking this answer is straightforward: $1 \div 9 = 0.111\ldots$

EXAMPLE 9.8: ***Using an Equation to Change a Repeating Decimal to a Common Fraction***

Find the fractional form of 3.727272….

Solution:

Let x represent the repeating decimal: $x = 3.7272\ldots$. There are two repeating digits. This dictates that x be multiplied by the second power of 10. We use $100x$ as the second term: $100x = 372.7272\ldots$.

EXAMPLE 9.8: ***Using an Equation to Change a Repeating Decimal to a Common Fraction (Continued)***

Subtracting as before:

$$\begin{array}{r} 100\,x = 372.7272... \\ -\,x = \quad 3.7272... \\ \hline 99\,x = 369.000 \end{array} \quad \Rightarrow \quad \frac{1}{99}\cdot 99\,x = \frac{1}{99}\cdot 369 \quad \Rightarrow \quad x = \frac{369}{99} = \frac{41}{11}$$

This checks since 41 ÷ 11 = 3.7272…

Solving Radical Equations

When solving radical equations, the entire side of the equation must be raised to a power – not simply the individual terms.

Equations in which the unknown occurs in a radicand are called *radical equations*.

To solve a radical equation it must be arranged so that the radical with the unknown is the only term on the left hand side of the equation. Then the radical is eliminated by raising both sides of the equation to the index of the radical. This is normally a squaring action, as most radical equations contain square roots.

Generally, when solving radical equations, it is necessary to check the answer, as a solution may not exist. In our examples, equations will have solutions.

EXAMPLE 9.9: ***Solving Radical Equations***

Solve for the value of x.

$$\sqrt{3x} = 6$$

Solution:

$$(\sqrt{3x})^2 = 6^2$$

$$3x = 36 \quad \Rightarrow \quad \frac{3x}{3} = \frac{36}{3}$$

$$x = 12$$

(Peform the check)

EXAMPLE 9.10: ***Solving Radical Equations***

Solve for the value of x.

$$\sqrt{2x+3} = 7$$

Solution:

$$(\sqrt{2x+3})^2 = 7^2$$

$$2x+3 = 49$$

$$2x+3-3 = 49-3$$

$$2x = 46$$

$$\frac{2x}{2} = \frac{46}{2}$$

$$x = 23$$

(Perform the check)

EXERCISES

9.1 Solve for the variable:

a) $x-10=1$ **b)** $x+2=6$ **c)** $\frac{1}{5}x=2$

d) $6x=42$ **e)** $\sqrt{x}=3$ **f)** $x^2=144$

g) $4-x=19$ **h)** $40=10-x$ **i)** $\sqrt{x-2}=6$

j) $x-5=5$ **k)** $10x=1250$ **l)** $\frac{1}{4}x=1$

m) $2x+4=14$ **n)** $25-x=13$ **o)** $3x^2=48$

p) $3\sqrt{y+3}=2\sqrt{3y-12}$ **q)** $\sqrt{x+3}-2=6$ **r)** $x-1.6=0.1x+1.8$

s) $\sqrt{x}+1=4$ **t)** $\sqrt{z}-5=20$ **u)** $\sqrt{L-3}=8$

v) $4\sqrt{z+3}-2=6$

9.2 Find the fractional form of the following repeating decimals:

a) 0.777... **b)** 2.1313...

9.2 Solving Inequalities in x

Sometimes mathematical inequalities must be solved.

The procedure is the same as solving an equation with a few exceptions. Symbolically, the possible types of inequalities are given in Table 9.1.

TABLE 9.1: **Statements of Inequalities in x**

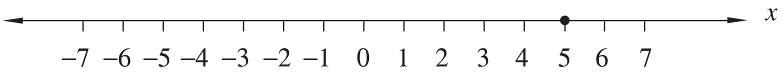
$x > a$	x is greater than a
$x < a$	x is less than a
$x \geq a$	x is greater than or equal to a
$x \leq a$	x is less than or equal to a
$a < x < b$	x is greater than a and less than b
$a \leq x \leq b$	x is greater than or equal to a and less than or equal to b

Equations like those introduced so far have one or at most two solutions—a *unique solution* in x. Inequalities, on the other hand, can have an *infinite number of solutions*. We refer to the solution of an inequality as a *solution set*. For example, $x > a$ has as its solution set all values lying to the right of a on the real number line. $x \leq a$ has as its solution set all values lying on and to the left of a on the number line.

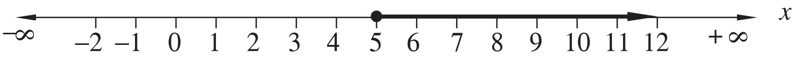
The representation of the solution set of an inequality can be shown in a *graph* or given by *interval notation*. Graphing is perhaps the most intuitive method. Before solving inequalities, we first describe both of these methods of stating an inequality's solution. With the aid of a graphed solution, we can more easily determine the interval notation. For this reason we consider the graphed solution first.

Graphing an Inequality

Graphing the unique solution to a single-variable equation on the number line is straightforward. For example, $x = 5$ is a single point and is graphed as

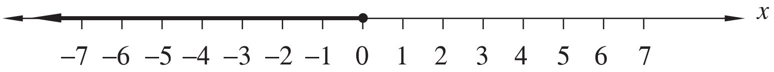


For a single variable *inequality*, such as $x > 5$, the graph is part of a line:



The dot is open to indicate that 5 is *not* in the solution set. The arrow pointing right indicates that the solution set is the set of all real numbers greater than 5. This is an example of a *strict inequality* (*x is greater than but not equal to 5*); the values sought are strictly greater than 5.

Now consider the graph of $x \leq 0$:



A graph is incorrect if the wrong kind of dot is used.

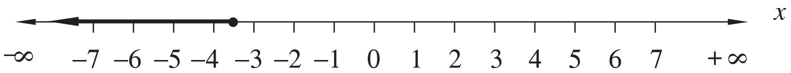
The arrow pointing left indicates that the solution set is the set of all real numbers less than zero. However, the dot is closed to indicate that zero is also included in the solution set (*x is less than or equal to zero*). This is not a strict inequality.

All values from a solution set should satisfy an inequality. The simplest way to check a solution is to substitute a number from its graph into the inequality. Also substitute the endpoint to be sure the right type of dot is used.

Example 9.11: ***Graphing the Solution to an Inequality***

Graph $x \leq \frac{-7}{2}$.

Solution:



The solid dot indicates that $\frac{-7}{2}$ is included in the solution set and is not simply a boundary.

Set and Interval Notation

Inequalities shown in graphs are represented non-graphically with set notation or interval notation. *Set notation* expresses a solution in braces, similar to the way elements in a set are listed. The solution to Example 9.11 can be expressed in set notation as: $\{x \mid x \le -3\frac{1}{2}\}$. This is read as "the set of all values of x such that x is less than or equal to $-3\frac{1}{2}$."

An alternative to set notation is the more familiar *interval notation*. An interval on the number line contains the solution set of an inequality. Intervals are classified as *open* or *closed*, or *half-open* (or *half-closed*.) They can be infinite or finite in length. Use the following guidelines for graphing and writing a solution to an inequality in interval notation:

Half-open is the same as half-closed.

- *Closed* intervals are indicated by inequalities such as $a \le x \le b$ and $a \ge x \ge b$. Their graphs have filled dots at a and b to show that the endpoints are included. The corresponding interval notation takes brackets: $[a, b]$.
- *Open* intervals are indicated by strict inequalities such as $a < b$ and $a > b$. Their graphs have unfilled dots at a and b to show that the endpoints are not included. The corresponding interval notation takes parentheses: (a, b).
- An interval can be *half-open* or *half-closed*, in which case, its graph will have a filled dot at one endpoint and an unfilled dot at the other. The corresponding interval notation will have a parenthesis enclosing one endpoint and a bracket enclosing the other: $[a, b)$ or $(a, b]$.
- An interval that includes "half" the number line (for example, $x < 1$) has an open "endpoint" at infinity, ∞, or negative infinity, –∞. Both ∞ and –∞ are enclosed by a parenthesis, *never* a bracket: $(-\infty, a]$ or $[a, \infty)$, for example.

Another way to represent a solution set is by set notation.

Parentheses are used for ***open*** *intervals. Brackets are used for* ***closed*** *intervals.*

The entire number line is represented in interval notation as (–∞, ∞).

The next example illustrates all of the interval types as they may appear in a solution to an inequality. Each is written in conjunction with its graphical representation.

EXAMPLE 9.12: ***Interval Notation***

Consider the eight inequality expressions shown:

$$x < 1 \qquad x > 1 \qquad x \le 1 \qquad x \ge 1$$

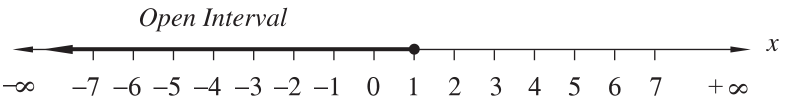
$$0 < x < 1 \qquad 0 \le x \le 1 \qquad 0 \le x < 1 \qquad 0 < x \le 1$$

The solution sets for these inequalities lie on the number line either to the left of 1 or to the right of 1. They may be the short interval between 0 and 1, or they may cover "half" the number line by going either to positive infinity or to negative infinity. They may include either 0 or 1 or both 0 and 1.

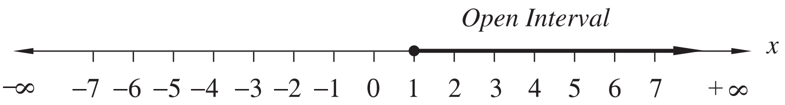
The graph for each inequality is given and its interval notation is shown.

Example 9.12: ***Interval Notation (Continued)***

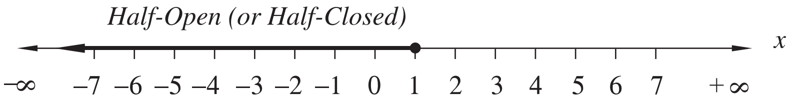
$x < 1$
$(-\infty, 1)$



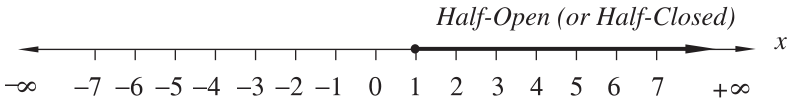
$x > 1$
$(1, \infty)$



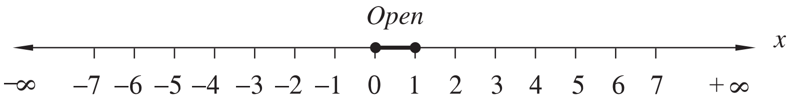
$x \leq 1$
$(-\infty, 1]$



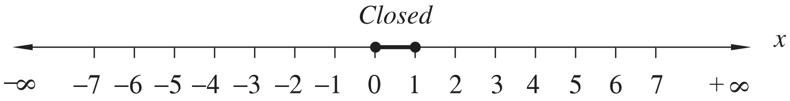
$x \geq 1$
$[1, \infty)$



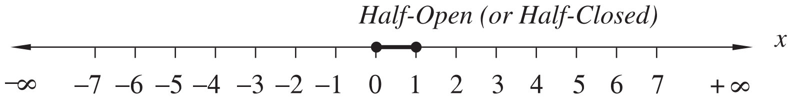
$0 < x < 1$
$(0, 1)$



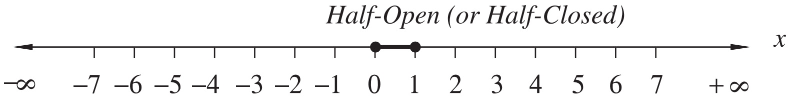
$0 \leq x \leq 1$
$[0, 1]$



$0 \leq x < 1$
$[0, 1)$



$0 < x \leq 1$
$(0, 1]$



The process of solving an inequality is similar to that of solving equations. The appropriate inverse operations are employed to isolate the variable, as in the following example.

Example 9.13: ***Solving an Inequality***

Solve for x: $3x + 21 \leq x - 7$.

Solution:

$$3x - x + 21 \leq x - x - 7$$

$$2x + 21 - 21 \leq -7 - 21$$

$$\frac{1}{2} \cdot 2x \leq \frac{1}{2}(-28)$$

EXAMPLE 9.13: ***Solving an Inequality (Continued)***

$$x \le -14$$

The interval notation for this solution set is (–∞, –14].

Example 9.13 shows an almost identical process to that used to solve the equation $3x + 21 = x - 7$. The only difference is in the solution. Instead of $x = -14$, we have $x \le -14$. Any value from the interval beginning at −14 and going toward negative infinity will satisfy the inequality. For example, $x = -15$ substituted into the original problem keeps the inequality true.

$$3x + 21 \le x - 7$$

$$3(-15) + 21 \overset{?}{\le} -15 - 7$$

$$-24 \le -22 \text{ (true)}$$

$$\therefore \text{ solution checks}$$

The notable difference between solving inequalities and solving equations arises when both sides of an inequality must be either multiplied or divided by a negative number. This very important exception is stated in the following rule.

Rule for Inequalities

Whenever both sides of an inequality are either multiplied or divided by a negative, the inequality must be reversed: < becomes > and vice versa.

Consider the inequalities $x - 6 > 8$ and $6 - x > 8$. We solve the first exactly as we would the equation $x - 6 = 8$, except the solution is an interval of the number line:

$$x - 6 > 8$$

$$x - 6 + 6 > 8 + 6$$

$$x > 14 \qquad \text{interval notation: } (14, \infty)$$

When solving the second inequality, the negative sign in front of the x causes us to invoke the rule for inequalities:

$$6 - x > 8$$

$$6 + (-6) - x > 8 + (-6)$$

$$-x > 2$$

$$(-1)(-x) < (-1) \cdot 8 \qquad \text{inequality reverses}$$

$$x < -8 \qquad \text{interval notation: } (-\infty, -8)$$

Notice that the > symbol was reversed when multiplying by the –1.

EXAMPLE 9.14: ***Rule for Inequalities***

Find the solution set of the inequality and express the answer in interval notation. Graph the solution.

$$2x + 4 \le 5x - 3$$

EXAMPLE 9.14: ***Rule for Inequalities (Continued)***

Solution:

$$2x+4 \le 5x-3$$

$$2x+4-4 \le 5x-3-4$$

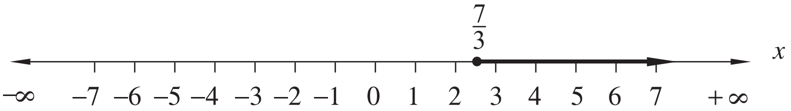
$$2x \le 5x-7$$

$$2x-5x \le 5x-5x-7$$

$$-3x \le -7$$

$$(\tfrac{-1}{3})(-3x) \ge (\tfrac{-1}{3})(-7) \qquad \text{reverse inequality sign}$$

$$x \ge \frac{7}{3} \qquad \text{interval notation: } [\tfrac{7}{3}, \infty)$$



In the next chapter we discuss how to graph linear equations in two variables, x and y.

EXERCISES

9.3 Draw the number line for each interval. State whether the interval is an open interval, closed interval, or a half-open (or half-closed) interval.

a) $[1,6]$ **b)** $(4,7)$ **c)** $(-3,0]$ **d)** $[1,5)$

e) $(-6,-4)$ **f)** $(3,8]$ **g)** $[-3,-1]$ **h)** $(2,5]$

9.4 Solve for the value of x in each inequality. Give the answer using interval notation and show the solution on a number line.

a) $2x+6 < x+4$ **b)** $-3x+20 \ge 2x-5$

c) $\dfrac{-3x}{5}-2 < \dfrac{-x}{5}$ **d)** $-4x < -6x+8$

e) $10x-12 \le 5x+3$ **f)** $\dfrac{3}{2}x+4 \ge \dfrac{1}{2}x$

g) $-7x+14 > 4x-8$ **h)** $x+1 \le 4x-5$

10

GRAPHING LINEAR EQUATIONS

We now expand our discussion of the single-variable linear equation to the general *linear equation* in two variables, x and y. Some examples of linear equations are

$$x + y = 0 \qquad y = -3x \qquad x = -4 \qquad y = \frac{2}{3}$$

The *graph* of a linear equation is a line. Although drawing a line is a simple geometric construction, *graphing* a line requires knowledge of algebraic concepts.

10.1 The Cartesian Plane

The Cartesian plane is a two-dimensional space formed by intersecting, in a perpendicular fashion, two number lines at their zeros. The number line placed horizontally is called the *x-axis* and the number line placed vertically is called the *y-axis* of the Cartesian plane. Since the number lines are of infinite length the resulting plane is of infinite area. Consequently, we only draw the portion of the plane immediately surrounding the point of intersection of the number lines.

The idea of plotting mathematical relationships using x, y-coordinates called ordered pairs was formulated around 1630 by Pierre de Fermat and René Descartes, mathematicians who worked separately on the same concept.

Naming Points

Figure 10.1 shows the Cartesian plane, which is also called the (x, y)-*coordinate plane*. Notice how drawing lines through each integer value on both the x- and y-axes forms a grid. Thus drawn, the intersections mark *points* in the plane. Each point defines a unique location in the plane with an x component (or *abscissa*) and a y component (or *ordinate*), called the x- and *y-coordinates* of a point. The two coordinates are given together as the *ordered pair* (x, y). The point of intersection of the zeros has a special name, the *origin*, (0, 0).

In any sketch of a Cartesian plane, the origin is labeled as (0, 0) *and the horizontal and vertical axes x and y, respectively.*

Plotting Points

The Cartesian plane or two-dimensional space is often called "2-space."

The x value in an ordered pair is positive in the right half of the plane and negative in the left half of the plane. Similarly, the second number in an ordered pair, the y value, is positive in the upper half of the plane and negative in the lower half. Thus, the ordered pair (3, 4) is the point in two-dimensional space that lies 4 units above the number 3 on the x-axis and 3 units to the right of the number 4 on the y-axis. It is at this location that we plot the point (3, 4).

The Cartesian plane itself is divided into *quadrants*, which are naturally defined by the boundaries of the axes. The quadrants are commonly numbered I, II, III, and IV, starting in the upper right of the plane and proceeding counterclockwise.

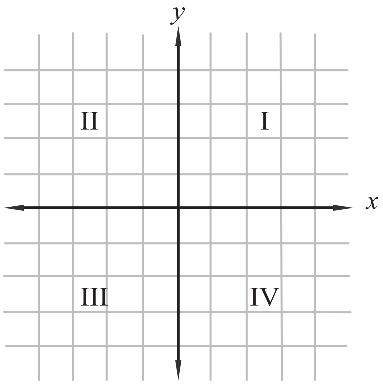


FIGURE 10.1 ***The Cartesian Plane***

EXAMPLE 10.1: ***Plotting Points on the Cartesian Plane***

Plot the points (0, 0), (5, 2), (0, 2), (–3, 4), (– 4, 0), (–1, –3), and (6, – 4) on the Cartesian plane.

Solution: Observe the location of the plotted points.

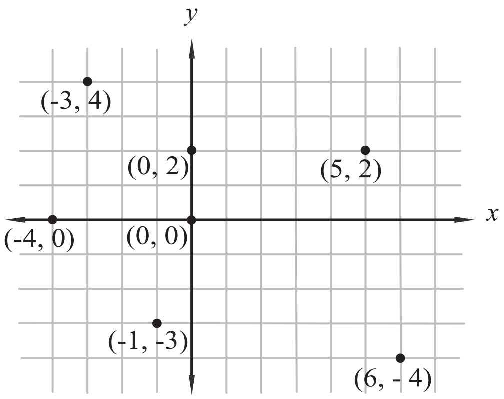


FIGURE 10.2

For any point on the x-axis, the value of the y-coordinate is zero. This is because the y-coordinate indicates the number of units a point lies above or below the x-axis. If a point is *on* the x-axis, it is zero units above or below the x-axis.

Example 10.1: *Plotting Points on the Cartesian Plane (Continued)*

Likewise, for any point *on* the y-axis, the value of the x-coordinate is zero. This is because the x-coordinate indicates the number of units a point lies to the left or right of the y-axis. If a point is on the y-axis, it is zero units left or right of the y-axis. We will return to this idea when we consider equations of different types of lines.

Example 10.2: *Quadrant Identification*

State the quadrant or axis where each of the following points lie.

$$(0,0) \quad (5,2) \quad (0,2) \quad (-3,4) \quad (-4,0) \quad (-1,3) \quad (6,-4)$$

Solution:

$(0,0)$ is the origin. $(5,2)$ is in Quadrant I. $(0,2)$ is on the y-axis. $(-3,4)$ is in Quadrant II. $(-4,0)$ is on the x-axis. $(-1,-3)$ is in Quadrant III. $(6,-4)$ is in Quadrant IV.

EXERCISES

10.1 For the following points, identify the x- and y-coordinates:

a) $(-5, 5)$ **b)** $(4, 20)$ **c)** $(\sqrt{3}, -7)$ **d)** $(-9, -13)$ **e)** $(380, -74)$

10.2 Give the ordered pair for each point marked on the coordinate plane:

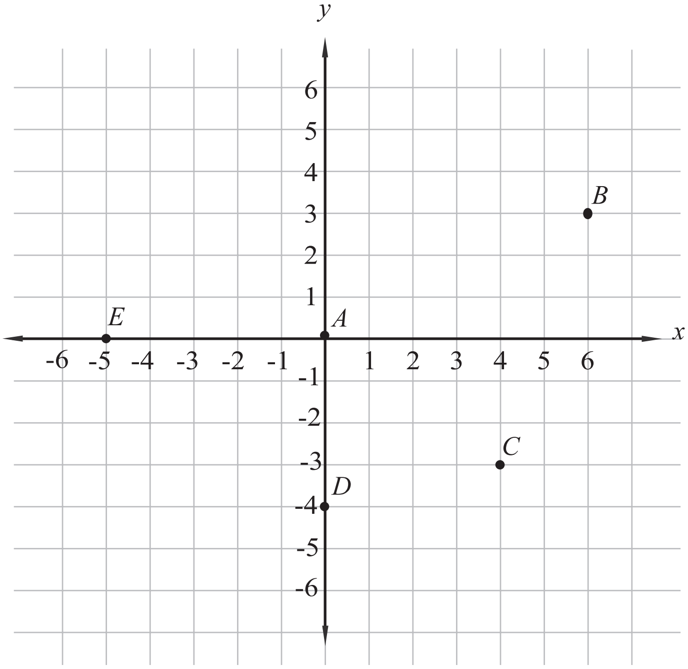


Figure 10.3

10.3 Plot the following points on the Cartesian plane. Make sure each point is labeled with the correct ordered pair.

a) (0, 0)
b) (1, 5)
c) (0, −7)
d) (2, 3)
e) (3, 2)
f) (8, 0)
g) (−6, −1)
h) (4, −8)
i) (−7, −7)
j) (−5, 5)

10.4 State the quadrant or axis where the following points or signs of points lie.

a) (1, 2)
b) (−8, 6)
c) (−20, −35)
d) (3.41276, −1.36)
e) (0, 3)
f) (342, −17)
g) (5, −5)
h) (−4, 0)
i) (−5, −5)
j) (10, 21)
k) (0, 0)
l) (−, +) or $(-x, +y)$*
m) (+, −) or $(+x, -y)$*
n) (−, −) or $(-x, -y)$*
o) (+, +) or $(+x, +y)$*

* x and y are positive real numbers

10.2 Graphing Points of a Line

Our discussion of the methods for graphing linear equations in two-dimensional space begins with the line $y = x$. This equation describes the set of points with identical x- and y-coordinates. Notice that the origin (0, 0) is such a point. So are (6, 6), (−4, −4), $(2\frac{2}{3}, 2\frac{2}{3})$ and (−171, −171). Clearly, there are infinitely many such points in two-space.

Some of these points are plotted in Figure 10.4.

Use graph paper when learning to graph lines.

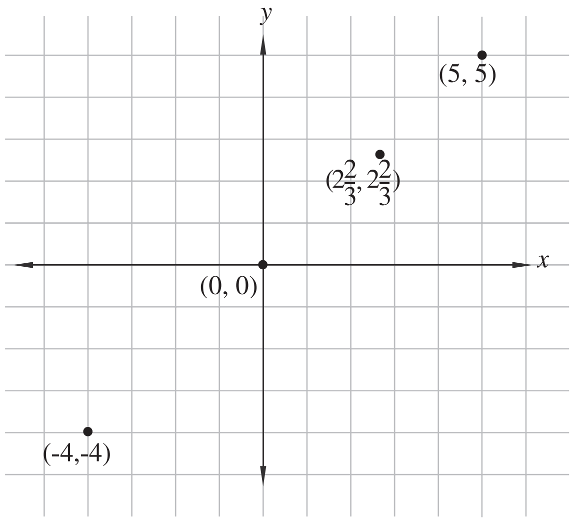


FIGURE 10.4 ***A plot of points with equal coordinates***

The line containing these points and all other points with equal x- and y-coordinates is given by the equation $y = x$ and shown in Figure 10.5.

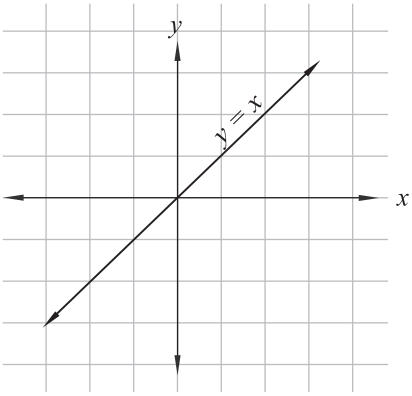


FIGURE 10.5 ***Graph of y = x***

Making a Table of (x, y)-values

The traditional method for graphing lines and other figures is to generate a set of ordered pairs in the form of a table of (x, y)-values. Begin by choosing an arbitrary value for x and substituting this value into the equation to produce the corresponding value for y. For a line, of course, only two points are required, but extra points are often chosen just as a check. Nonlinear equations require several carefully chosen points.

Two points in a plane define a line.

EXAMPLE 10.3A: ***Using a Table to Make a Graph***

Graph the line $y = 2x$ by generating a table of ordered pairs.

Solution:

x	$y = 2x$
–2	–4
–1	–2
0	0
1	2
2	4

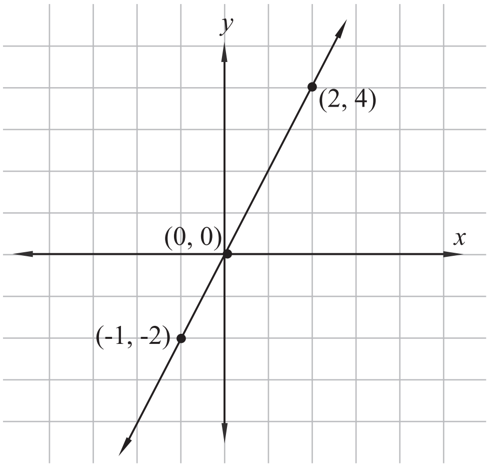


FIGURE 10.6

A good idea for graphing a line is to write three or four ordered pairs in a table of (x, y)-values, using negative, zero, and positive x-values to determine points (x,y).

From geometry we know that two points, and only two points, are needed to construct a line. By graphing more than two points we are more likely to detect a possible error in any of the pairs. An incorrect computation would result in a point lying off the line.

EXAMPLE 10.3B: *Using a Table to Make a Graph*

Make a table of (x, y)-values for the line $y = -x - 3$. Graph the line.

Solution:

Taking care to watch the signs of the coefficients, the following table is generated and the points plotted on the graph:

x	$y = -x - 3$
−3	0
−2	−1
0	−3
1	−4
2	−5

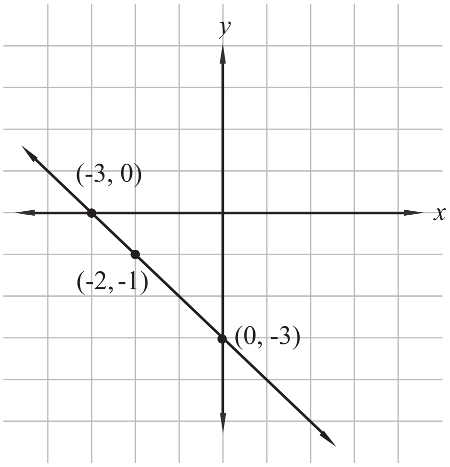


FIGURE 10.7

EXERCISES

10.5 Draw a graph to determine if the following points describe a line.

a) (0, 0), (–5, –5), (7,8), (–1, –1)
b) (–1, –1), (0, 1), (1, 3), (2, 5)
c) (1, 1), (0, 2), (-1, 3), (–2, 0)
d) (0, –1), (1, –3), (–1, 3), (2, –5)

10.6 Graph the following lines by first making a table and then using the points to draw a graph.

a) $y = 4x$
b) $y = x - 3$
c) $y = x + 1$
d) $y = 2x - 8$
e) $y = -\frac{1}{2}x + 5$
f) $y = -\frac{x}{4} - 2$
g) $y = -5x + 3$
h) $y = 2x + 4$
i) $y = 10x - 20$

10.7 For the line formed by $y = -10x - 3$, state whether the following points lie on the line. (Hint: Substitute x and y values into equation and see if equality holds.)

a) (0, –3)
b) (1, 3)
c) (–2, 17)
d) (–4, –43)
e) $(\frac{1}{2}, -8)$

10.3 The Slope of a Line

The *slope* of a line is an indication of a line's steepness. It also indicates whether the line is rising or falling as it progresses from left to right. Between any two distinct points, the change in y, or vertical displacement of the line, is called its *rise*. The corresponding change in x, or horizontal displacement, is called its *run*. The slope of a line is the ratio of the rise to the run, and by convention it is usually represented by the letter m. This ratio can be calculated between any two distinct points on a line. For a given line, the slope does not change, regardless of which two points are used.

Consider the points labeled $P(x_1, y_1)$ and $P(x_2, y_2)$ on the line in Figure 10.8.

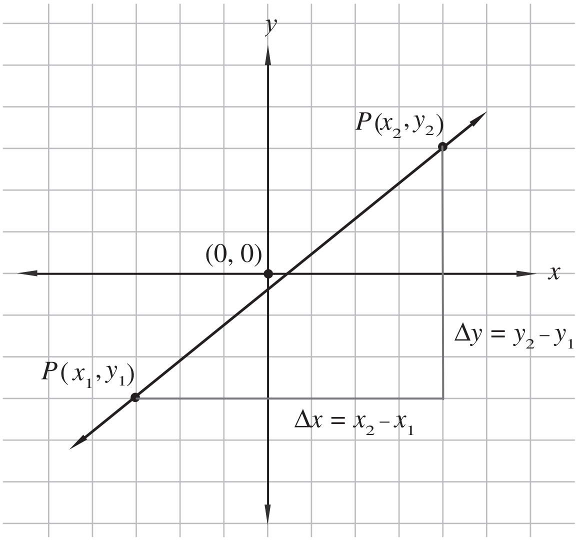


FIGURE 10.8 *Slope of a line*

$$m = \frac{\Delta y}{\Delta x}$$

$P(x_1, x_2)$ is a common way to label a point.

The slope, m, is calculated as follows:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Reversing the order of the subscripts does not change the computation. That is,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1(y_1 - y_2)}{-1(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}$$

However, corresponding subscripts must be lined up, or the sign of the slope will be in error.

Changes in y and x are indicated by Δy and Δx, read "delta y and delta x." The Greek letter delta, Δ, signifies the amount by which a variable changes.

EXAMPLE 10.4: ***Finding Slope Given Two Points***

Find the slope m of a line containing the points (–6, –1) and (2, 8).

Solution: Let $(x_1, y_1) = (2, 8)$ and $(x_2, y_2) = (-6, -1)$

Then $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-8}{-6-2} = \frac{-9}{-8} = \frac{9}{8}$$

Slope: Positive, Negative, Zero, and Undefined

A line with a *positive slope* rises as it progresses from left to right. A line with a *negative slope* falls from left to right. A horizontal line has *zero slope*; it neither rises nor falls. In terms of rise over run, a line with zero slope is said to have zero rise, so its $\Delta y = 0$. A vertical line has an undefined (or infinite) slope. In terms of rise over run, a line with infinite slope is said to have zero run, so its $\Delta x = 0$.

Point-Slope and Slope-Intercept Forms of a Line

The *point-slope form* of a line is derived from the slope formula. We begin by considering a line with a known point called $P(x_1, y_1)$ and any other arbitrary point called $P(x, y)$. The slope of this line is $m = \frac{y - y_1}{x - x_1}$ as shown in Figure 10.9.

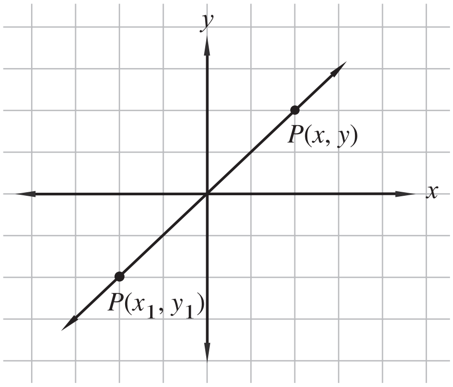


FIGURE 10.9 ***Deriving point-slope form of a line***

The point-slope form of a line is derived by multiplying both sides of the slope equation by the quantity $(x - x_1)$ and rearranging terms:

$$m = \frac{y - y_1}{x - x_1}$$

$$m(x - x_1) = \frac{(y - y_1)}{(x - x_1)}(x - x_1)$$

Point-slope form of a line.

$$\therefore \qquad y - y_1 = m(x - x_1)$$

The point-slope form of the line allows us to find the equation of a line given one point and the line's slope, as illustrated in Example 10.5.

EXAMPLE 10.5: *Using Point-Slope Form Given m and One Point*

Write an equation for a line with slope −3 passing through point (−5, 7).

Solution: Using the equation for the point-slope form of the line,

$y - y_1 = m(x - x_1)$, with $(x_1, y_1) = (-5, 7)$ and $m = -3$, we get $y - 7 = -3(x - (-5))$

Point-slope form: $y - 7 = -3x - 15$

We can further simplify the point-slope equation by rearranging terms as follows, giving the equation in *slope-intercept* form:

$$y = -3x - 15 + 7$$

$$y = -3x - 8$$

The equation $y = -3x - 8$ is the slope-intercept form of the line. This form is very useful for graphing linear equations.

If two points on a line are given, the equation of the line can be found again using point-slope form as demonstrated in Example 10.6.

EXAMPLE 10.6: *Using Point-Slope Form Given Two Points*

Write an equation for a line passing through points (6, −2) and (−4, 4).

Solution: This example illustrates a common applied problem in mathematics in which several pieces of data are known (in this case, the coordinates of two points), and an equation that models the data is sought. The solution involves finding the slope between the two points and then using one of the data points to establish the equation of the line.

STEP 1: Find slope, with $(x_1, y_1) = (6, -2)$ and $(x_2, y_2) = (-4, 4)$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = m = \frac{4 - (-2)}{-4 - 6} = \frac{6}{-10} = -\frac{3}{5}$$

STEP 2: Substitute one of the points and the value of m found in the first step into the point-slope form of the line. Using (6, −2) gives

$$y - (-2) = -\frac{3}{5}(x - 6)$$

$$y + 2 = -\frac{3}{5}(x - 6) \qquad \text{Point-slope form}$$

$$y = -\frac{3}{5}x + \frac{18}{5} - 2 \qquad \text{Rearranging and simplifying}$$

$$y = -\frac{3}{5}x + \frac{13}{5} \qquad \text{Slope-intercept form}$$

In general, the *slope-intercept form* of a linear equation is written as:

Slope-intercept Form of a Line

$$y = mx + b$$

where m is the slope of the line, b is the y-intercept, and variables x and y are the point coordinates that make up the line.

The y-intercept, b, is where the line crosses the y-axis. The y-intercept occurs where the x-value is zero.

The slope-intercept form of a line is particularly useful for graphing purposes. One can look at an equation in this form and using just the values given for m and b, graph the line. Since the constant b represents the coordinate of the point where the line intersects the y-axis, it represents the point $(0,b)$. Notice that if we substitute 0 for x into $y = mx + b$, the resulting value is the y-intercept, b:

$$y = mx + b = m(0) + b = 0 + b = b$$

Thus, when $x = 0$, $y = b$, so one point on the graph of $y = mx + b$ is $(0, b)$. We can graph this point *directly from the equation* by plotting the point $(0, b)$ on the y-axis as shown in Figure 10.10.

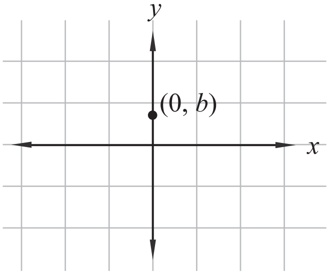


FIGURE 10.10 ***y-intercept of a line***

Now that the value of b is known, use the slope m to draw the graph. Since we can read this value for a particular line directly from the equation, we can use it as a kind of map to find a second point of the line. The two points are sufficient to draw the graph of the line. In general terms, letting $m = \frac{\Delta y}{\Delta x}$, the second point is found by moving Δy units vertically from $(0, b)$ and then moving Δx units horizontally as illustrated in Figure 10.11.

Given an equation for a line in the slope-intercept form, y = mx + b, begin graphing the line at the point (0, b) and use the slope, m, as a map to second point.

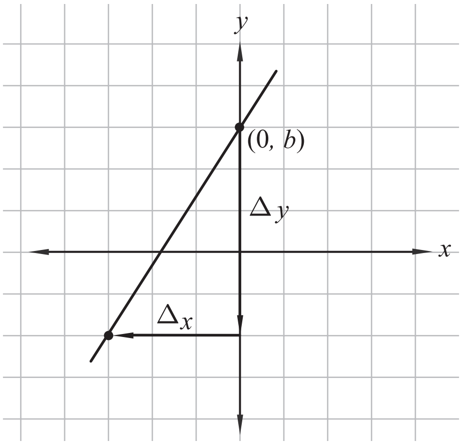


FIGURE 10.11 ***Using slope m to graph a line from y-intercept* (0, *b*)**

Vertical and Horizontal Lines

As described earlier, the slope of a horizontal line is zero and that of a vertical line is undefined. How are these facts reflected in the slope-intercept form of a line? Consider horizontal lines first.

Since the slope m of a horizontal line is zero, the slope-intercept form of a horizontal line would reduce to $y = 0 \cdot x + b = b$.

Thus, a horizontal line intersects the y-axis at $(0, b)$, as would any line whose slope is defined. The equation of a horizontal line ($y = b$) indicates that the y-coordinate is equal to b for *all* values of x as demonstrated in Figure 10.12.

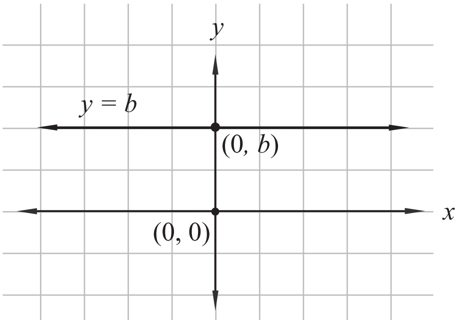


FIGURE 10.12 ***Horizontal line, y = b***

The equation, $y = -2$ represents the line running parallel to the x-axis and intersecting the y-axis at $(0, -2)$. Other points on the line include $(-7, -2)$, $(8, -2)$, and $(29, -2)$. Regardless of the value of x, the y value is -2. The line is horizontal and its slope is 0.

The x-axis is a horizontal line whose equation is y = 0.

The slope of a vertical line is *undefined*, owing to the fact that between any two points on a vertical line there is no change in the direction of x.

Recall that a fraction whose denominator is zero is undefined. The symbol for this is the same as for infinity, ∞.

That is, $$m = \frac{\Delta y}{\Delta x} = \frac{\Delta y}{0} = \infty.$$

As such, we cannot fit the equation of a vertical line into slope-intercept form. Analogous to horizontal lines ($y = b$), the equation of a vertical line is $x = a$. Here, a is the *x-intercept* of the line, as well as the value of the x-coordinate for all values of y as illustrated in Figure 10.13.

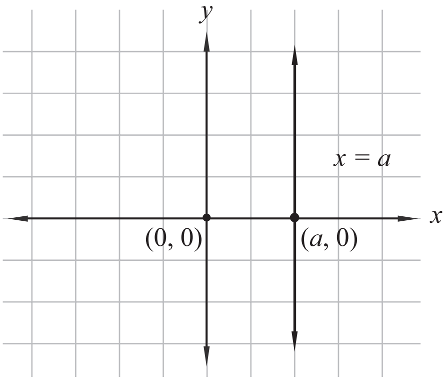


FIGURE 10.13 ***Vertical line, x = a***

The y-axis is a vertical line whose equation is x = 0.

For example, $x = 10$ is a linear equation in which no y appears. This is the line intersecting the x-axis at (10, 0) and running parallel to the y-axis. Other points on this line include (10, –9), (10, –1), and (10, 13). Regardless of the value of y, the x value is 10.

EXERCISES

10.8 Find the slope of each line containing the indicated points.

a) (3, 4) (7, 12)

b) (–1, 2) (5, 5)

c) (–8, –6) (–2, 6)

d) (40, 5) (–13, 5)

e) (x_1, y_1) (x_2, y_2)

10.9 State whether the indicated points form lines that rise (left to right); fall (left to right); do neither; or only rise.

a) (0, 0) (6,2)

b) (–1, 3) (0, 0)

c) (0, 0) (–7, –1)

d) (13, 15) (–3, 15)

e) (–17, 7) (–14, 2)

10.4 Applying Linear Equation Forms to Graphs

We now look at how to recognize and use the different forms of linear equations to graph lines.

EXAMPLE 10.7: ***Graphing a Line from Slope-Intercept Form, Using Only m and b***

Graph the line of the equation $y = x - 4$. What is the x-intercept of the line? (The point of intersection with the x-axis.)

Solution:

This equation is in slope-intercept form: $y = mx + b$ where slope $m = 1$ and y-intercept $b = -4$. To graph the equation from this information alone, begin at (0, –4), marking the point. A slope of 1 indicates a ratio of rise over run equal to $\frac{1}{1}$. From (0, –4), move 1 unit up and 1 unit to the right. Mark a second point. Connect the two points thus plotted to produce the graph of the line $y = x - 4$. Label the line with the equation, as shown.

EXAMPLE 10.7: *Graphing a Line from Slope-Intercept Form, Using Only m and b (Continued)*

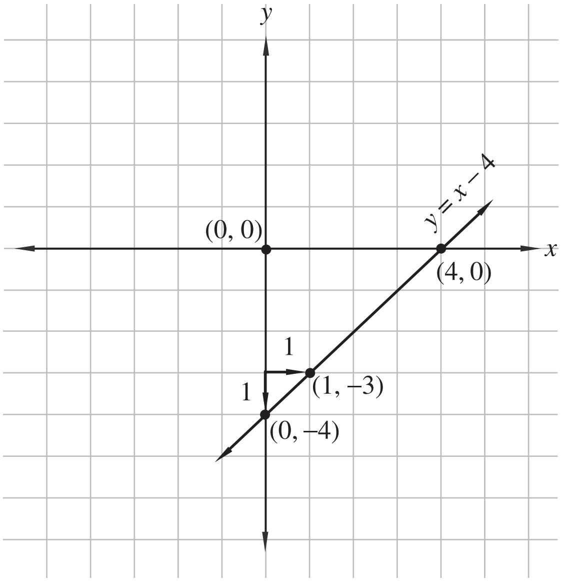


FIGURE 10.14

Notice that the x-intercept occurs at $y = 0$. Substituting $y = 0$ gives $0 = x - 4$, $x = 4$, so the x-intercept occurs where $x = 4$ at the point (4, 0) as can be seen on the graph.

EXAMPLE 10.8: *Putting a Linear Equation in Slope-Intercept Form to Graph It*

Graph the line $3x + 5y = 10$.

Solution: Although we could make a table of (x, y) values to graph this line, a quicker method is to put the equation in slope-intercept form and graph from this form.

$$3x + 5y = 10$$

$$5y = -3x + 10$$

$$y = \frac{-3}{5}x + \frac{10}{5}$$

$$y = \frac{-3}{5}x + 2$$

Now that the equation is in slope-intercept form, we can see that the y-intercept is $b = 2$, with slope $m = \frac{-3}{5}$. Begin by plotting the point $(0, 2)$ and then move *down* 3 units and *right* 5 units and plot the point $(5, -1)$. The reason for moving down is that the change in y is negative. Connecting these two points gives us the graph of the line as shown in Figure 10.15.

EXAMPLE 10.8: ***Putting a Linear Equation in Slope-Intercept Form to Graph It (Continued)***

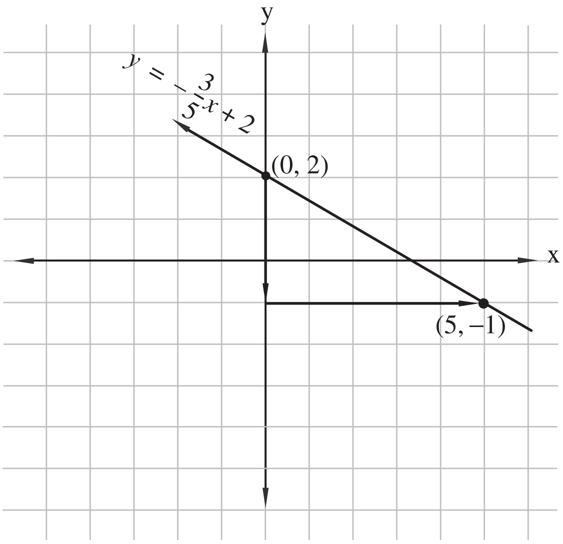


FIGURE 10.15

EXAMPLE 10.9: ***Finding an Equation of a Line from Its Graph***

Use the graph in Figure 10.16 to find the equation of the line in slope-intercept form.

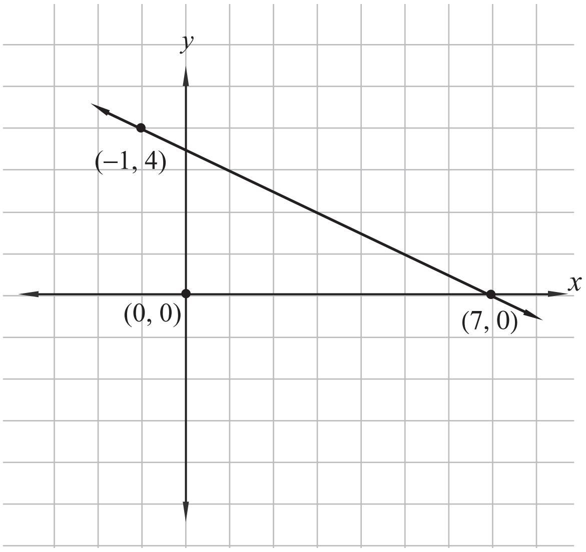


FIGURE 10.16

Solution: Choose two points and calculate the slope *m.*

Let $(x_1, y_1) = (7,0)$ and $(x_2, y_2) = (-1,4)$

Then
$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{0-4}{7-(-1)} = \frac{-4}{8} = -\frac{1}{2}$$

EXAMPLE 10.9: *Finding an Equation of a Line from Its Graph (Continued)*

Substitute either (7, 0) or (–1, 4) into $y = mx + b$ and solve for b.

Choosing (7, 0) gives $0 = -\frac{1}{2}(7) + b = \frac{7}{2} + b$. Hence, $b = \frac{7}{2}$ or $3\frac{1}{2}$.

Since $m = -\frac{1}{2}$ and $b = 3\frac{1}{2}$, the equation of the line in slope-intercept form is $y = -\frac{1}{2}x + 3\frac{1}{2}$.

EXAMPLE 10.10: *Finding Two Points on a Line Given the Equation*

Name two points on the line $y = -2x + 7$.

Solution:

One point is clear from the equation: It is (0, 7), since the y-intercept is 7. To find a second point, choose a value of x that is easy to substitute into the equation; say $x = 1$:

$$y = -2x + 7 \quad \Rightarrow \quad -2(1) + 7 = -2 + 7 = 5 \quad \Rightarrow \quad y = 5 \text{ when } x = 1$$

Thus, a second point on the graph of the line $y = -2x + 7$ is (1, 5).

Be sure to write the coordinates in correct order after finding them.

EXERCISES

10.10 For each equation, identify both the slope and the point where the graph intersects the y-axis.

a) $y = \frac{2}{3}x - 5$

b) $y = -10x + \frac{9}{5}$

c) $y = 4x + (-17)$

d) $y = 16 - 3x$

e) $y = -22 + 8x$

f) $y = x$

g) $y = -x + 14$

h) $y = 2$

i) $y = 36 - 11x$

j) $y = \frac{-2x}{9} + 1$

10.11 Given a point and the slope, graph each line and write its equation in slope-intercept form.

a) (0, 0), $m = 4$

b) (0, –2), $m = 5$

c) (0, 4), $m = -\frac{1}{2}$

d) (1, 1), $m = -1$

e) (–2, 5), $m = 0$

f) (1, 0), $m = -7$

g) (0, 0), $m = \frac{4}{3}$

h) (–3, 2), $m = 1$

i) (–4, –3), $m = 2$

j) (6, 0), $m = \frac{1}{-5}$

10.12 Write the following equations in slope-intercept form, then find two points on the line.

a) $12x - \frac{3}{4}y = 3$

b) $x - y = 9$

c) $5x + 2y = 10$

d) $32x - 8y = 24$

e) $9x - \frac{3}{2}y = 6$

f) $-x - y = -4$

g) $2x + y = 3$

h) $14x - 2y = -34$

i) $3x + 7y = 2$

j) $-\frac{1}{2}x - \frac{3}{4}y = \frac{1}{8}$

10.13 Use the graphs in Figures 10.17A – 10.17F to find the equations of the lines in slope-intercept form.

a)

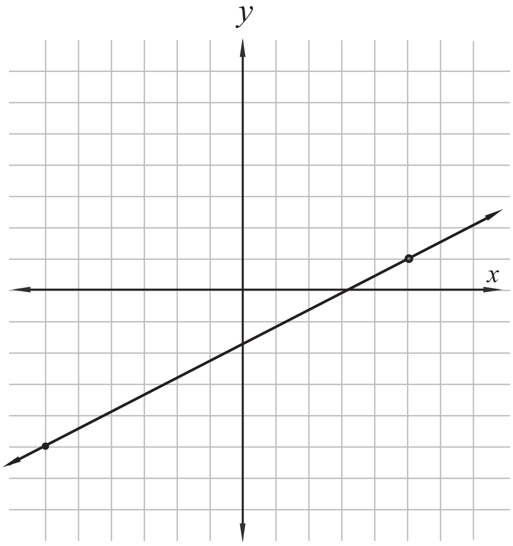


FIGURE 10.17A

b)

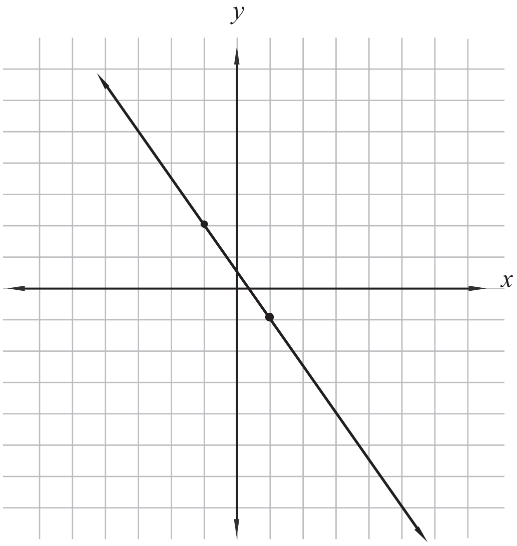


FIGURE 10.17B

c)

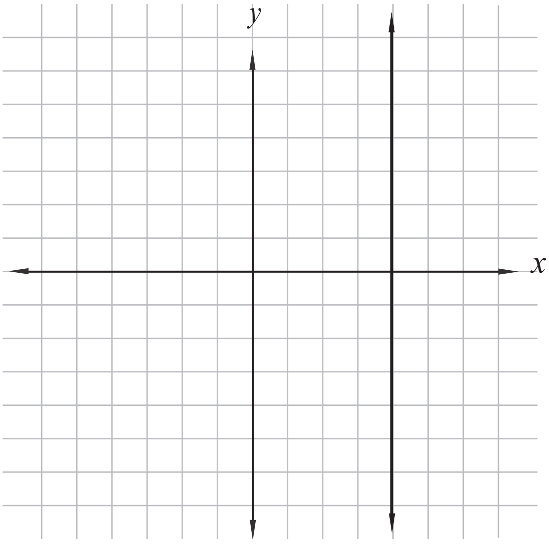


FIGURE 10.17C

d)

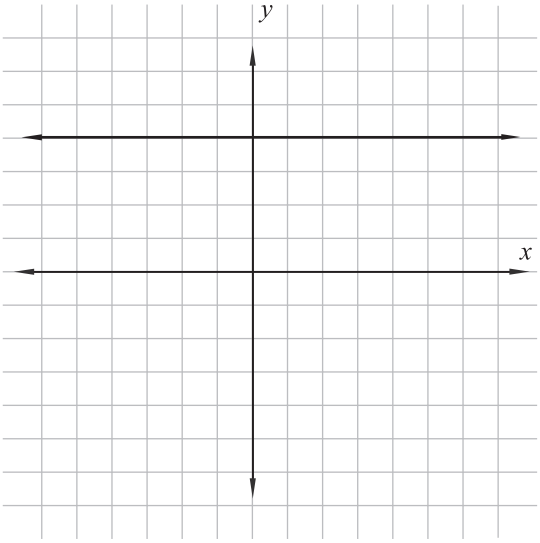


FIGURE 10.17D

e)

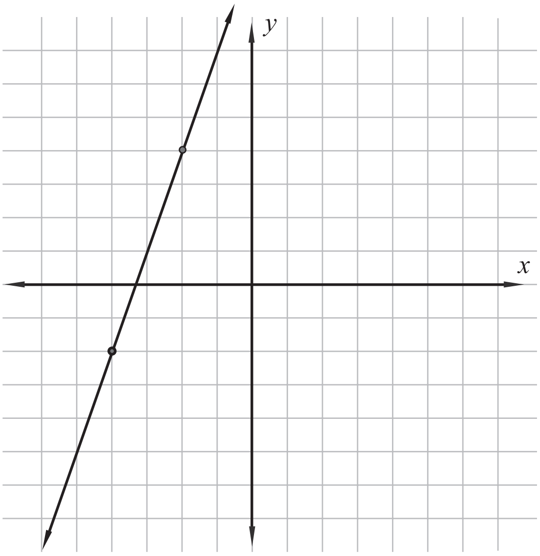


FIGURE 10.17E

f)

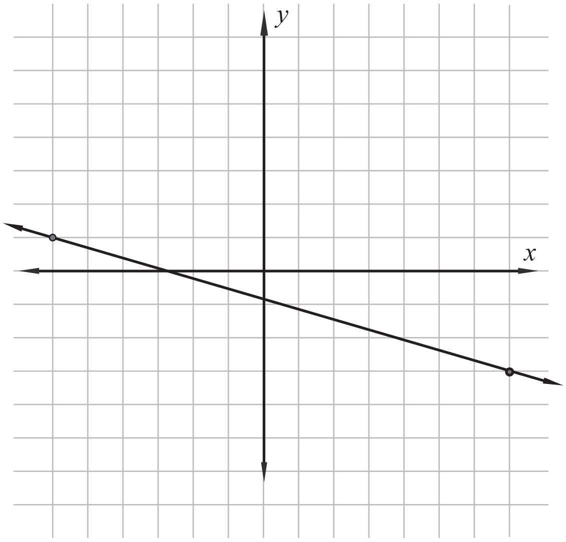


FIGURE 10.17F

11

TRANSFORMING AND SOLVING SHOP FORMULAS

Shop and other scientific formulas are usually written with one variable on the left side of the equation and all other variables and constants on the right side. When the variable whose value is sought is not isolated, the formula must be rearranged, or *transformed*. In Chapter 9 we saw how to do this in the simplest case—that of a single-variable equation with all other terms constant. Now we consider equations with many variables. The procedure for isolating the variable of interest is the same: relevant operational inverses are applied in succession until the equation or formula is in the desired configuration.

11.1 Literal Equations

Variables represent physical entities that can be *quantified*, or given numerical values. Some common variables in technical mathematics are r, d, t, s, ω, and v, as well as C, I, P, R, θ, α, and β. Equations composed of this kind of variable are called *literal equations*. For example, the distance formula, $d = rt$, is a literal equation describing distance traveled, d, as a function of rate, r, and time, t. Another example of a literal equation is the well-known formula for determining circumference of a circle, $C = 2\pi r$.

$d = rt$ is a linear equation in which the slope is r and the y-intercept is 0.

EXAMPLE 11.1: ***Distance Formula***

The distance formula, $d = rt$, is the formula for finding distance d traveled by an object moving at a constant rate r over time t. Given that an object's rate is 40 m/sec, how much time is required for the object to travel 200 m?

Solution:

The basic formula is $d = rt$. To solve the problem, we need to transform the equation so that the variable t is isolated. We transform $d = rt$ by applying the multiplicative inverse of r to both sides of the equation:

$$\frac{1}{r}\cdot d = \frac{1}{\cancel{r}}\cdot \cancel{r}\,t \qquad \rightarrow \qquad \frac{d}{r} = t \quad \text{or} \quad t = \frac{d}{r}$$

EXAMPLE 11.1: *Distance Formula (Continued)*

Variables appear in italic type while units are in plain type.

Substitution of values into $t = \frac{d}{r}$ gives

$$t = \frac{200\ \cancel{\text{m}}}{40\ \frac{\cancel{\text{m}}}{\text{sec}}} = 5\ \text{sec}$$

EXAMPLE 11.2: *Percentage Formula*

Memorization of all three formulas is not necessary since simple transformations of the first formula yields each of the others.

The variations of the percentage formula were given in Chapter 5 using the variables P for percentage, R for rate, and B for base. Starting with $P = RB$, transform the formula to solve for R and B directly.

Solution:

$$P = RB \rightarrow \qquad P \cdot \frac{1}{B} = R\cancel{B} \cdot \frac{1}{\cancel{B}} \qquad \rightarrow \qquad R = \frac{P}{B}$$

$$P = RB \rightarrow \qquad \frac{1}{R} \cdot P = \frac{1}{\cancel{R}} \cdot \cancel{R}B \qquad \rightarrow \qquad B = \frac{P}{R}$$

EXAMPLE 11.3: *Surface Area Formula*

The total surface area (S) of a right-circular cylinder is given by the formula

$$S = \pi D\left(h + \frac{D}{2}\right)$$

where D is the diameter of the cylinder's round cross-section, h is the cylinder's height, and $\pi = 3.1415...$ Transform this formula so that it can be used to solve for h and find the height of a right-circular cylinder with $S = 50$ square inches and $D = 1$ inch.

Solution:

First, isolate the portion of the expression where h is found, the expression in parentheses. Do this by cancelling the πD multiplier through the use of its multiplicative inverse, $\frac{1}{\pi D}$:

$$S = \pi D\left(h + \frac{D}{2}\right)$$

$$\frac{1}{\pi D} \cdot S = \frac{1}{\pi \cancel{D}} \cdot \pi \cancel{D}\left(h + \frac{D}{2}\right) \qquad \rightarrow \qquad \frac{S}{\pi D} = h + \frac{D}{2}$$

Now isolate h by subtracting $\frac{D}{2}$ from both sides of the equation:

$$\frac{S}{\pi D} - \frac{D}{2} = h + \frac{D}{2} - \frac{D}{2} \qquad \rightarrow \qquad h = \frac{S}{\pi D} - \frac{D}{2}$$

Example 11.3: *Surface Area Formula (Continued)*

The left and right sides are reversed in the final formula so that the variable we are solving for appears on the left-hand side of the equation.

Substituting $S = 50$ square inches and $D = 1$ inch into the transformed formula yields:

$$h = \frac{S}{\pi D} - \frac{D}{2}$$

$$h = \frac{50}{\pi(1)} - \frac{1}{2}$$

$$h = 15.4 \text{ inches}$$

EXERCISES

11.1 Transform the equation to solve for the indicated variable.

a) $125a^3 = \frac{5a}{x^2}$ (solve for a)

b) $3(x+2y) = 2(5x-y)$ (solve for x)

c) $10(abc)^2 = 1440$ (solve for a)

d) $L = \sqrt{\frac{T}{3}}$ (solve for T)

e) $3j^2 = 5E - 13$ (solve for E)

f) $K = \frac{1}{2}mv^2$ (solve for v)

g) $15 - 2j^2 - 5V = 3$ (solve for V)

h) $\left(\frac{a}{b}\right)^3 = 125b^3$ (solve for a)

i) $F = k\frac{QQ'}{r^2}$ (solve for k)

j) $3P + 4H + 10 = 6P - 8H + 4$ (solve for P)

k) $T^2 = 4\pi^2\left(\frac{L}{g}\right)$ (solve for L)

l) $3x^2 + 4xy - (2x+y)^2 = 11y^2 - (5x^2 + y^2)$ (solve for x)

m) $\frac{(xy)^2}{x} = 36y^4 + y^2$ (solve for x)

n) $v = v_0 - at$ (solve for t)

o) $P = I^2R$ (solve for I)

p) $V = V_0\left(\frac{P_1}{P_2}\right)\left(\frac{T_2}{T_1}\right)$ (solve for V_0)

q) $V = 2\pi r^2h + 2\pi rh$ (solve for h)

r) $P_1V_1 = P_2V_2$ (solve for P_2)

s) $\frac{90a^2}{3a^2} = \frac{18a^2 - 8}{3a^2 - 2}$ (solve for a)

t) $PV = nRT$ (solve for T)

u) $V = \pi r^2h$ (solve for h)

11.2 Applications of Literal Equations in Shop Mathematics

The remainder of this chapter is dedicated to illustrating numerous practical applications of literal equations in shop mathematics. Being able to transform these and other equations to fit particular situations is an essential skill.

Direct Current (DC) Electrical Formulas

The two most commonly used formulas in direct current (DC) applications are Ohm's law and the power formula. Power, whether electrical power or mechanical power, is the rate at which energy is consumed or work is performed. When discussing electrical formulas, the standard unit of power is the *watt*, an SI unit. Mechanical power is given in terms of watts or in the English unit, *horsepower*.

Recall from Chapter 7 that 1 watt = 1 newton-meter per second $\left(\frac{\text{N-m}}{\text{sec}}\right)$, or 1 joule per second $\left(\frac{\text{J}}{\text{sec}}\right)$. Notice that, like the horsepower units, the watt is composed of work (force times distance) per unit time.

A newton-meter is also called a joule.

Ohm's law is the basic formula for deriving fundamental electrical formulas in electronics and electrical engineering. It is stated

Ohm's law

$$V = IR$$

where V is voltage in units of volts, I is current in units of amperes (amps), and R is resistance in units of ohms (symbolically represented as Ω).

The *power formula* is stated:

Power formula

$$P = I^2R$$

where P is power in units of watts, I is current in units of amperes, and R is resistance in units of ohms.

The algebraically equivalent forms of Ohm's law are

Sometimes E is used as the symbol to represent voltage, not to be confused with E when used for energy!

$$I = \frac{V}{R} \quad \text{from} \quad V \cdot \frac{1}{R} = I\cancel{R} \cdot \frac{1}{\cancel{R}} \rightarrow I = \frac{V}{R}$$

and

$$R = \frac{V}{I} \quad \text{from} \quad \frac{1}{I} \cdot V = \frac{1}{\cancel{I}} \cdot \cancel{I}R \rightarrow R = \frac{V}{I}$$

Substituting each of these into the basic power formula yields the other forms of the power formula:

$$P = I^2R = \left(\frac{V}{R}\right)^2 R = \frac{V^2}{R^{\cancel{2}}}\cancel{R} = \frac{V^2}{R}$$

and

$$P = I^2R = I^{\cancel{2}}\left(\frac{V}{\cancel{I}}\right) = IV,$$

respectively. The ease with which these formulas are rearranged shows the benefit of knowing how to transpose a formula. Memorization of just two formulas, $V = IR$ and $P = I^2R$, allows us to derive many others, depending on what is needed.

$$V = IR \qquad I = \frac{V}{R} \qquad R = \frac{V}{I}$$

Forms of Ohm's law

$$P = I^2R \qquad P = \frac{V^2}{R} \qquad P = IV$$

Forms of power formula

EXAMPLE 11.4: ***Rearrangement of Electrical Formulas***

Suppose a 100-ohm resistor is used in a DC circuit with a 9-volt battery. How much power is dissipated by the resistor?

Solution: R and V are known. We seek P. Of the three power formulas, the second leads directly to the solution:

$$P = \frac{V^2}{R} = \frac{(9\text{ volts})^2}{100\ \Omega} = \frac{81\text{ volts}^2}{100\ \Omega} = 0.81\text{ watt}$$

However, if we know only the two basic formulas, $V = IR$ and $P = I^2R$, we can find a "path" from the known data (R and V) to what is unknown (P). That path is

$$R\text{ and }V \rightarrow I \qquad I\text{ and }R \rightarrow P$$

We need to rearrange $V = IR$ to $I = \frac{V}{R}$, then substitute the data:

$$I = \frac{V}{R} = \frac{9\text{ volts}}{100\Omega} = 0.09\text{ ampere}$$

We can now use $P = I^2R$ to find the power dissipated by the resistor:

$$P = (0.09\text{ ampere})^2(100\Omega) = (0.0081\text{ ampere}^2)(100\Omega) = 0.81\text{ watt}$$

Transforming Thread Translation Ratios

The movement of the jaw on a vise or the carriage on a lathe is termed *translation*. Consider the formula used to calculate threads per inch on a lead screw:

$$t_i = \frac{n_R n_T}{S}$$

where t_i = number of threads per inch
n_R = number of threads per revolution
n_T = number of threads (single or multiple)
S = distance moved by thread (inches per revolution)

This formula can be used to calculate any of the four variables, provided the other three are known. Suppose, as in the following example, we need to determine S on a certain lead screw.

EXAMPLE 11.5: *Thread Translation*

Given the formula for calculating the number of threads per inch, $t_i = \frac{n_R n_T}{S}$, determine how far a single-threaded lead screw on a vise moves after one revolution of the crank if the lead screw has 4 threads per inch.

Solution: Since S is unknown, the thread per inch formula can be transformed to $S = \frac{n_R n_T}{t_i}$ by multiplying both sides of the original formula by S and dividing both sides of the original formula by t_i. So

$$S = \frac{(1 \text{ rev})(1 \text{ thread})}{4 \frac{\text{threads}}{\text{inch}}} = 0.25 \text{ inches}$$

Similarly, the formula could be rearranged to solve for n_R, the number of thread revolutions, to arrive at $n_R = \frac{St_i}{n_T}$, or to solve for the number of threads, $n_T = \frac{St_i}{n_R}$.

Worm Gear Ratios

Many circumstances arise in which the speed of a machine's output gear must be reduced. When high reduction rates are required, worm gear sets are often used, singly or in multiples. Calculation of speed ratios of worm gear sets involves the *threading* of the worm. A *single-thread worm* will turn the worm gear the distance of one tooth each time the worm turns one revolution. The single-thread worm is treated as a one-tooth gear. A *double-thread worm* moves the gear two teeth per revolution and is treated as a two-tooth gear. The same idea holds for triple-thread worm gears, quadruple-thread, and so on.

The familiar formula for worm gear calculations is:

$$\frac{\omega_W}{\omega_G} = \frac{N_G}{N_W}$$

where

ω_G = speed of gear (usually in revolutions per minute, rpm)

ω_W = speed of worm (usually in revolutions per minute, rpm)

N_G = number of teeth on gear

N_W = number of threads on worm

Example 11.6: *Finding Speed of a Worm Gear*

A triple-thread worm meshes with a 39-tooth gear. At what speed does the worm have to rotate to turn the gear 60 rpm?

Solution: All known values of the other variables are substituted into the gear speed formula, which is then solved for the unknown quantity. Note that “triple-thread” means $N_W = 3$. We seek ω_W, which we isolate as follows:

$$\cancel{\omega_G} \cdot \frac{\omega_W}{\cancel{\omega_G}} = \omega_G \cdot \frac{N_G}{N_W}$$

$$\omega_W = \frac{N_G}{N_W}\omega_G$$

Substitute known values into the resulting equation:

$$\omega_w = \frac{(39\ \cancel{\text{teeth}})(60\,\text{rpm})}{3\ \cancel{\text{threads}}} = 780\,\text{rpm}$$

Teeth and threads of the screw mesh at a rate of 1 tooth per 1 thread.

The units of teeth and threads cancel because each represents length: that length being the distance between gear teeth and the distance between the threads. These distances have to match to allow the gears and teeth to mesh.

Example 11.7: *Determining Number of Teeth on a Worm Gear*

A single-thread worm turning at 2400 rpm drives another gear to rotate at 60 rpm. How many teeth are on the gear?

Solution: Solve the gear speed formula for the unknown quantity, N_G and substitute known values:

$$\frac{\omega_W}{\omega_G} \cdot N_W = \frac{N_G}{\cancel{N_W}} \cdot \cancel{N_W}$$

$$N_G = \frac{N_W \omega_W}{\omega_G} = \frac{(2400\ \cancel{\text{rpm}})(1\ \text{tooth})}{60\ \cancel{\text{rpm}}} = 40\ \text{teeth}$$

N_G is on the right of the original formula, but for the transformed formula, we move it to the left.

In these examples we have identified the unknown variable. However, it is often up to a worker to determine for which variable to solve and to set up the formula accordingly.

Horsepower Formulas

Power is the rate at which work is performed or energy is consumed. Work and energy are measured in units of foot-pounds (ft · lb) in the English system, and so power is measured in foot-pounds per minute $\left(\frac{\text{ft}\cdot\text{lb}}{\text{min}}\right)$ or foot-pounds per second $\left(\frac{\text{ft}\cdot\text{lb}}{\text{sec}}\right)$. The standard unit of power in the English system is *horsepower*. The unit of horsepower is abbreviated “hp.” The variable representing horsepower is printed in italics, *hp*. Hence, the formula for calculating horsepower is

About 200 years ago, James Watt determined that the average horse can lift 550 pounds about 1 foot per second. This became the definition for horsepower.

$$hp = 550\ \frac{\text{ft}\cdot\text{lb}}{\text{sec}}$$

$$hp = \frac{Fs}{550t}$$

Formula for calculating horsepower

where hp = amount of horsepower $\frac{\text{ft} \cdot \text{lb}}{\text{sec}}$

F = force (lb)

s = distance (ft)

t = time (sec)

This formula shows that time and horsepower vary inversely with respect to one another. The next examples illustrate how the horsepower formula is rearranged for the unknown variable.

Example 11.8: ***Finding Horsepower, Given Force, Distance, and Time***

A steel beam for a skyscraper weighs 1500 lb. How much horsepower is developed when a crane lifts the beam 35 feet in 20 seconds?

Solution:

$$hp = \frac{Fs}{550t} = \frac{(1500\ \cancel{\text{lb}})(35\ \cancel{\text{ft}})}{\left(550\ \frac{\cancel{\text{ft}} \cdot \cancel{\text{lb}}}{\cancel{\text{sec}} \cdot \text{hp}}\right)(20\ \cancel{\text{sec}})} = 4.8\ \text{hp}$$

The least accurate piece of data is 20 sec, which has one significant figure. Therefore, the answer can be given reliably as 5 hp. Ordinarily, units are not shown in calculations except when clarity is essential. Units, however, should always be attached to final results.

In Example 11.8, units for all quantities are clearly written out and cancellation is shown to emphasize that formulas do not magically come out with the proper units. $550\ \frac{\text{ft} \cdot \text{lb}}{\text{sec}}$ may seem like an unusual term, but it is a combination of all factors needed to arrive at a desired type of unit, in this case, horsepower.

Problems also arise in which horsepower is known and the amount of time required to accomplish a certain task is unknown. The following examples illustrate this.

Example 11.9: ***Finding Time, Given Horsepower***

Suppose a car weighing 1675 pounds has a 150-hp engine. How many seconds are needed for the car to travel $\frac{1}{4}$ mile?

Solution:

Convert miles to feet: $\left(\frac{1}{4}\ \cancel{\text{mile}}\right)\left(\frac{5280\ \text{feet}}{\cancel{\text{mile}}}\right) = 1320\ \text{feet}$

The horsepower formula must be transformed so the variable t is on the left hand side of the equation. Multiply both sides of the equation by t. Then divide both sides of the equation by hp:

$$hp \cdot t = \frac{Fs}{550\cancel{t}} \cdot \cancel{t}$$

EXAMPLE 11.9: ***Finding Time, Given Horsepower (Continued)***

$$\frac{1}{\cancel{hp}}\cdot\cancel{hp}\,t=\frac{1}{hp}\cdot\frac{Fs}{550}$$

$$t=\frac{Fs}{550hp}=\frac{(1675\,\cancel{\text{lb}})(1320\,\cancel{\text{ft}})}{\left(550\,\frac{\cancel{\text{ft}}\cdot\cancel{\text{lb}}}{\text{sec}\cdot\cancel{\text{hp}}}\right)(150\,\cancel{\text{hp}})}=26.8\text{ sec}$$

EXAMPLE 11.10: ***Finding Load, Given Horsepower***

How much weight is a 25-hp winch able to lift, if its cable moves at a velocity of 2 $\frac{\text{ft}}{\text{sec}}$?

Solution: Since velocity is given, rather than distance and time, substitute v for $\frac{s}{t}$:

$$hp=\frac{Fs}{550t}=\frac{Fv}{550}$$

Recall that velocity is distance divided by time.

Reverse the formula so load F is on the left:

$$\frac{Fv}{550}=hp$$

Multiply both sides of the equation by 550 and divide both sides of the equation by v:

$$\frac{\cancel{550}}{\cancel{v}}\cdot\frac{F\cancel{v}}{\cancel{550}}=\frac{550}{v}\cdot hp\ \Rightarrow F=\frac{550\,hp}{v}$$

Substitute hp = 25 hp and $v = 2\frac{\text{ft}}{\text{sec}}$:

$$F=\frac{550hp}{v}=\frac{\left(550\,\frac{\cancel{\text{ft}}\cdot\text{lb}}{\cancel{\text{sec}}\cdot\cancel{\text{hp}}}\right)(25\ \cancel{\text{hp}})}{2\,\frac{\cancel{\text{ft}}}{\cancel{\text{sec}}}}=6875\text{ lb}$$

550 $\frac{\text{ft}\cdot\text{lb}}{\text{sec}\cdot\text{hp}}$ comes from the fact that one horsepower has 550 $\frac{\text{ft}\cdot\text{lb}}{\text{sec}}$.

Significant figures in the answer should not exceed two. Hence, the answer is 6900 lb.

In Example 11.10, notice that the formula for velocity is $v = \frac{s}{t}$, similar to the formula for rate of speed, $r = \frac{d}{t}$. Distance is written as s when a rate of velocity (v), rather than a rate of speed (r), is described. The difference between "velocity" and "rate of speed" is technical, and it is often not apparent in the formula. The difference is that "velocity" includes both magnitude and direction, while "rate of speed" is only a magnitude.

Horsepower of Pumps and Motors

T for "torque" must not be confused with T for "number of turns of crank" for indexing an angle on a machine.

Motors are sometimes rated in terms of *torque* (T) at a given number of revolutions per minute (*rpm*). Torque, or moment of force, is a measure of the tendency of a force to rotate a body about an axis. The axis is called the *lever arm*, or the *moment arm*, and has units of length. The units of torque are lb·ft, with lb as the force unit and ft as the lever arm length.

The horsepower formula can be adapted to situations involving torque (T) as follows:

$$hp = \frac{T \cdot rpm}{5250}$$

The variable for rate in the torque version of the horsepower formula is *rpm*. A numerical value for rpm is substituted in place of the variable *rpm*.

Details of the conversion of the first horsepower formula to this one are given in the following equation:

$$hp = \frac{T}{550 \frac{\text{ft} \cdot \text{lb}}{\text{sec} \cdot \text{hp}} \cdot 60 \frac{\text{sec}}{\text{min}}} \cdot \frac{2\pi}{t} = \frac{T \cdot rpm}{5250}$$

The factor of 2π (approximately 6.28) is needed to convert the rotational distance on a circle to linear distance.

Example 11.11: ***Estimating Torque for an Electric Motor***

Suppose a motor is required to have a minimum starting torque of 20 ft·lb, Will the 3-horsepower motor in the warehouse provide sufficient torque at 1800 rpm to start the motor?

Solution: Rearrange the horsepower formula so torque is on the left side, and substitute the known values:

$$T = \frac{5250\,hp}{rpm} = \frac{(5250)(3\text{ hp})}{1800\text{ rpm}} = 8.5\text{ ft} \cdot \text{lb}$$

The value is less than 20 ft·lb, so the 3-horsepower motor cannot provide enough torque. A different motor must be found.

The horsepower required to drive a hydraulic pump is given by

$$hp = \frac{Fs}{550t} = \frac{Pq}{6600}$$

where P = pressure in psi $\left(\frac{\text{lb}}{\text{in}^2}\right)$ and q = flow rate $\left(\frac{\text{in}^3}{\text{sec}}\right)$. The factor 6600 arises from converting inches to feet (550 · 12 = 6600). Similar to the torque formula, the flow rate (q) in the pump formula includes variable t.

EXAMPLE 11.12: ***Finding Horsepower of a Pump***

Find the horsepower rating of a pump required to deliver 462 $\frac{\text{in}^3}{\text{sec}}$ at 2000 $\frac{\text{lb}}{\text{in}^2}$.

Solution: Substitute the values given into the formula as written:

$$hp = \frac{Pq}{6600} = \frac{\left(2000\,\frac{\text{lb}}{\text{in}^2}\right)\left(462\,\frac{\text{in}^3}{\text{sec}}\right)}{6600\,\frac{\text{in}\cdot\text{lb}}{\text{sec}\cdot\text{hp}}} = 140\ \text{hp}$$

Notice that this is a simplified formula for pump horsepower, which does not take into account any fluid friction or pump inefficiencies.

EXAMPLE 11.13: ***Finding Flow Rate of a Pump***

How many gpm can a 400-hp pump deliver at 3000 psi?

Solution: Since the units of flow rate in this example are gpm, or $\frac{\text{gal}}{\text{min}}$, we perform a rather lengthy factor-label conversion to change $\frac{\text{gal}}{\text{min}}$ to $\frac{\text{in}^3}{\text{sec}}$. It will take us from $\frac{\text{gal}}{\text{min}}$ to $\frac{\text{gal}}{\text{sec}}$ to $\frac{\text{ft}^3}{\text{gal}}$ to $\frac{\text{in}^3}{\text{ft}^3}$ as follows:

$$1\ \text{gpm} = \left(\frac{1\ \cancel{\text{gal}}}{\cancel{\text{min}}}\right)\left(\frac{1\ \cancel{\text{min}}}{60\ \text{sec}}\right)\left(\frac{7.48\ \cancel{\text{ft}^3}}{1\ \cancel{\text{gal}}}\right)\left(\frac{(12\ \text{in})^3}{\cancel{\text{ft}^3}}\right) = 215\,\frac{\text{in}^3}{\text{sec}}$$

Transform the horsepower formula for pumps using the new constant:

$$q = \frac{(hp)(215)}{P} = \frac{(400)(215)}{3000} = 29\ \text{gpm}$$

Be familiar with abbreviations for common units, such as psi for $\frac{lb}{in^2}$ and gpm for $\frac{gal}{min}$.

Machine Shop Problems

Formula transformations encountered in typical machine shop problems are illustrated with several examples.

uhp is the amount of horsepower required to cut metal at a rate of 1 $\frac{in^3}{min}$.

Other abbreviations sometimes encountered are "inches per minute" and "feet per minute" –ipm and fpm, respectively.

EXAMPLE 11.14: ***Depth of Cut***

The amount of horsepower required at the cutter (hp_c) for turning a metal workpiece on a lathe is given as

$$hp_c = (uhp)(C)(12)(V)(f)(d)$$

where

uhp = unit horsepower
C = correction factor for the feed
V = cutting speed (feet per minute)
f = feed (inches per minute)
d = depth (inches)

What are the units for correction factor C in Example 11.14?

The factor 12 is needed to keep units straight (12 in = 1 ft). It converts V from feet per minute to inches per minute.

EXAMPLE 11.14: *Depth of Cut (Continued)*

Determine the appropriate depth of cut d when unit horsepower uhp for mild machine steel (AISI 1020) is 0.58 hp, correction factor C is 0.96 for a feed $f = 0.015\ \frac{\text{in}}{\text{min}}$, cutting speed $V = 100.0\ \frac{\text{ft}}{\text{min}}$ and $hp_c = 1.5$ hp.

Solution: Transform the formula so that d is on the left hand side:

$$d = \frac{hp_c}{(uhp)(12CVf)}$$

$$d = \frac{1.5\text{ hp}}{(0.58\text{ hp})(0.96)\left(12\frac{\text{in}}{\text{ft}}\right)\left(100.0\frac{\text{ft}}{\text{min}}\right)\left(0.015\frac{\text{in}}{\text{min}}\right)} = 0.15\text{ in}$$

EXAMPLE 11.15: *Cutting Speed*

The correct spindle speed of a lathe in rpm is obtained from the formula

$$N = \frac{12V}{\pi D}$$

where

$N =$ spindle speed (revolutions per minute)

$V =$ cutting speed (feet per minute)

$D =$ diameter of milling cutter (inches)

To minimize rounding error, use π from calculator keypad, rather than using 3.14, but give answer in the correct number of significant digits.

Solve for cutting speed when spindle speed is 850 rpm and diameter of the workpiece is 0.750 inch.

Solution: Transpose the formula to solve for V, rather than N:

$$N \cdot \frac{\pi D}{12} = \frac{\cancel{12}V}{\cancel{\pi}\,\cancel{D}} \cdot \frac{\cancel{\pi}\,\cancel{D}}{\cancel{12}}$$

$$V = \frac{N\pi D}{12}$$

Substitute the given values:

$$V = \frac{\pi N D}{12} = \frac{(\pi)(850)(0.750\,\cancel{\text{in}})}{12} = 167\text{ feet per minute}$$

EXAMPLE 11.16: *Spindle Speed*

The table feed rate f_m for a milling operation is

$$f_m = f_t n_t N$$

EXAMPLE 11.16: *Spindle Speed (Continued)*

where

f_m = table feed rate (inches per minute)
f_t = feed rate (inches per tooth)
n_t = number of cutting teeth on cutter
N = spindle speed (revolutions per minute)

Another abbreviation for inch per tooth is ipt.

Solve for spindle speed N (the speed of rotation of the cutter in rpm) when $f_m = 35\ \frac{\text{ft}}{\text{min}\cdot\text{rev}}$, $f_t = 0.010\ \frac{\text{in}}{\text{tooth}}$, and $n_t = 10$ teeth.

Work out the units in Example 11.16 to see that the final answer is in rpm.
Hint: n_t is the number of teeth on one revolution of the cutter.

Solution: Transform the formula, so N is on the left:

$$f_t n_t N = f_m$$

$$\frac{1}{\cancel{f_t}\cancel{n_t}}\cdot \cancel{f_t}\cancel{n_t}N = \frac{1}{f_t n_t}\cdot f_m$$

$$N = \frac{f_m}{f_t n_t} = \frac{35}{(0.010)(10)} = 350 \text{ rpm}$$

EXAMPLE 11.17: *Indexing an Angle*

In the use of a Cincinnati Universal Dividing Head, one turn of the crank rotates the spindle $\frac{1}{40}$ revolution. Since there are 360° in one revolution, one turn of the crank moves the spindle $\frac{360^\circ}{40} = 9^\circ$. When a specific angle measure D is to be indexed, the formulas for number of turns of the crank T are:

Positioning a workpiece at a precise angle or interval of rotation for a machining operation is called indexing.

$$T = \frac{D^\circ}{9^\circ} \qquad \text{(for angle portion in degrees)}$$

$$T = \frac{D'}{(9^\circ)\left(\frac{60'}{1^\circ}\right)} = \frac{D'}{540'} \qquad \text{(for angle portion in minutes)}$$

$$T = \frac{D''}{(9^\circ)\left(\frac{60'}{1^\circ}\right)\left(\frac{60''}{1'}\right)} = \frac{D''}{32{,}400''} \qquad \text{(for angle portion in seconds)}$$

These formulas are only for the Cincinnati Universal Dividing Head. Similar formulas for T can be derived for other machines, based on the distance the spindle turns for one revolution of the crank for the particular machine.

How many crank turns are required to index 14°20′?

Solution: Since 14°20′ has both degree and minute measure, we must add the first two T formulas:

Assume that degrees and minutes are accurate to at least tenths.

$$T = \frac{D^\circ}{9^\circ} + \frac{D'}{540'} = \frac{14}{9} + \frac{20}{540} = 1.6 \text{ turns (or } 1\tfrac{3}{5} \text{ turns)}$$

Note that the units in each fraction cancel, leaving quantities that represent number of turns of the crank. These we can add.

Hydraulic Formulas

Pressure is defined as a force exerted on an object divided by the area over which that force is exerted.

Fundamental formula of hydraulics

$$P = \frac{F}{A}$$

***P** for pressure must not be confused with the variable for power.*

Due to hydrostatic pressure, two reservoirs joined by a pipe each have equal pressures when in a state of equilibrium. Therefore,

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}$$

at each of the reservoirs. Transforming this relationship through division of each side by F_2 and multiplication of each side by A_1 leads to *Pascal's law*:

Pascal's law

$$\frac{F_1}{F_2} = \frac{A_1}{A_2}$$

The following examples show simple applications of Pascal's law.

EXAMPLE 11.18: ***Applying Pascal's Law***

The fundamental unit for pressure in the metric system is the Pascal, abbreviated Pa.

$$1 Pa = 1\frac{N}{m^2}$$

The hydraulic jack shown in Figure 11.1 experiences a force, $F_1 = 150$ lb acting on a 1-inch diameter piston with area A_1. The fluid flows to a 4-inch diameter piston with area A_2 and exerts a force F_2. Solve for F_2.

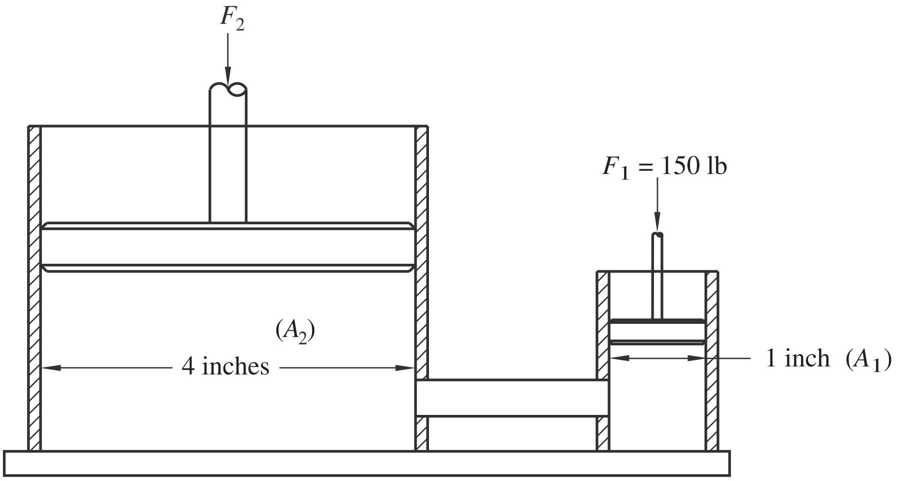


FIGURE 11.1

Solution: Transpose the formula for Pascal's law so F_2 is isolated on the left-hand side of the equation:

$$\frac{F_1}{F_2} = \frac{A_1}{A_2}$$

EXAMPLE 11.18: *Applying Pascal's Law (Continued)*

Multiply both sides by A_2:

$$\frac{F_1}{F_2}A_2 = \frac{A_1}{\cancel{A_2}}\cancel{A_2}$$

Multiply both sides by F_2:

$$\frac{F_1A_2}{\cancel{F_2}}\cancel{F}_2 = A_1F_2$$

So $$F_1A_2 = F_2A_1$$

Reverse the sides of the equation so that F_2 is on the left:

$$F_2A_1 = F_1A_2$$

Isolate F_2 by using the multiplicative inverse of A_1:

$$F_2\cancel{A}_1\cdot\frac{1}{\cancel{A_1}} = F_1A_2\cdot\frac{1}{A_1}$$

Thus, $F_2 = \frac{F_1A_2}{A_1}$. Recalling that a cross-sectional area of a round pipe is equal to πr^2 gives

$$F_2 = \frac{F_1A_2}{A_1} = \frac{F_1\cancel{\pi}r_2^2}{\cancel{\pi}r_1^2} = \frac{F_1r_2^2}{r_1^2},$$

where r_1 and r_2 are the radii of the respective pistons. Substitute $r = \frac{D}{2}$, where D is piston diameter, with the given value of D for each piston:

$$F_2 = \frac{F_1r_2^2}{r_1^2} = \frac{(150\text{ lb})\left(\frac{4\text{ in}}{2}\right)^2}{\left(\frac{1\text{ in}}{2}\right)^2} = \frac{(150\text{ lb})\left(4\,\cancel{\text{in}^2}\right)}{0.25\,\cancel{\text{in}^2}} = 2400\text{ lb}$$

The cross-sectional area of a round pipe or piston is $A = \pi r^2 = \pi\left(\frac{d}{2}\right)^2$. *Area calculations are reviewed in Chapter 16.*

EXAMPLE 11.19: *Hydraulic Hoist*

How much weight can a single hydraulic hoist with a 6-inch diameter working piston lift if the hydraulic pressure is 80 psi?

Solution: Rearrange $P = \frac{F}{A}$ to $F = PA$ and substitute values:

$$F = (80)\left(\pi r^2\right) = 80\pi\left(\frac{6}{2}\right)^2 = 2262\text{ lb}$$

Flow rate is given in units of volume per unit time.

Another formula frequently encountered in hydraulics problems calculates flow rates. It is stated as:

$$q = Av$$

where

$$q = \text{flow rate}\left(\text{gpm}, \frac{\text{in}^3}{\text{sec}}, \text{ etc.}\right)$$

$$A = \text{cross-sectional area of pipe } (\text{m}^2, \text{ in}^2, \text{ etc.})$$

$$v = \text{velocity}\left(\text{fpm}, \frac{\text{m}}{\text{sec}}, \text{ etc.}\right)$$

Example 11.20: ***Water Flow Rate***

How fast does water flow in a 150-mm diameter pipe if 75 gallons of water are delivered every minute?

Solution: First, convert all variables to a consistent system of units. For SI, convert to m^3 and sec:

$$\#\text{m} = (150 \text{ mm})\left(\frac{1\text{m}}{1000 \text{ mm}}\right) = 0.150 \text{ m}$$

$$\#\frac{\text{m}^3}{\text{sec}} = \left(\frac{75\,\cancel{\text{gal}}}{1\,\cancel{\text{min}}}\right)\left(\frac{3.79\,\cancel{\text{liter}}}{1\,\cancel{\text{gal}}}\right)\left(\frac{1\text{m}^3}{1000\,\cancel{\text{liter}}}\right)\left(\frac{1\,\cancel{\text{min}}}{60 \text{ sec}}\right) = 0.00474\ \frac{\text{m}^3}{\text{sec}}$$

Transform the flow equation $q = Av$ to solve for v:

$$v = \frac{q}{A} = \frac{0.00474\,\frac{\text{m}^3}{\text{sec}}}{\pi\left(\frac{0.150 \text{ m}}{2}\right)^2} = \frac{0.00474\,\frac{\text{m}^3}{\text{sec}}}{0.01767 \text{ m}^2} = 0.268\ \frac{\text{m}}{\text{sec}} \text{ or } 268\ \frac{\text{mm}}{\text{sec}}$$

This formula does not take into account other factors affecting flow, such as pipe friction. Nonetheless, it is useful for estimation. Friction factors can be found in many handbooks and depend on the fluid and the pipe material.

Always take care that the units of the various quantities in any formula agree. For example, if an area is given in meters squared (m^2), the velocity should be given—or converted as necessary—to meters per second $\left(\frac{\text{m}}{\text{sec}}\right)$.

Hydraulic Cylinders

Hydraulic cylinders are used throughout industry to move heavy loads quickly and precisely, as on an assembly line press or in an airplane's wing flaps. A generic, single-headed hydraulic cylinder is shown in Figure 11.2.

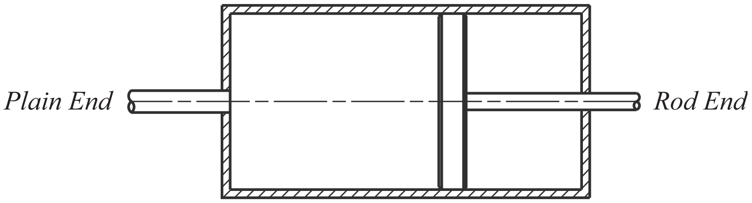


Figure 11.2 ***Single-headed hydraulic cylinder***

Because the hydraulic cylinder shown is single-headed, it has both a *plain end* and a *rod end*. Calculations depend on which end is considered. On the rod end, the rod itself is occupying space and taking up free area on the piston, area that would otherwise be available for the pressure to act over.

Many hydraulic cylinder problems are solved with two formulas, one for force and the other for flow rate. Consider the formula for force (F) first. For the rod end of a single-headed cylinder:

$$F = P\frac{\pi(D^2 - d^2)}{4}$$

where D = diameter of piston

d = diameter of rod

For the plain end of a single-headed cylinder, $d = 0$, since there is no rod. Hence,

$$F = P\frac{\pi}{4}(D^2 - d^2) = P\frac{\pi}{4}\left(D^2 - 0^2\right) = P\frac{\pi D^2}{4}$$

That is, $F = P\frac{\pi D^2}{4}$ for force on the plain end of a single-headed cylinder.

EXAMPLE 11.21: ***Force of a Hydraulic Cylinder***

Determine the force developed by a 4-inch hydraulic cylinder operated with150 psi line pressure at the plain end.

Solution: Substituting the known values into the force equation for the plain end of the hydraulic cylinder yields

$$F = P\frac{\pi}{4}D^2 = \frac{\left(150.0\ \frac{\text{lb}}{\text{in}^2}\right)\pi\left(4.0\text{ in}\right)^2}{4} = 1885\text{ lb}$$

The speed (v) at which the piston of a hydraulic cylinder moves is proportional to the flow rate (q) of the hydraulic fluid. Hence, for round hydraulic cylinders:

Flow rate in the rod end of a hydraulic cylinder.

$$q = Av = \frac{\pi(D^2 - d^2)v}{4}$$

where q = flow rate

D = diameter of cylinder

d = diameter of piston rod

v = velocity of piston

As with the force equations, the plain end of a hydraulic cylinder has no rod to consider, so $d = 0$, and the formula becomes

Flow rate in the plain end of a hydraulic cylinder

$$q = \frac{\pi D^2 v}{4}$$

EXAMPLE 11.22: *Velocity of a Hydraulic Cylinder*

A *double rod end piston* has a rod coming out of both sides of the piston, meaning that the hydraulic cylinder does not have a plain end. Find the velocity of a 3.0-inch double rod end piston with a rod diameter of 0.75 inch and a flow rate of 230 $\frac{\text{in}^3}{\text{sec}}$.

Solution:

Reverse the flow formula so that velocity (v) is on the left side:

$$\frac{\pi(D^2 - d^2)v}{4} = q$$

Solve for v:

$$\frac{4}{\pi(D^2 - d^2)} \cdot \frac{\pi(D^2 - d^2)v}{4} = \frac{4}{\pi(D^2 - d^2)} \cdot q$$

$$v = \frac{4q}{\pi(D^2 - d^2)}$$

Now substitute the given values and solve the formula:

$$v = \frac{4q}{\pi\left(D^2 - d^2\right)} = \frac{4 \cdot 230}{\pi\left(3.0^2 - 0.75^2\right)} = 34.7\ \frac{\text{in}}{\text{sec}}$$

The ability to adapt equations and formulas through transformation of variables is essential for success in technological fields. Without the capability to transform an equation such as formulas for worm gear ratios or the various power laws, a worker is at the mercy of formula tables, as if every situation were a "special case" that had to be memorized! We have shown that many "special cases" are quickly derived from basic formulas using algebra. Working through the algebra to transform a formula for a particular purpose builds an intuition of why and how mechanical and electrical systems work. It is also quicker than having to find the information in a handbook.

If a hydraulic cylinder is not round, the correct area formula must be substituted.

EXERCISES

ELECTRICAL PROBLEMS

11.2 Using the formula, $V = IR$, where V is voltage in volts, I is current in amperes, and R is resistance in ohms, determine:

a) The resistance of an incandescent lamp when 110 volts are applied with a current of 0.5 ampere.

b) What would the resistance be if the current were increased to 0.75 ampere and the voltage remained 110 volts?

11.3 Using the formula, $P = I^2R$, where P is power in watts, I is current in amperes, and R is resistance in ohms, determine:

a) How many watts are developed for a current of 10 amperes flowing through a resistance of 8 ohms?

b) What would the resistance be if the power were increased to 1000 watts for a 10-ampere current?

c) How much resistance is needed to operate a 1500-watt electric oven with a current of 12.5 amperes?

11.4 Using the formula $P = IV$, where P is power in watts, I is current in amperes, and V is voltage in volts, determine:

a) How much power is required to operate a 12-ampere soldering iron at 110 volts?

b) How much current is needed to operate a 1000-watt iron at 110 volts?

11.5 The formula for the sum of the separate resistances in a parallel circuit containing 3 resistors is: $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. What size resistor is needed for R_3 if $R_1 = 2$ ohms, $R_2 = 6$ ohms, and the total circuit resistance is 1.33 ohms?

11.6 The impedance Z of an electric circuit is given by $Z = \sqrt{\left(R^2 + X^2\right)}$ where R is the resistance in ohms and X is the reactance in ohms.

a) Calculate the impedance of a circuit when $R = 342$ ohms and $X = 5.69$ ohms.

b) What value of R is needed if the desired impedance is 600 ohms and the reactance remains at 5.69 ohms?

11.7 The voltage at the low end of a transmission line is given in the formula: $V_L = V_G - R_1 I_1$ where V_G is voltage of the generator, I_1 is the current in amperes, and R_1 is the total resistance of the line in ohms.

a) Solve for V_L when $V_G = 44{,}000$ volts, $I_1 = 150$ amperes, and $R_L = 49.5$ ohms.

b) Solve for I_1 when $V_G = 44{,}000$ volts, $V_L = 30{,}000$ volts, and $R_L = 49.5$ ohms.

The horsepower formulas in problems Problems 11.8 – 11.13 use the following variables:

P = pressure (psi)
T = torque (ft · lb)
t = torque (inch · lb)
F = weight or force (lb)
V = velocity (ft per minute)
A = piston area (square inches)
L = stroke (ft)
N = shaft speed (rpm)
N_s *= strokes per minute (twice rpm)*
kw = kilowatts
R = length (ft)

HORSEPOWER PROBLEMS

11.8 The formula for indicated horsepower *ihp* is: $ihp = \frac{PLAN_s}{33{,}000}$.

a) Determine the indicated horsepower of a steam engine running at 166 strokes per minute with a mean effective pressure of 110 psi, an 8-inch diameter piston, and a stroke length of 16 inches.

b) Solve for P under the same conditions if the *ihp* = 50 hp.

11.9 The formula for brake horsepower *bhp* is: $bhp = \dfrac{2\pi NFR}{33{,}000}$.

a) A motor is placed on a test stand to determine its brake horse power. Its speed N is 250 rpm; its length of brake arm R is 2.5 feet; and it carries a weight F of 130 lb. Determine the brake horsepower.
b) Solve for F under the same conditions when $bhp = 10$ hp.

11.10 Horsepower can be found using the formula: $hp = \dfrac{2\pi TN}{33{,}000}$ when torque has units of ft·lb.

a) An engine running at $N = 360$ rpm on a test stand has a torque measurement, $T = 500$ ft·lb. Solve for horsepower.
b) Solve for torque when $hp = 50$ and $N = 720$ rpm.

11.11 Horsepower can be found using the formula: $hp = \dfrac{2\pi tN}{396{,}000}$ when torque has units of in·lb.

a) Find the horsepower of a miniature gasoline engine used in a model plane with $t = 6.875$ in·lb when $N = 12{,}000$ rpm.
b) Determine how much torque is provided by a 0.40-cubic-inch displacement (cid) model gas engine generating 0.59 hp at 11,000 rpm.

11.12 Horsepower can be found using the formula: $hp = \dfrac{FV}{33{,}000}$.

a) Calculate the horsepower necessary to lift a 300-lb casting at a rate of 50 feet per minute.
b) How much weight can a 0.75-hp device lift at a rate of 2 feet per second?

11.13 Horsepower can be found using the formula: $hp = \dfrac{kw}{1.34}$.

a) An electric drive motor in a steel mill uses 670 kw at full load. What is the horsepower of this motor?
b) What is the kilowatt rating of a 5-hp motor?

HORSEPOWER OF PUMPS AND MOTORS PROBLEMS

11.14 The horsepower required to drive a hydraulic pump is given in the formula: $hp = \dfrac{pq}{1715}$ where p is pressure in psi and q is flow rate in gpm.

a) Find the horsepower required to deliver 1500 gpm at 2000 psi.
b) Given 400 hp and 350 psi, what flow rate can be achieved?

11.15 If p is given in psi and q is given in cubic feet per second, then horsepower can be calculated using the formula: $hp = 0.2618\,pq$.

a) Find the horsepower required to pump 95 cubic feet per second at 5 psi.

b) Given 200 hp and a 75-cubic-feet-per-second flow rate, what pressure can be achieved?

MACHINE SHOP PROBLEMS

11.16 The full depth of the thread on an external American National Standard Unified Thread is given in the formula: $D = \frac{0.61343}{N}$ where D = full depth of thread in inches and N = number of threads per inch.

a) Calculate the full depth of the thread on a bolt having 11 threads per inch.

b) Determine the number of threads per inch when the full depth of thread is 0.3067 inch.

11.17 The outside diameter of a spur gear can be determined from the following formula: $D_0 = \frac{N}{P} + \left(2 \times \frac{1}{P}\right)$.

a) Calculate the outside diameter of a spur gear when diametral pitch P is 14 and number of teeth N is 42.

b) If $D_0 = 2.2$ and $N = 42$, find the value for P.

11.18 The diameters of blanks for drawing plain cylindrical shells can be obtained by using the following formula: $D = \sqrt{d^2 + 4dh}$.

a) Find diameter D when diameter of the finished shell d is 1.5 inches and height h is 2 inches.

b) Find h when $d = 2$ inches and $D = 4$ inches.

11.19 Load capacity P in pounds of a round-wire, helical compression spring is expressed in the formula $P = \frac{\pi d^3 r}{8K_a D}$.

a) Solve for P when $d = 0.192$ inch, $r = 5000$ psi, $K_a = 1.19$, and $D = 2.0$ inches.

b) Solve for d when $P = 75$ lb, $r = 5000$ psi, $K_a = 1.19$, and $D = 2.0$ inches.

11.20 The length L of a roll of belting is given in the formula: $L = \pi\left(\frac{D+d}{2}\right)n$ where D = outside diameter, d = inside diameter, and n = number of turns of belting.

a) Solve for L when $D = 37\frac{1}{4}$ inches, $d = 7\frac{5}{8}$ inches, and $n = 79\frac{1}{3}$ turns of belting.

b) Solve for D when $L = 2500$ inches, $d = 7\frac{5}{8}$ inches, and $n = 79\frac{1}{3}$ turns of belting.

11.21 The formula for the length L of a pulley belt is:

$$L = 2C + \frac{\pi}{2}(D + d) + \frac{(D - d)^2}{4C}$$

where C = center distance, D = diameter of larger pulley, and d = diameter of smaller pulley. For the figure shown:

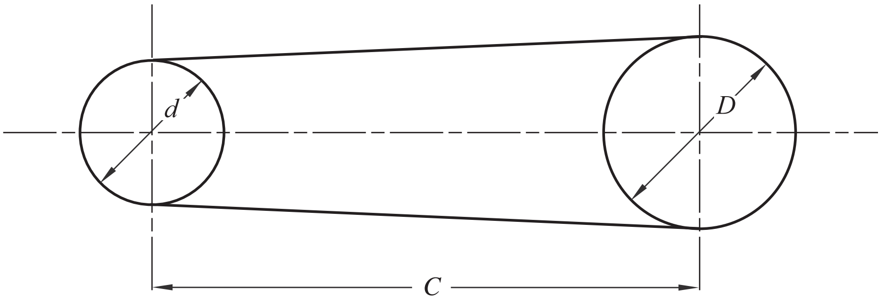


FIGURE 11.3

a) What belt length is needed for an 18-inch pulley and a 12-inch pulley spaced 85 inches center to center.

b) What belt length would be needed if the smaller pulley were changed to an 8-inch pulley?

11.22 The following formulas are used to determine the best wire size W when measuring American National Standard Unified threads with 3 wires and a micrometer: $W = 0.57735P$ or $W = \frac{0.57735}{N}$ where P = thread pitch in inches and N = threads per inch.

a) Find the best wire size when $P = 0.1250$ inch.

b) Find the best wire size when $N = 8$.

c) Find W when P is 0.2500 inch.

11.23 The horsepower required at the cutter for turning a metal workpiece on a lathe is given as: $hpc = 12\,(uhp)\,CVfd$ where: uhp = horsepower required to cut 1 cubic inch per minute (unit horsepower), C = correction factor for the feed, V = cutting speed in feet per minute, f = feed in inches per revolution, and d = depth of cut in inches.

a) Find the horsepower necessary at the cutter when the unit horsepower (uhp) for mild machine steel (AISI 1020) is 0.58, $C = 0.96$, $f = 0.015$ inch per revolution, $V = 100$ fpm, and $d = 0.150$ inch.

b) Solve for V under the same conditions with $hpc = 0.09375$.

11.24 The total horsepower required to drive a lathe is given by the formula $hpm = \frac{hpc}{e_m}$, where hpc = horsepower at the cutting tool, hpm = horsepower at the driving motor, and e_m= mechanical efficiency factor.

a) Solve for the horsepower of the driving motor of a lathe when $hpc = 1.36$ and $e_m = 0.90$.
b) Solve for the efficiency when $hmp = 1.75$ and $hpc = 1.50$.

11.25 A milling machine must be set up such that the milling cutter will operate at the proper cutting speed. The formula used is: $N = \frac{12V}{\pi D}$ where N = required spindle rpm, V = required cutting speed in feet per minute (fpm), and D = diameter of the milling cutter in inches.

a) Calculate the spindle speed in rpm for a 0.500-inch diameter cutter when $V = 100$ fpm.
b) Solve for the cutting speed when the spindle speed is 850 rpm and the cutter is 0.750 inch diameter.
c) Solve for the diameter of the workpiece in a lathe when the spindle speed is 925 rpm and the cutting speed is 100 fpm.

11.26 The tap size for an American National Standard Unified Coarse thread is found by using the formula $D = T - \frac{0.812}{N}$ where D = size of tap drill in inches, T = nominal diameter of the tap or thread, and N = number of threads per inch.

a) Find the size of the tap drill for $\frac{5}{8}$-inch. American National Standard Unified Coarse thread with 11 threads per inch.
b) Determine the number of threads per inch when $D = 0.8985$ inch and $T = 1$ inch.

11.27 A formula used to determine cutting speed corrected for the effect of vibration is: $V = V_o F_f F_d$ where V_o = cutting speed without vibration, F_f = feed correction factor, and F_d = depth of cut.

a) Solve for V when $V_o = 100$ fpm, $F_f = 0.74$, and $F_d = 0.87$.
b) Solve for the feed correction factor when $V = 75$ fpm, $V_o = 100$ fpm, and $F_d = 0.87$.

11.28 The speed N of a milling machine in rpm is calculated using the formula $N = \frac{12V}{\pi D}$ where V = cutting speed in feet per minute, and D = diameter of the milling center in inches.

a) Solve for N when V = 64.4 fpm and $D = 3$ inches.
b) Solve for V when $N = 100$ rpm and $D = 3$ inches.

11.29 The table feed rate f_m in inches per minute for a milling operation is given by the formula $f_m = f_t n_t N_t$ where f_t = inches per tooth, n_t = number of cutting teeth on the cutter, and N = spindle speed in rpm.

a) Solve for the table feed rate when $f_t = 0.010$ inch per tooth, $n_t = 10$, and $N = 350$ rpm.

b) Solve for N when the table feed rate is 25 inches per minute and $f_t = 0.010$ inch per tooth, and $n_t = 10$.

11.30 The differential pulley is the chain type pulley used in repair shops to lift heavy objects manually. The chain engages sprockets on 2 different diameter pulleys. The amount of force P needed to pull on the chain to lift a given weight W is given by the formula $P = \frac{W(R-r)}{2R}$, where R is the radius of the large sprocket and r is the radius of the small sprocket.

a) How much force is needed to lift a 950-lb object when $R = 8$ inches and $r = 7$ inches?

b) Using 100 lb of force under the same conditions, how much weight can be lifted?

11.31 The minimum safe strength required in a taper pin is given by the formula $S = \frac{1.27PR}{Dd^2}$, where S = minimum safe strength in psi, P = load in pounds, R = length of lever arm in inches, D = diameter of shaft in inches, and d = diameter of taper pin in inches at center line of shaft.

a) What is the minimum safe strength required of a taper pin to secure a lever to a 2-inch diameter shaft when $d = \frac{3}{8}$ inch, $P = 50$ lb, and $R = 30$ inches.

b) Find the shaft diameter under the same conditions when $S = 5000$ psi.

HYDRAULIC PROBLEMS

11.32 The volumetric flow rate q of a fluid through a pipe is related to the pipe's cross-sectional area A and the flow velocity v. The formula is $q = Av$.

a) How many cubic inches per second flow through a pipe with cross sectional area equal to 0.4418 square inches and flow velocity of 12 inches per second?

b) What cross sectional area is needed to accommodate 10 cubic inches per second and flow velocity of 12 inches per second?

11.33 The minimum wall thickness t for a metallic pipe or tank subjected to internal pressure is given by the formula $t = \frac{Dp}{2S}$ where D = inside diameter, p = internal gauge pressure, and S = allowable tensile stress of the wall material.

a) Calculate the minimum wall thickness for a 3-foot diameter steel tank designed to store fluid at 350 psig (pounds per square inch gauge pressure). Use $S = 55{,}000$ psi.

b) Solve for p under the same conditions when the wall thickness is 0.125 inch.

11.34 The flow velocity V in feet per second of water in a pipe is given by the formula $V = C\sqrt{\frac{hD}{L+54D}}$, where C = flow coefficient (usually obtained from a handbook table), D = pipe diameter in feet, h = total pressure "head" in feet, and L = length of pipe in feet.

a) Find the discharge velocity from a 12-inch pipe, 1 mile long working under a head of 100 feet. Use $C = 48$.

b) Solve for D under the same conditions if the discharge velocity is 3 feet per second.

11.35 The flow velocity in pipes of unequal diameters is given in the formula $v_1 = \frac{A_2 v_2}{A_1}$, where v_1 = flow velocity in first pipe, v_2 = flow velocity in second pipe, A_1 = cross-sectional area of first pipe, and A_2 = cross-sectional area of second pipe. Find the flow velocity of a liquid flowing from the second pipe when the flow velocity in the first pipe is $5\frac{\text{in}}{\text{sec}}$, $A_1 = 2\ \text{in}^2$, and $A_2 = 2\ \text{in}^2$.

11.36 Flow Q in gallons per minute (gpm) through a pipe with inside diameter d in inches and flow velocity V in feet per second is given by the formula $Q = 2.44d^2V$.

a) How many gpm of water are discharged from a 1.5-inch inside diameter pipe if the flow velocity is 10 feet per second?

b) What is the flow velocity in the same pipe if the discharge is 100 gpm?

11.37 Barlow's formula gives the wall thickness required for tubes and pipes. It is: $t = \frac{pD}{2S}$, where t = wall thickness, p = hydraulic pressure, D = outside diameter of pipe, and S = maximum allowable stress in the pipe wall. Calculate the stress in the wall of a 1.5-inch outside diameter pipe with a $\frac{1}{8}$-inch wall thickness when operating at 500 psig.

11.38 A 1-inch diameter pneumatic cylinder is to be used on a tool where it is expected to exert a 50-lb clamping force. Can this cylinder do this job if the air pressure supplied to the cylinder is 90 psi?

11.39 An oil-hydraulic cylinder is to be attached to a drill press to feed the drill through the workpiece automatically. A force of 1200 lb will have to be overcome to perform this task. The hydraulic pressure available will be 500 psi. Calculate the diameter of the cylinder required.

12

RATIO AND PROPORTION

Ratios and proportions appear in many practical applications of mathematics. They are used to express relationships between two or more quantities.

12.1 Statements of Comparison

Ratio

A *ratio* is a division relationship that expresses a statement of comparison between two quantities of the same units, such as ft/ft, lb/lb, or of different units, such as \$/hour, feet/second, or miles/gallon. The slash symbol (/) stands for *per* in discussions of ratios or proportions. It also indicates *rate*, and again, rate and ratio both involve division. Hence, *the ratio of quantity a to quantity b* can be expressed as a/b or $\frac{a}{b}$, the common fraction of a divided by b. A ratio can also be expressed with a colon (:). Thus $a : b$ is also the ratio of a to b.

A ratio is a fraction.

Because a ratio is a fraction, all rules for fractions apply to ratios as well. For example, suppose the ratio of an adult's height to a child's height is 6:4. This ratio can be reduced to 3:2, or it can be expressed as the fraction $\frac{6}{4}$, with the reduced form $\frac{3}{2}$ being the preferred form. Thus, a ratio might be an improper fraction, such as $\frac{3}{2}$, or it might be a proper fraction, such as $\frac{2}{3}$. In the latter case we would be expressing the ratio of the child's height to the adult's height.

EXAMPLE 12.1: ***Ratio as Comparison***

The statement, "One hammer weighs twice as much as another hammer," means that the hammer weights are in a 2:1 ratio.

The statement, "Spur gear A has 60 teeth and spur gear B has 180 teeth," indicates that the gear ratio of spur gear A to spur gear B is 60:180, or $\frac{60}{180}$, which reduces to $\frac{1}{3}$, or 1:3.

Simple Proportion

A *proportion* is a statement of equality between ratios. Symbolically, we write

$$a:b::c:d\,,$$

read "*a* is to *b* as *c* is to *d*." In fraction form, this is the equation

$$\frac{a}{b}=\frac{c}{d}$$

The *a* and *d* terms are called the *extremes* of the proportion, and the *b* and *c* terms are the *means*. Through algebra we see that the equality of a proportion derives from the process of *cross multiplication*, the multiplication of both sides by the denominator term of each side:

$$\frac{a}{\cancel{b}}\cancel{b}d=\frac{c}{\cancel{d}}b\cancel{d}\Rightarrow ad=bc$$

The idea that the product of the extremes equals the product of the means is given algebraically by the rule of proportions:

Rule of Proportions

$$\frac{a}{b}=\frac{c}{d}\Rightarrow ad=bc$$

Example 12.2: ***Rule of Proportions***

The proportion "3 is to 5 as 6 is to 10" is expressed as 3 : 5 :: 6 : 10, or $\frac{3}{5}=\frac{6}{10}$.

This checks, since $3 \cdot 10 = 6 \cdot 5$.

A proportion must translate into a true equality. For example, we would not write 5 : 8 :: 4 : 7, since $\frac{5}{8}\neq\frac{4}{7}$.

Whenever three terms in a proportion are known, the remaining term can be found by solving the equation algebraically. In the following example, the same proportion is shown four times, each time with one of the four parts given as an unknown.

Example 12.3: ***Finding the Unknown in a Proportion***

Solve for the unknown in each equation representing a single proportion.

$$\frac{a}{12}=\frac{3}{8}\qquad\frac{4.5}{b}=\frac{3}{8}\qquad\frac{4.5}{12}=\frac{c}{8}\qquad\frac{4.5}{12}=\frac{3}{d}$$

Solution:

In the first and third proportions, there is no need to cross multiply, since the unknown is in the numerator. Only one step is needed to isolate the unknown:

$$\frac{a}{12}=\frac{3}{8}\quad\Rightarrow\quad a=\frac{(3)(12)}{8}=\frac{36}{8}\quad\Rightarrow\quad a=4.5$$

EXAMPLE 12.3: *Finding the Unknown in a Proportion (Continued)*

$$\frac{4.5}{12}=\frac{c}{8} \quad \Rightarrow \quad c=\frac{(4.5)(8)}{12}=\frac{36}{12} \quad \Rightarrow \quad c=3$$

We make use of cross multiplication in the other two equations:

$$\frac{4.5}{b}=\frac{3}{8} \quad \Rightarrow \quad 3b=(4.5)(8) \quad \Rightarrow \quad b=\frac{36}{3} \quad \Rightarrow \quad b=12$$

$$\frac{4.5}{12}=\frac{3}{d} \quad \Rightarrow \quad 4.5d=(3)(12) \quad \Rightarrow \quad d=\frac{36}{4.5} \quad \Rightarrow \quad d=8$$

EXAMPLE 12.4: *Application of Proportion*

A car manufacturer builds a scale model of a concept car to investigate the vehicle's aerodynamic properties and optimize final body design. This model reduces all the true dimensions of the actual vehicle by a factor of 4; that is, each measurement of the full-sized vehicle is cut down (divided) by 4. Thus, 4 m on the real car scales down to 1 m on the model. What is the ratio of the model car length to the real car length? What is the length of a model for a car that is 6 m in length?

Solution: The ratio of the real car length to the model car length is 4:1. The proportion to find the length of the model for a car having a length of 6 m is

$$\frac{4\,\text{m}}{1\,\text{m}}=\frac{6\,\text{m}}{x}$$

Units are identical throughout, so we may write more simply $\frac{4}{1}=\frac{6}{x}$.

Solving, $$4x = 6$$

$$x=\frac{6}{4}=1.5\text{ m}$$

The units in a proportion must correspond on the left and right sides of the equation, as illustrated in Example 12.5.

EXAMPLE 12.5: *Finding the Unknown in a Proportion*

An object travels 45 feet in 10 seconds. How far does it travel in 1 second, assuming it is moving at a constant rate?

Solution:

Letting *d* be the unknown distance, the proportion is

$$\frac{45\text{ ft}}{10\text{ sec}}=\frac{d}{1\text{ sec}}$$

Then, $$d=\frac{(45\,\text{ft})(1\,\text{sec})}{(10\,\text{sec})}=4.5\,\text{ft}$$

Simple Gear Ratios

Gear speed is usually measured in revolutions per minute (rpm). Speed ratios and relative number of turns between meshing gears are related to the number of teeth on the circumference of the contacting gears. Since the teeth on one gear must mesh exactly with the teeth on the second gear, the speed and turn ratios of the shafts attached to the gears are in proportion to the number of teeth on each of the gears. This is illustrated in Example 12.6.

EXAMPLE 12.6: ***Simple Gear Ratios***

Gear A has 36 teeth and gear B has 22 teeth as shown in Figure 12.1. How many revolutions will gear B make if gear A makes 5 turns?

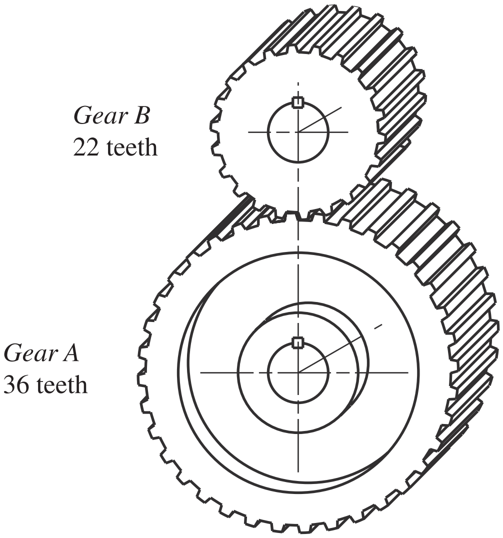


FIGURE 12.1

Solution:

The proportion of gears A and B can be expressed as:

$$\left(\begin{matrix}\text{\# of teeth}\\ \text{on gear A}\end{matrix}\right)\times\left(\begin{matrix}\text{\# of turns}\\ \text{of gear A}\end{matrix}\right)=\left(\begin{matrix}\text{\# of teeth}\\ \text{on gear B}\end{matrix}\right)\times\left(\begin{matrix}\text{\# of turns}\\ \text{of gear B}\end{matrix}\right)$$

Accordingly, by rearranging the equation, the number of turns of gear *B* becomes:

$$\text{\# of turns of gear B}=\frac{(\text{\# of teeth on gear A})(\text{\# of turns of gear A})}{(\text{\# of teeth on gear B})}$$

$$\text{\# of turns of gear B}=\frac{(36)(5)}{(22)}$$

$$=8.18\text{ turns, or }8.18\text{ revolutions}$$

Compound Ratio and Proportion

A ratio involving a product of two or more simple ratios is called a *compound ratio*. Situations involving compound ratios arise when more than one variable in a problem affects the proportion. In general, solving a compound proportion requires a series of ratio multiplications.

EXAMPLE 12.7A: ***Compound Ratio***

Find the cost of a single sheet of paper that is sold for \$50 in boxes containing 10 reams. A ream of paper contains 500 sheets.

Solution:

This problem involves three variables: cost, number of reams, and number of sheets in a ream. Thus, there are three ratios: cost per box, reams per box, and sheets per ream. We seek *cost per sheet*; we are given cost per 10 reams. One approach to solving this problem is to apply the factor-label method learned in Chapter 7.

$$\#\frac{\$}{\text{sheet}} = \left(\frac{\$50}{\cancel{\text{box}}}\right)\left(\frac{1\ \cancel{\text{box}}}{10\ \cancel{\text{reams}}}\right)\left(\frac{1\ \cancel{\text{ream}}}{500\ \text{sheets}}\right) = \frac{\$0.01}{\text{sheet}}$$

The cost is 1 penny per sheet.

The product of two or more ratios is a compound ratio.

Using the *factor-label* method, the ratios (factors) are arranged so that, when multiplied, the units in the denominator of one ratio cancel the units in the numerator in the following ratio, resulting in the desired units.

EXAMPLE 12.7B: ***Compound Ratio***

A shipment of 8 steel bars that are 6 feet long by 4 inches wide by 1 inch thick weighs 652.8 lb. How many pounds will a shipment consisting of 7 steel bars that are 8 ft long by 3 inches wide by 2 inches thick weigh?

Solution:

Let X represent the unknown quantity (the weight of the second shipment) and write the ratio of the unknown quatity to the known weight of the first shipment:

$$\frac{X\ \text{lb}}{652.8\ \text{lb}}$$

Write the ratio of the number of bars in second shipment to number of bars in the first shipment:

$$\frac{7\ \text{pieces}}{8\ \text{pieces}}$$

Write the ratio of the length of the bars in the second shipment to length of bars in first shipment:

EXAMPLE 12.7B: *Compound Ratio (Continued)*

$$\frac{8\text{ feet}}{6\text{ feet}}$$

Write the ratio of the width of the bars in the second shipment to width of bars in first shipment:

$$\frac{3\text{ inches}}{4\text{ inches}}$$

Write the ratio of the thickness of the bars in the second shipment to thickness of bars in the first shipment:

$$\frac{2\text{ inches}}{1\text{ inch}}$$

Write the proportion as the ratio of the weights equated to the product of the ratios describing the bars. Note that the terms of the second shipment are the *numerators* of each ratio while the terms of the first shipment are the *denominators*.

$$\frac{X}{652.8\text{ lb}} = \frac{7\text{ pieces}}{8\text{ pieces}} \cdot \frac{8\text{ feet}}{6\text{ feet}} \cdot \frac{3\text{ inches}}{4\text{ inches}} \cdot \frac{2\text{ inches}}{1\text{ inch}}$$

Write as a single ratio equated to a ratio:

$$\frac{X}{652.8\text{ lb}} = \frac{7\cdot 8\cdot 3\cdot 2}{8\cdot 6\cdot 4\cdot 1}$$

Cancel terms:

$$\frac{X}{652.8\text{ lb}} = \frac{7\cdot \cancel{8}\cdot \cancel{3}\cdot \cancel{2}}{\cancel{8}\cdot \underset{2}{\cancel{6}}\cdot \underset{2}{\cancel{4}}\cdot 1}$$

Multiply factors:

$$\frac{X}{652.8\text{ lb}} = \frac{7}{4}$$

Cross-multiply:

$$4\cdot X = 7\cdot 652.8\text{ lb}$$

Solve for X:

$$4X = 4569.6\text{ lb}$$

$$\frac{4X}{4} = \frac{4569.6\text{ lb}}{4}$$

$$X = 1142.4\text{ lb}$$

Mean Proportional

The *mean proportional* occurs in a proportion in which the b and c terms in $\frac{a}{b} = \frac{c}{d}$ are the same. For example, in the proportion $\frac{8}{4} = \frac{4}{2}$, the number 4 is the mean proportional between 8 and 2.

In general, the mean proportional of two numbers is found by multiplying the numbers, then taking the square root of the product. For example the mean proportional between 16 and 4 is simply $\sqrt{(16)(4)} = \sqrt{64} = 8$. So, $\frac{16}{8} = \frac{8}{4}$.

EXAMPLE 12.8A: ***Mean Proportional***

How do we read the equation $\frac{27}{x} = \frac{x}{3}$? What is the x term called? Solve for x.

Solution:

This equation is read “27 is to x as x is to 3.” The x term is the *mean proportional* because the denominator of the first fraction is equal to the numerator of the second fraction. Solving for x:

$$(27)(3) = x^2$$

$$81 = x^2$$

$$x = \sqrt{81} = 9$$

Often the positive principal root is the only practical answer, as a negative quantity may not have a physical meaning.

EXAMPLE 12.8B: ***Mean Proportional***

What is the mean proportional of 12 and 3?

Solution:

$$\frac{12}{x} = \frac{x}{3}$$

$$36 = x^2$$

$$x = \sqrt{36} = 6$$

EXAMPLE 12.8C: ***Mean Proportional***

Given the mean proportional between the extremes 16 and x is 12, find the value of x.

Solution:

$$\frac{16}{12} = \frac{12}{x}$$

$$16x = 12^2 = 144$$

$$x = \frac{144}{16} = 9$$

EXERCISES

12.1 Write the ratio of 4 to 2 and reduce.

12.2 Write the ratio of 3 to 12 and reduce.

12.3 Write the ratio of 8 to $\frac{1}{2}$ and reduce.

12.4 Write the ratio of $\frac{1}{2}$ to $\frac{1}{4}$ and reduce.

12.5 Write the ratio of 27 to 3 and reduce.

12.6 Write the ratio of 4 to 16 and reduce.

12.7 Write the ratio of 16 to 48 and reduce.

12.8 Write the ratio of 1 to 9 and reduce.

12.9 Write the ratio of 15 to 3 and reduce.

12.10 Write the ratio of 12 to $\frac{1}{2}$ and reduce.

12.11 Write the ratio of $\frac{3}{4}$ to $\frac{1}{2}$ and reduce.

12.12 What is the ratio of 3 feet to 6 inches?

12.13 What is the ratio of 5 quarts to 4 pints?

12.14 What is the ratio of 1 foot to 1 mile?

12.15 Write the ratio of 2 miles to 25 kilometers.

12.16 Write the ratio of 8 ounces to 1 kilogram.

12.17 One foot has 12 inches. How many inches are in 5 feet?

12.18 Eight taps cost \$12. How much will 20 taps cost?

12.19 An electric motor running at 1750 rpm has a 5-inch diameter pulley. What size pulley is required on the driven shaft to turn it at 875 rpm?

12.20 What is the ratio of 50 to 10?

12.21 Solve for x in the proportion: $\frac{4}{32} = \frac{x}{16}$.

12.22 Solve for z in the proportion: $\frac{3}{\frac{1}{2}} = \frac{18}{z}$.

12.23 Two numbers are in the ratio of 3 to 4. The lesser number is 60. What is the greater number?

12.24 Five taper reams cost \$12. How much would 1 dozen taper reams cost?

12.25 A machinist turns 56 pieces in 8 hours. How much time is needed to turn 14 pieces?

12.26 The gas tank in Lisa's car holds 12 gallons. Lisa filled the tank. After driving 200 miles, Lisa refilled the tank by adding 7.5 gallons of gas. How many miles per gallon does Lisa's car get?

12.27 When gear *A* in Figure 12.1 makes $8\frac{1}{2}$ revolutions, how many revolutions will gear *B* make?

12.28 The ratio of teeth in gears *A* and *B* is $\frac{3}{5}$. How many teeth does gear *B* have if gear *A* has 15 teeth?

12.29 A spur gear has 73 teeth and its pinion has 52 teeth. When the pinion has made 5.3 turns, how many turns has the spur gear made?

12.30 A motor running at 675 rpm has a pinion gear with 12 teeth. The number of teeth in the driven gear is the same as its rpm. How many teeth does the driven gear have?

12.31 Two spur gears are in mesh. One gear has 25 teeth and rotates at 81 rpm. How many teeth does the other gear have if its number of teeth and rpm are the same?

12.32 A length of railroad rail 30 feet long weighs 90 pounds per yard. How much will a length of rail 9 feet 8 inches weigh?

12.33 An automobile has a rear axle with a 3.52 ratio. In high gear (direct drive) how many rpm do the rear wheels achieve when the motor is running at 3800 rpm?

12.34 A pump that discharges 4 gallons per minute can fill a tank in 24 hours. How much time would be required to fill the tank if the pump could discharge 12 gallons per minute?

12.35 An electric motor running at 800 rpm drives through a speed reducer whose output shaft turns at 125 rpm. What is the reduction ratio of the speed reducer?

12.36 A pattern for a casting weighs 5.25 pounds and is made from a material having a density of 31.5 $\frac{\text{lb}}{\text{ft}^3}$. How much would a casting made from this pattern weigh if the casting metal density is 425 $\frac{\text{lb}}{\text{ft}^3}$?

12.37 Railroad grades (slopes) are measured as the number of feet of rise per 100 feet of horizontal run. A rise of 2 feet over a 100-foot run is designated as a 2% grade. What is the rise of a 1% grade over a distance of 1 mile (5280 feet)?

12.38 What is the mean proportional between 16 and 4?

12.39 What is the mean proportional between 2 and 18?

12.40 What is the mean proportional between 20 and 5?

12.41 What is the mean proportional between $\frac{3}{4}$ and $\frac{4}{27}$.

12.42 The mean proportional between 25 and x is 125. What is the value of x?

12.43 An *equally pitched roof* has both sides at the same angle with the horizontal. Two buildings have equally pitched roofs as shown in Figure 12.2. The span of the first roof is 160 ft and the vertical height of the second roof is 10 ft. What is the span of the second roof if its span equals the vertical height of the first roof?

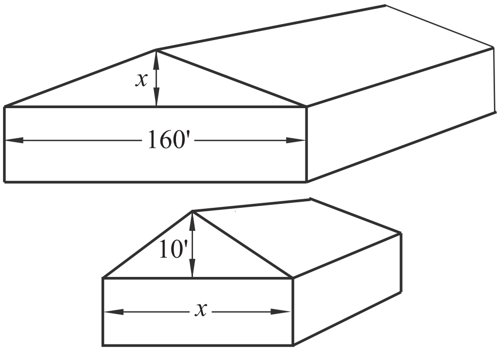


FIGURE 12.2

12.44 Two automobile engines have equal ratios of horsepower to cubic inch displacement. One has a displacement of 288 cubic inches while the second has a rated horsepower of 200. If the horsepower of the first engine equals the displacement of the second engine, what is the displacement of the second engine?

12.45 Fifteen workers assemble 72 motors in 6 days, working 8 hours per day. How many motors could 20 workers assemble working 10 hours per day for 5 days?

12.46 Three fork truck drivers can unload 5 boxcars in 10 hours. How many boxcars could be unloaded using 4 fork truck drivers working 6 hours?

12.47 Four workers can carpet a 4000-square-foot house in 8 hours. How many hours would 6 workers need to carpet a 5000-square foot house?

12.48 Ten lengths of 6-inch channel iron 8 ft long with a linear density of 13 $\frac{\text{lb}}{\text{ft}}$ weigh 1040 lb. How many pounds will 16 lengths of 6-inch channel iron 10 ft long with a linear density of 10.5 $\frac{\text{lb}}{\text{ft}}$ weigh?

12.49 Three pumps running at 250 gpm can fill a reservoir in 6 hours. How many hours are required for two pumps running at 300 gpm to fill the reservoir at 75% capacity?

12.50 An automobile final assembly line staffed by 232 workers produces 464 cars during an 8-hour shift. How many cars can be produced in 10 hours if the assembly line is staffed by only 200 workers?

12.51 A painting contractor bid to paint the exterior of a house for \$2800 using 6 painters each earning \$11.10 per hour and working 8 hours for each of 3 days. What would the contractor's quote be if instead she used 9 painters each earning \$10.50 per hour and working 8 hours for each of 2 days?

12.52 A cement block wall for a small factory can be laid up in 5 days with 7 masons working 8 hours each day. How many days would be required to complete 60% of the wall using 5 masons working 10 hours each day?

12.2 Mixture Proportions

A common application of ratio and proportion involves determining quantities of different components that make up a mixture. In technical applications, "percent" and "part" refer to *relative proportions* of quantities being considered (for example, metals in an alloy or elements in a compound). Water, for instance, is commonly referred to as having two parts hydrogen and one part oxygen. This means that every oxygen atom in a water molecule is joined with two hydrogen atoms (H_2O). Manufacturing engineers often refer to scrap rate in terms of percent. A 10 percent scrap rate, for example, indicates that for every 100 pounds of raw material put into a manufacturing process, 10 pounds become scrap and 90 pounds become useful product.

In scientific and other technical applications, quantities of components are frequently given in terms of relative mass (weight) or relative volume.

As another example, consider tinsmith solder, which contains 59 percent tin and 41 percent lead by weight. Therefore, 100 pounds of tinsmith solder has 59 pounds of tin and 41 pounds of lead. Two hundred pounds of tinsmith solder would have $(200)(0.59) = 118$ pounds of tin and $(200)(0.41) = 82$ pounds of lead. We could also say that tinsmith solder is 59 parts tin and 41 parts lead by weight. The base of 100 parts is implied by this phrasing.

When using percents, remember that the sum of all individual percents must add up to 100 percent – one whole.

Similarly, a volume of one liter (1000 ml) of a 20-percent sodium hypochlorite solution used in wastewater treatment contains 200 ml of sodium hypochlorite and 800 ml of water. Likewise, 100 gallons of a solution of sodium hyperchlorite contains 20 gallons of pure sodium hyperchlorite and 80 gallons of pure water.

EXAMPLE 12.9: *Volume Mixture*

A certain shade of green paint is a mixture of 2 parts blue and 3 parts yellow paint. How many gallons of blue and how many gallons of yellow are needed to make 12 gallons of green paint?

Solution: First note that the smallest quantity (in whole gallons) of green paint we could produce using this mixture is 5 gallons (2 gal blue + 3 gal yellow = 5 gal green). Arbitrarily choose the amount of blue paint in the larger mixture to be the unknown. That is, let x = gal of blue paint. Set up the proportion as

EXAMPLE 12.9: ***Volume Mixture (Continued)***

$$\frac{2 \text{ gal blue}}{5 \text{ gal green}} = \frac{x \text{ gal blue}}{12 \text{ gal green}}$$

Solving for x:

$$x = \frac{24}{5} = 4.8 \text{ gal blue paint}$$

The amount of yellow paint required is found by subtracting 4.8 gal blue from the total:

$$12 - 4.8 = 7.2 \text{ gal yellow paint}$$

Therefore, we need 4.8 gal blue and 7.2 gal yellow to mix 12 gallons of the green paint.

Notice that we could have begun by determining how much yellow paint is needed. To solve the problem this way, let x = gal of yellow paint and set up the following proportion:

$$\frac{3 \text{ gal yellow}}{5 \text{ gal green}} = \frac{x \text{ yellow}}{12 \text{ gal green}} \quad \Rightarrow \quad x = \frac{(3)(12)}{5} \text{ gal yellow paint}$$

$$\Rightarrow \quad x = 7.2 \text{ gal yellow paint,}$$

which checks with 12 – 7.2 = 4.8 gallons of blue paint found in the first part of this example.

Use a base amount of 100 to take advantage of percent terminology. Here, percent can mean "per 100 lb."

EXAMPLE 12.10: ***Weight Mixture***

Suppose a project requires 400 pounds of tinsmith solder. How much tin and how much lead are required?

Solution:

As previously stated, tinsmith solder is 59% tin and 41% lead by weight. Let x represent the unknown amount of tin, set up the proportion, and solve:

$$\frac{x}{400} = \frac{59}{100}$$

$$100x = (59)(400) = 23,600$$

$$x = \frac{23,600}{100} = 236 \text{ pounds of tin}$$

Therefore, 400 – 236 = 164 pounds of lead are needed.

EXERCISES

12.53 The local hardware store sells custom-colored paint. A certain orange color needed to match the color on a piece of machinery consists of 1.5 parts red paint and 2 parts yellow paint. How many gallons of red and how many gallons of yellow paint are needed to make 28 gallons of orange paint?

12.54 Consider an alloy composed of 5 parts copper, 3 parts lead, and 1 part tin. How many pounds of each metal will be needed to cast a 240-pound part?

12.55 A solder is composed of 42% lead and 58% tin. How many pounds of solder can be made from 40 pounds of tin and how many pounds of lead are required?

12.56 A tool steel has 1 part molybdenum, 12 parts chrome, 1 part carbon, and the balance iron to make 100 parts. How much of each element is required to produce 95 pounds of the tool steel?

12.57 A 50-ounce sample of a jeweler's alloy contains 18 ounces of pure gold. How much of a base metal must be added to make an alloy that will be 28% pure gold?

12.58 A bell is made from a metal containing 25 parts copper to 12 parts tin. How many pounds of copper and how many pounds of tin are needed to produce a bell weighing 935 pounds?

12.59 A casting has 4 parts copper, 3 parts lead, and 2 parts tin. How many pounds of each metal are needed to produce a casting weighing 184 pounds?

12.60 How many pounds of lead must be added to change 840 pounds of solder consisting of 44% tin and 56% lead to a mixture that is 35% tin and 65% lead?

12.61 A metal used to make bearings (Babbitt) has 84% tin, 10% antimony, 5% copper, and 1% lead. How many pounds of each metal are needed to make a bearing weighing 25.5 pounds?

12.3 Tapers and Other Tooling Calculations Requiring Proportions

In technical work, objects that have a gradual decrease in width or diameter are referred to as "*having a taper*" or, simply, as *tapers*. Tapers are frequently encountered with machine tools. Twist drills, end mills, arbors, lathe centers, and other small tools and machine parts have tapered shanks, which fit into spindles, sockets, or adaptors, providing an accurate alignment of the centers of the driving spindle and the tool. The tool's taper also provides a convenient friction method for holding the tool in the spindle. The taper of a tool must be identical to the taper of its holder to provide the best friction fit.

Tapers are classified, in general, as *self-holding* and *self-releasing*. The self-holding type are the smaller tapers (2° to 3°), in which the shank of the tool becomes so firmly seated in its socket that there is considerable frictional resistance to prevent the tool from rotating relative to the socket. Tapers of the self-releasing type are larger than the tapers of the self-holding type. A milling machine spindle having a taper of 3.5 inches per foot is an example of a self-releasing taper. Self-releasing tapers require a positive locking device to prevent slipping, but they are easier to remove than the self-holding variety. Tapers are also classified as *conical* and *flat*. Figures 12.3A and 12.3B illustrate the difference between conical and flat tapers. Conical tapers have a circular cross-section and are found on taper shanks of cutting tools, machine tool spindle noses, taper pins, tapered bearings, and so on. The conical taper provides an excellent method of alignment for external and internal surfaces. Flat tapers have a rectangular cross-section, as seen in the shape of a "drift," which is used to remove tapered tools from spindles. Flat tapers are actually inclined planes, which are used as adjusting wedges and keys. Tapered keys are used to hold two components together, while drift keys or drift pins are used to extract drills from sockets.

While many standards for self-holding tapers exist, the most commonly used ones in the United States are the American National, Morse, and Brown & Sharpe tapers. The exact dimensions and specifications for standard tapers are tabulated in engineering handbooks such as Machinery's Handbook.

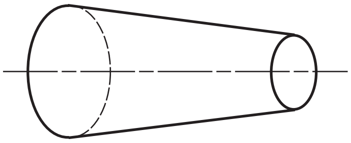


FIGURE 12.3A
Conical taper

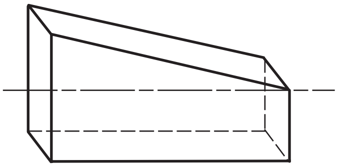


FIGURE 12.3B
Flat taper

The determination of "amount of taper" is a problem in proportion. It involves the large and small dimensions at the ends of the tapers and the overall length of the piece. In general, a proportion can be written for taper calculations as follows:

$$\frac{\text{large end dimension} - \text{small end dimension}}{\text{overall length}} = \frac{\text{taper}}{\text{unit of length}}$$

Any dimensional units of length, such as inch, foot, centimeter, and meter, can be used in taper calculations as long as the units are reported consistently. For example, T_{pi} means *taper per inch*. A ratio of $\frac{\text{1 inch of taper}}{\text{5 inches of length}}$ has $T_{pi} = 0.2$.

Since 1 foot has 12 inches, we observe that the relationship between taper per foot (T_{pf}) and taper per inch (T_{pi}) is

$$T_{pf} = 12T_{pi} \text{ or } T_{pi} = \frac{T_{pf}}{12}$$

$$T_{pf} = \frac{12\ \cancel{\text{in}}.}{\text{ft}} \cdot \frac{\text{taper}}{\cancel{\text{in}}} \Rightarrow T_{pf} = 12T_{pi}$$

Similarly, we can refer to taper per meter, taper per millimeter, and so on.

Figures 12.4 and Figures 12.5 show geometric interpretations of tapers for both conical and flat tapers.

Conical Taper

Segments *AH* and *DG* in Figure 12.4 can be easily measured, giving total conical taper $\frac{AH - DG}{L}$. Another way to express the taper is $\frac{AB + FH}{L}$, since $AB + FH = AH - DG$.

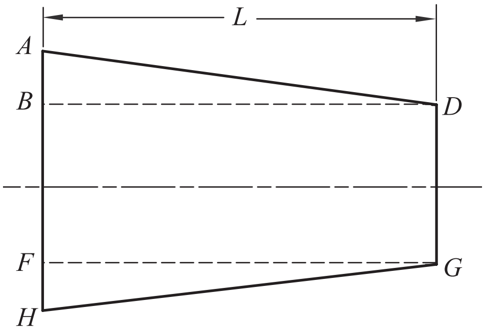
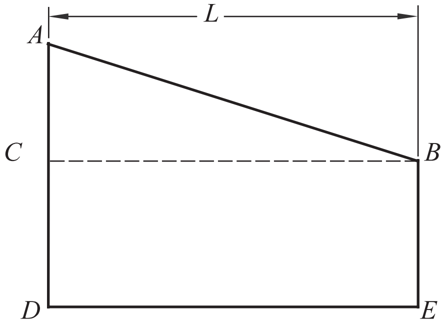


FIGURE 12.4 ***Geometry of a conical taper***

Flat Taper

The total flat taper is the length of the segment *AC* over *L* in Figure 12.5. To calculate taper, notice that $AC = AD - BE$. The segments *AD* and *BE* can be easily measured. Written algebraically, the taper is $\frac{AD - BE}{L}$.



Be careful to notice that the flat taper has only one sloping side while the conical taper is sloped on both the top and the bottom.

FIGURE 12.5 ***Geometry of a flat taper***

EXAMPLE 12.11A: ***Conical Taper Calculation***

A conical taper is 4 inches long. The large diameter is 1 inch and the small diameter is $\frac{3}{4}$ inch. Calculate the taper per inch.

Solution:

$$T_{pi} = \frac{\text{large end dimension} - \text{small end dimension}}{\text{overall length}} = \frac{1 - \frac{3}{4}}{4} = \frac{0.25}{4} = 0.0625$$

EXAMPLE 12.11B: *Conical Taper Calculation*

A taper plug is measured on an optical comparator. The large end has a diameter of 0.9831 inch, the small end has a diameter of 0.8229 inch, and the length is 6.0000 inches. Calculate the taper per inch.

Solution :

$$T_{pi} = \frac{\text{large end dimension} - \text{small end dimension}}{\text{overall length}} = \frac{0.9831 - 0.8229}{6.0000} = 0.0267$$

Technically, the dimensions in the solutions to Examples 12.11A and 12.11B are $\frac{\text{inch}}{\text{inch}}$. When numerator and denominator units are the same, the taper is usually reported simply as a number. Otherwise, specific units, such as $\frac{\text{inch}}{\text{foot}}$, are indicated.

EXAMPLE 12.12: *Conversion from T_{pi} to T_{pf}*

Convert the answer from Example 12.11B from taper per inch to taper per foot.

Solution:

$$T_{pf} = 12T_{pi} = (12)(0.0267) = 0.3204\frac{\text{inch}}{\text{foot}}$$

EXAMPLE 12.13: *Solving for a Dimension in a Taper*

Determine the small diameter *d* of the conical taper shown in Figure 12.6 when

$$D = 1.2206 \text{ inches}, L = 8.30 \text{ inches, and } T_{pf} = 1.1000\frac{\text{inches}}{\text{foot}}$$

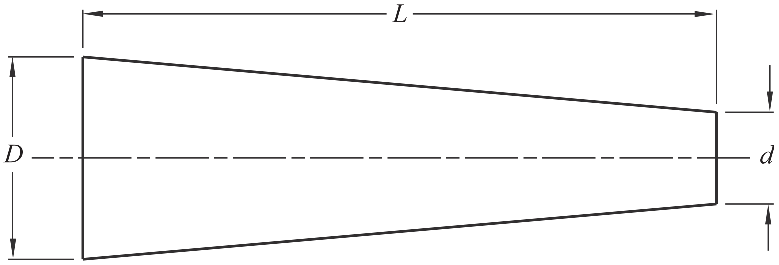


FIGURE 12.6

Solution: First, convert taper per foot to taper per inch as follows:

$$T_{pi} = \frac{T_{pf}}{12} = \frac{1.1000}{12} = 0.0917$$

Then use the proportion, $T_{pi} = \frac{\text{taper}}{\text{inch}} = \frac{D-d}{L}$, substituting known values:

EXAMPLE 12.13: ***Solving for a Dimension in a Taper (Continued)***

$$0.0917 = \frac{1.2206 - d}{8.3000}$$
$$(0.0917)(8.3000) = 1.2206 - d$$
$$0.7611 = 1.2206 - d$$
$$d = 1.2206 - 0.7611 = 0.4595 \text{ inch}$$

Notice that since the information was provided to four decimal place accuracy, all intermediate calculations, and thus the final answer, were rounded to four decimal places.

A common method of turning tapers is called the *offset-center method*. The workpiece is placed between the headstock and the tailstock centers. When both of the centers are in line, the movement of the cutting tool is parallel to the axis of the work. In this position a cylindrical piece would be cut. If, however, the tailstock center is moved out of axial alignment the work will then be tapered as the tool cuts from end to end because the axis of the workpiece is at an angle with the movement of the tool. The total amount of taper will depend on how much the tailstock center is offset. The formula for taper turning by the offset-center method is:

$$\text{offset} = \frac{(\text{taper per inch})(\text{length of workpiece})}{2}$$

EXERCISES

12.62 A taper is 4 inches long. The large diameter is 1 inch and the small diameter is $\frac{3}{4}$ inch. Calculate the taper per inch.

12.63 A 6-inch long taper plug is measured on an optical comparator. The large diameter is 0.9831 inch and the small diameter is 0.8229 inch. Calculate the taper per inch.

12.64 Convert the answer to problem 12.63 to taper per foot.

12.65 Calculate the diameter of the large end of the taper shown Figure 12.6 when the small diameter is 0.77800 inch, the length is 3.25000 inches, and the taper per foot is 3.25000 inches.

12.66 Determine the taper per foot for the taper shown in Figure 12.7.

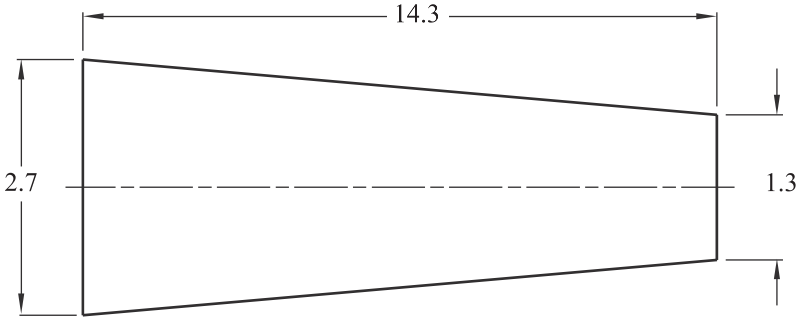


FIGURE 12.7 ***Taper with dimensions in feet***

12.67 Determine the taper per inch for the taper shown in Figure 12.8.

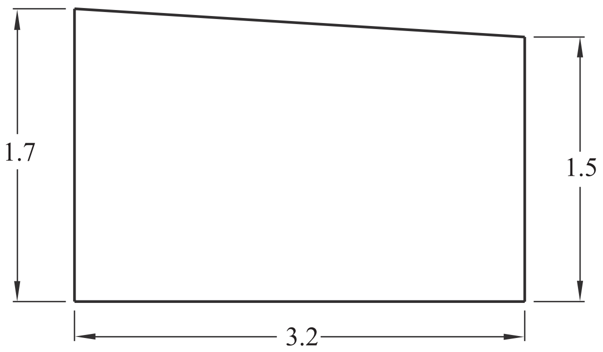


FIGURE 12.8 ***Taper with dimensions in feet***

12.68 The taper shown in Figure 12.9 has a 0.4000 inch taper per foot. Solve for the length.

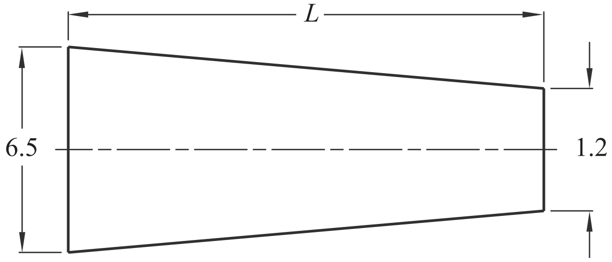


FIGURE 12.9 ***Taper with dimensions in inches***

12.69 In the taper gauge shown in Figure 12.10, solve for the length, L, when $D = 1.2556$ inches, $d = 1.0553$ inches, and the taper per foot is 1.2050 inches.

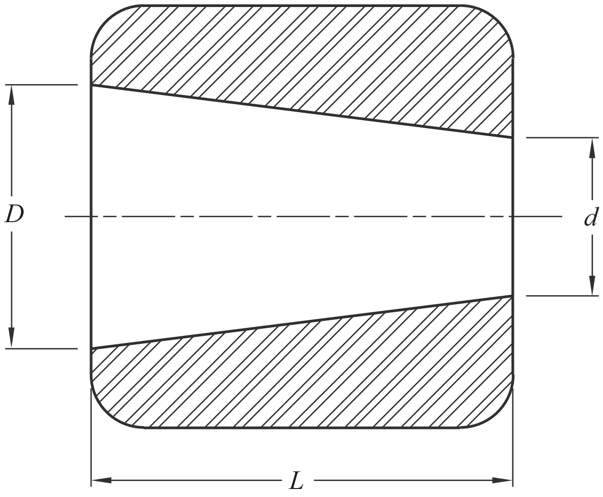


FIGURE 12.10

12.70 Solve for the small diameter d in the taper gauge shown in Figure 12.10 when $D = 1.3025$ inches, $L = 1.2558$ inches, and the taper per foot is 1.9250 inches.

12.71 Solve for the taper per inch in the taper plug and ring gauge set shown in Figure 12.11 when dimension $A = 1.125$ inches, $B = 2.000$ inches, $L = 5.000$ inches, $D = 1.250$ inches, and $d = 0.750$ inch.

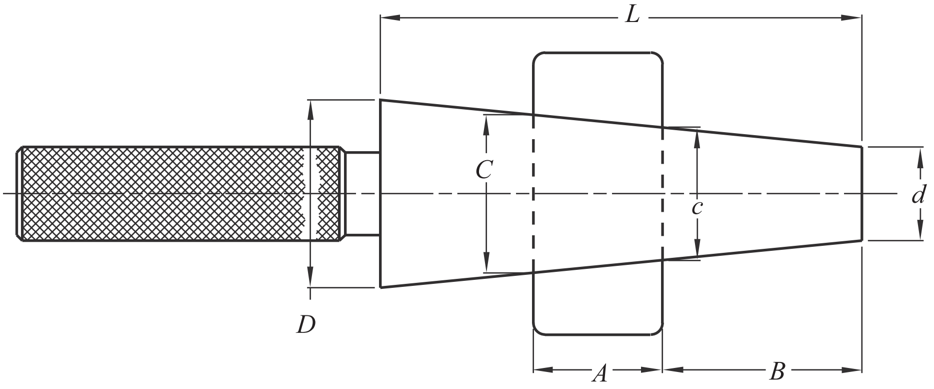


FIGURE 12.11

12.72 Solve for C shown in Figure 12.11 when dimension $A = 1.125$ inches, $B = 2.000$ inches, $L = 5.000$ inches, $D = 1.250$ inches, and $d = 0.750$ inch.

12.73 In the taper plug and ring set shown in Figure 12.12 the taper per foot is 0.3125 inch. If d were machined 0.0210 inch larger, how much farther would the taper plug advance into the ring gauge, the taper per foot remaining constant?

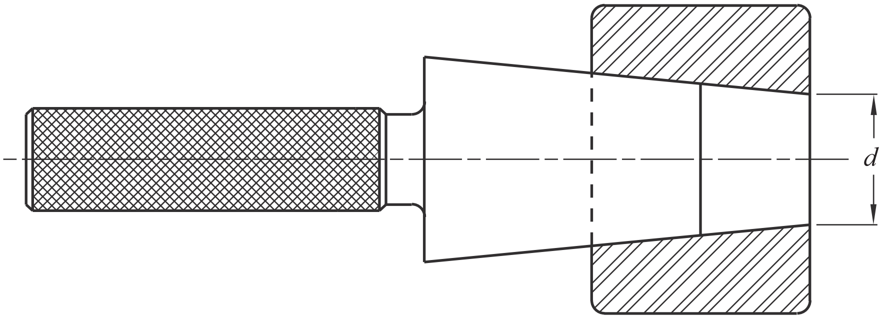


FIGURE 12.12

12.74 In Figure 12.12 the taper per foot is 0.875 inch. How much must dimension d be increased to permit the taper plug to advance 0.011 inch, the taper per foot remaining constant?

12.75 The shaft of a golf club is 38 inches long, $\frac{5}{8}$-inch diameter at the large end, and $\frac{3}{8}$-inch diameter at the small end. What is the taper per inch?

12.76 A 200-foot tall smokestack at a power plant is 19-foot diameter at the base and 8-foot diameter at the top. What is the taper per foot?

12.77 A taper may be turned in a lathe by shifting the tailstock a distance off-center equal to half the taper in inches. A shaft 3 foot long is to have a taper from a diameter of 4 inches on one end to a diameter of 3.500 inches on the other end. Find the amount of offset in inches.

12.78 A taper plug that is 7.500 inches long is to be turned in a lathe by the offset-center method. The taper is 4 inches long and the taper per foot is 0.750 inch. Determine the offset of the tailstock.

12.79 Calculate the tailstock offset to cut a 0.625-inch-per-foot taper on a workpiece that is 6.250 inches long.

12.80 What size taper would be cut on a 13-inch shaft if the tailstock were offset by 0.130 inch. Give the answer as taper per foot.

12.81 A shaft 3-foot-long is to have a 5-inch-long taper on the end cut with a 0.625-inch-per-foot taper. Calculate the offset necessary at the tailstock.

12.82 A 7-inch-long taper on the end of a 16-inch shaft is cut when the tailstock is offset 0.256 inch. What is the taper per foot?

12.83 How much offset is needed at the tailstock of a lathe to give a taper of $0.95\ \frac{\text{inch}}{\text{foot}}$ on a piece 27 inches long?

12.84 How much offset is needed at the tailstock of a lathe to give a taper of $0.95\ \frac{\text{inch}}{\text{foot}}$ on a piece 17 inches long?

12.85 A taper pin reamer has a taper of $0.275\ \frac{\text{inch}}{\text{foot}}$. The diameter of the small end is 0.187 inch and the length of the flutes is 4.500 inches. Find the diameter at the large end of the flutes.

12.86 How much is the offset of a tailstock to turn a taper on a piece 14 inches long if the tapered portion is to be 6.500 inches long, the large diameter of the shaft is 1.375 inches, and the small diameter is 0.915 inch?

12.87 The standard pipe thread taper is $\frac{3}{4}\ \frac{\text{inch}}{\text{foot}}$. How much is this taper expressed as $\frac{\text{inch}}{\text{inch}}$?

12.88 A taper wedge used by millwrights in leveling machines measures 3 inches wide, 14 inches long, and 1 inch thick at the butt and tapers to 0 inch at the edge. How far must the wedge be inserted under a drill press to raise it 0.325 inch?

12.89 In Figure 12.13, determine the distance *A*.

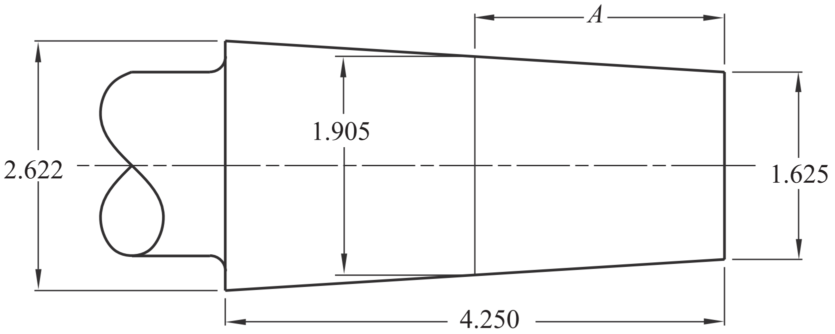


FIGURE 12.13 ***Taper with dimensions in inches***

12.4 Variation

Algebraic relationships involving proportion are of two types: *direct variation* and *inverse variation* (sometimes called *indirect variation*). In a *direct variation* relationship between two variables, an increase in one variable results in an increase in the other variable. Likewise, a decrease in one causes a decrease in the other. An *inverse variation*, on the other hand, behaves just the opposite of a direct variation. That is, in an *inverse variation* relationship between two variables, an increase in one variable results in a decrease in the other variable and a decrease in one causes an increase in the other.

Direct Variation

Some examples of quantities that are directly proportional, that is, in a direct variation relationship are:

- Assembly line speed and quantity of product manufactured.
- Rotational velocity of a fixed displacement pump and gallons per minute pumped.
- Voltage across a resistor and flow of electrical current.

In each of these instances, an increase in one variable causes an increase in the other, and likewise, a decrease in one causes a decrease in the other.

Direct variation between variables x and y is expressed algebraically as

$$y = kx^n, \text{ where } n > 0$$

Because x is directly proportional to y, the ratio $\frac{y}{x^n}$ is constant for all values of x and y. This is reflected in the second form of the relationship,

$$k = \frac{y}{x^n}$$

where k is called the *proportionality constant*.

Charles' law, as studied in a chemistry or physics class, is a good example of direct variation. This physical law states that under constant pressure P the volume V of a fixed mass of gas is directly proportional to its *absolute* temperature T. The direct variation of V and T is stated in the equation

Charles' Law

$$V = kT \qquad \text{or} \qquad \frac{V}{T} = K$$

According to Charles' Law, a balloon that is inflated in a warm room and then brought outdoors in cold winter weather will shrink. As modeled in the formula, balloon volume decreases as temperature decreases.

EXAMPLE 12.14: ***Direct Variation***

Use Charles' Law to find the proportionality constant for a balloon whose volume is 1 liter when the temperature is 20 degrees Celsius. Find the volume of the balloon when the temperature decreases to 0 degrees Celsius.

Solution:

Since absolute temperature must be used with Charles' Law, we convert 20 degrees Celsius to 20 + 273 = 293 Kelvin and 0 degrees Celsius to 0 + 273 = 273 Kelvin.

The proportionality constant, k, is found by substituting the known values into Charles' Law to find:

$$k = \frac{V}{T} = \frac{1 \text{ liter}}{293 \text{ Kelvin}} = 0.00341 \ \frac{\text{liter}}{\text{Kelvin}}$$

Using this value for k we can find that at 0 degrees Celsius the volume of the balloon decreases to:

$$V = kT = \left(0.00341 \ \frac{\text{liter}}{\cancel{\text{Kelvin}}}\right)\left(273 \ \cancel{\text{Kelvin}}\right) = 0.931 \text{ liter}$$

Another form of Hooke's law states the direct variation relationship between elasticity x of a spring and the force F needed to restore it: F = – kx.

Another example of direct variation comes from observing how stress affects different materials. The engineering concept of *stress* is related to the effect that loads, such as weight, have on various structures. In the study of strength of materials, one form of *Hooke's law* states that stress σ is directly proportional to *strain* ε within certain limits. The relationship between stress and strain is expressed as

σ and ε are the lower case Greek letters "sigma" and "epsilon," respectively.

$$\sigma = E\varepsilon$$

where E is the proportionality constant known as the *modulus of elasticity*, or *Young's modulus*. The value of E is different for each variety of steel, aluminum, brass, and all other homogeneous engineering materials. Notice how Hooke's law resembles the general equation for direct variation, $y = kx$.

Example 12.15: *Direct Variation – Stress and Strain*

Given a constant modulus of elasticity E = 29,000,000 psi (pounds per square inch) for a particular steel alloy used for bridge construction, how much stress σ does a bridge beam experience, due to a crossing truck of a certain weight, if the strain ε measured on the beam is 0.0003?

Solution:

Use scientific notation and substitute values for E and ε:

$$\sigma = E\varepsilon = (2.9\times10^{7}\ \text{psi})(3\times10^{-4}) = 8.7\times10^{3}\ \text{psi, or 8700 psi.}$$

If a heavier truck crosses over the bridge and causes a strain of 0.0005 on the same bridge beam, the stress increases proportionally, and the new stress is

$$\sigma = E\varepsilon = (2.9\times10^{7}\ \text{psi})(5\times10^{-4}) = 14.5\times10^{3}\ \text{psi, or 14.5 ksi.}$$

Stress is sometimes written in units of ksi (thousand pounds per square inch). Hence, 8700 psi is 8.7 ksi.

Transforming the formula to $E = \dfrac{\sigma}{\varepsilon}$ shows clearly that E is the ratio of stress to strain for the material in question. It is also the slope of the stress-strain curve. Figure 12.14 is a graph of Example 12.15. Slope E is positive, that is, it rises from left to right. In general, a positive slope in a linear graph of two variables (in this case, stress and strain) is the sign of direct variation. Observe that the variables either increase or decrease together. Thus, as the weight of the truck in Example 12.15 increases, so does the strain, with the result that the stress in the bridge beam increases proportionally. Conversely, as weight (strain) decreases, stress decreases.

Strain is caused by a load on a material. Stress is the reaction of a material to strain.

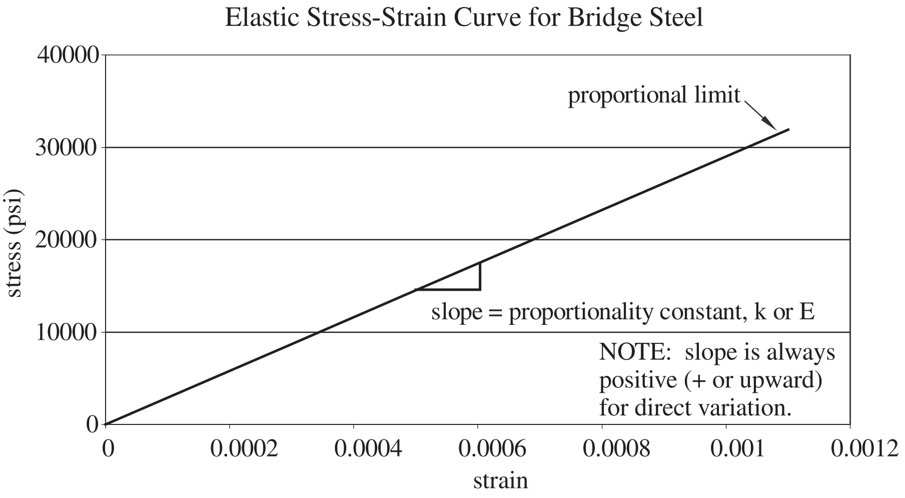


Figure 12.14 *Direct variation*

The slope of a material's stress-strain curve is equal to Young's modulus E for that material.

Strain is easily measured, so when given a material's modulus of elasticity, engineers can easily calculate the stress in a bridge beam. From this information engineers determine the size of beam required so that the bridge does not collapse when a heavy truck crosses.

Recall that π is the Greek letter pi and stands for the irrational number 3.1416….

EXAMPLE 12.16: ***Direct Variation – Area and Radius of a Circle***

The area A of a circle varies directly with r^2, the square of its radius. Algebraically, this statement is expressed $A = kr^2$. Mathematicians established long ago that the proportionality constant k in this relationship is π. So $A = \pi r^2$ is the formula for area of a circle. This is an example of direct variation involving an exponent.

What is the area of a circle whose radius is 5 cm?

Solution:

$$A = \pi r^2 = \pi\left(5^2\right) = \pi(25) = 78.5 \text{ cm}^2 \text{ (rounded to tenths)}$$

EXAMPLE 12.17: ***Direct Variation***

If y varies directly with the square of x, we can write $y = kx^2$. Suppose that $y = 36$ when $x = 2$. Find the value of the proportionality constant k and the value of y when $x = 4$.

Solution: The proportionality constant k is found by substituting the known values into the equation to find:

$$y = kx^2 \qquad \text{from which} \qquad k = \frac{y}{x^2} = \frac{36}{2^2} = \frac{36}{4} = 9$$

Using this value for k we can find values for y given any value of x. Accordingly, when $x = 4$, we find that:

$$y = kx^2 = (9)\left(4^2\right) = (9)(16) = 144$$

Notice that doubling the value of x causes a quadruple increase in the value of y.

Inverse Variation

Inverse (or *indirect*) *variation* results when an increase in one variable causes a decrease in another variable, or when a decrease in one variable results in an increase in another variable. Some examples of inverse variation are:

- Cutting speed of a tool and life of tool.
- Degree of job satisfaction and worker absenteeism.
- Oil production and price of gasoline.
- Traveling speed and travel time.

In each of these examples, an increase in one variable causes a decrease in the other, and likewise, a decrease in one causes an increase in the other.

Inverse proportion between y and x is expressed algebraically as

$$y = \frac{k}{x^n} \quad \text{or} \quad x^n y = k$$

where $n > 0$ and k is the proportionality constant.

Boyle’s law is a good illustration of inverse variation. This physical law states that at constant temperature, the volume of a fixed weight of gas is inversely proportional to its absolute pressure.

$$P = \frac{k}{V}$$

Boyle’s Law

Consider what happens to a balloon inflated at the top of a high mountain where the atmospheric pressure is low. Assuming that the temperature does not change, the balloon would shrink as the mountain is descended because the atmospheric pressure on the balloon would increase.

EXAMPLE 12.18: ***Inverse Variation – Boyle’s Law***

From Boyle’s Law, $P = \frac{k}{V}$, we know that the volume of a quantity of gas is inversely proportional to the pressure acting on it. If a gas occupies 740 cubic feet when under a pressure of 16 psi, how many cubic feet will the gas occupy when the pressure rises to 30 psi?

Solution:

The value of the proportionality constant is not known, so before we can determine how many cubic feet the gas will occupy when the pressure rises to 30 psi, we have to solve for *k* in the formula $P = \frac{k}{V}$.

k often needs to be calculated when solving proportion problems. Any known pair of data points can be used to find the value of k.

Substituting the values of the pair of known data (V = 740 ft^3 and P = 16 psi), we rearrange Boyles’ Law and solve for *k*.

$$P = \frac{k}{V} \Rightarrow 16\text{ psi} = \frac{k}{740\text{ ft}^3} \Rightarrow k = (16\text{ psi})(740\text{ ft}^3) = 11{,}840\text{ psi}\cdot\text{ft}^3.$$

Keep units in mind when working on proportionality problems. Notice that k is expressed in units of psi · ft^3.

Substituting k = 11,840 and rearranging Boyle’s Law to solve for V, the number of cubic feet occupied by the gas when the pressure rises to 30 psi becomes:

$$V = \frac{k}{P} = \frac{11{,}840\ \cancel{\text{psi}}\cdot\text{ft}^3}{30\ \cancel{\text{psi}}} = 395\text{ ft}^3$$

Thus the volume of this gas contracts from 740 cubic feet to 395 cubic feet when the pressure changes from 16 psi to 30 psi.

Figure 12.15 shows a graph of Example 12.18. We can readily see that the curve follows a downward trend. As pressure increases, the volume decreases. The downward direction of the graph indicates that the two variables are inversely related.

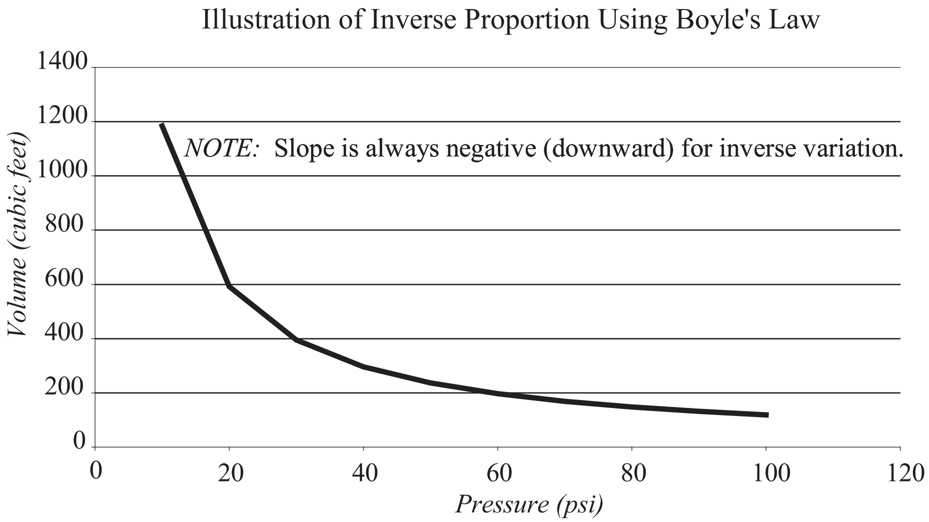


FIGURE 12.15 ***Inverse variation***

Observe that as the graph decreases toward zero pressure, the volume tends to infinity. No matter how high the pressure becomes, the volume never reaches zero. This is shown by the nonlinearity of the curve.

EXAMPLE 12.19: *Inverse Variation*

If y varies inversely with the square of x, we can write $y = \frac{k}{x^2}$. Suppose that $y = 36$ when $x = 2$. Find the value of the proportionality constant k and the value of y when $x = 4$.

Solution:

The proportionality constant k is found by substituting the known values into the equation: $36 = \frac{k}{2^2}$ from which $k = (36)(4) = 144$.

Using this value for k we can find values for y given any value of x. Accordingly, when $x = 4$, we find that

$$y = \frac{k}{x^2} = \frac{144}{4^2} = \frac{144}{16} = 9$$

Notice that doubling the value of x (from 2 to 4) causes a quadruple reduction in the value of y (from 36 to 9.)

This type of inverse proportion applies to many physical phenomena, such as light and sound intensity. It is known as the *inverse square law*.

Joint and Combined Variation

Many instances arise in shop and engineering calculations in which variation of more than two variables occurs. *Joint variation* occurs when all of the variables have a direct effect or all of the variables have an inverse effect. If both direct and inverse variation occur, the variation is called *combined variation*.

Problems involving combined and joint variation are solved just as we solved the other types of variation problems. In each case we determine the proportionality constant k from a set of known values for the variables. Once k is established, we can determine the value of an unknown variable by substituting k and the values given for the other variables into the formula.

EXAMPLE 12.20: ***Joint Variation – Kinetic Energy***

Kinetic energy is directly related to both mass m and velocity v of an object. In fact, kinetic energy varies directly with both the mass and the square of an object's velocity.

In algebraic symbols, kinetic energy E is expressed as $E = \frac{1}{2}mv^2$. The proportionality constant is always $\frac{1}{2}$ for kinetic energy calculations and the amount of energy is directly proportional to both the mass of the object and the square of the object's velocity.

An experiment could be conducted to show that $\frac{1}{2}$ is the correct value for the proportionality constant.

Coulomb's Law is a good illustration of combined variation. It states that the force F between two charged particles varies directly with the product of the charges q_1 and q_2 and inversely with the square of the distance d between the particles. Algebraically, this is written:

$$F = \frac{k\,q_1\,q_2}{d^2}$$

Coulomb's Law of Charged Particles

Similar to Coulomb's law, *Newton's universal law of gravitation* states that any two bodies attract each other with a force F that is proportional to the product of their masses m_1 and m_2, and inversely proportional to d^2, the square of the distance between them. Algebraically, this is written:

$$F = k\frac{m_1 m_2}{d^2}$$

Newton's Law of Gravitational Force

Experiments have shown that for gravitational force, k is an extremely small number in SI units:.

$$k = 0.0000000000667\ \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}$$

EXAMPLE 12.21: *Combined Variation — Newton's Law of Gravitational Force*

Find the gravitational force of attraction between two locomotives 4 meters apart if each locomotive has mass of 150,000 kg.

Solution:

It is easier to work with a very small number like k by changing it to scientific notation:

$$k = 6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \text{ and } m_1 = m_2 = 1.5\times10^5 \text{ kg}$$

Units are left out of the intermediate calculation. Prove that the final answer is in newtons.

Distance $d = 4$ m. Substituting all values into the formula gives

$$F = \left(6.67 \times 10^{-11}\right) \frac{\left(1.5 \times 10^5\right)\left(1.5 \times 10^5\right)}{4^2} = 0.094 \text{ N}$$

When a problem in proportion is encountered, a judgment has to be made as to which kind of variation is implied by the situation, direct or inverse. Determining the set up of a problem, particularly for inverse variation, is sometimes tricky. Remember that the characteristic trait of an inverse variation problem is that the magnitudes of variable values move in opposite directions: an increase in the x variable causes a decrease in the y variable and vice versa.

EXERCISES

12.90 Given that y is directly proportional to x, find the proportionality constant k for the indicated values of x and y.

a) $y = 5$ when $x = 10$ **b)** $y = 100$ when $x = 25$ **c)** $y = 72$ when $x = 9$

12.91 Given that y is inversely proportional to x, find the proportionality constant k for the indicated values of x and y.

a) $y = 5$ when $x = 10$ **b)** $y = 100$ when $x = 25$ **c)** $y = 72$ when $x = 9$

12.92 Suppose y is directly proportional to x with proportionality constant $k = 7$. Find y for the indicated value of x.

a) $x = 3$ **b)** $x = 8$ **c)** $x = 27$

12.93 Suppose y is inversely proportional to x with proportionality constant $k = 3$. Find y for the indicated value of x.

a) $x = 9$ **b)** $x = 15$ **c)** $x = 2$

12.94 Suppose y is inversely proportional to x^3 and that $y = 162$ when $x = 3$. Determine the value of the proportionality constant k and find the value of y for the indicated value of x.

a) $x = 6$ **b)** $x = 10$ **c)** $x = 4$

12.95 Suppose y is inversely proportional to z and directly proportional to x^2. If $y = 24$ when $x = 4$ and $z = 2$, determine the value of the proportionality constant k and find the value of y for the indicated values of x and z.

a) $x = 3$ and $z = 1$ **b)** $x = 5$ and $z = 2$ **c)** $x = 9$ and $z = 4$

12.96 The elongation of a guy wire varies directly with small loads. What is the elongation of the guy wire when the load is 2500 N if a load of 400 N causes an elongation of 8 mm?

12.97 According to Newton's second law of motion, a mass m acted upon by a force F has an acceleration a that is directly proportional to F and inversely proportional to m. Suppose a 25-kg mass initially at rest is acted upon by a 300-newton horizontal force and the resulting acceleration is $12\frac{\text{m}}{\text{sec}^2}$. Find the acceleration for a 50-kg mass initially at rest acted upon by a 300-newton horizontal force.

12.98 Coulomb's law states that the force F between two charged particles varies directly with the product of the charges q_1 and q_2 and inversely with the square of the distance d between the particles. Algebraically, this is expressed as $F = \frac{kq_1q_2}{d^2}$. Suppose two particles each having a charge of 1 coulomb exert a force of 9×10^9 newtons when separated by a distance of 1 meter.

a) Determine the value of the proportionality constant k.
b) What units does k have in this proportion?
c) If the charges are placed a distance of 2 meters apart, what is the resulting force?

13

OPERATIONS ON POLYNOMIALS

Expanding polynomials through multiplication is often a step in solving equations. The same is true of polynomial factorization, which, in a sense, is the reverse of polynomial expansion. Polynomial factoring may be simple, as in the case of extracting a common monomial term from each term in a polynomial; or it may be more involved, as when factoring a polynomial into a product of other polynomials. The mathematics of factoring is explained for each of these cases.

13.1 Expanding Algebraic Expressions

Combining expressions through multiplication is called *expanding an expression*. This process is often needed to clear parentheses; thus it involves the distributive property of multiplication over addition. Exponent rules are applied as needed.

Multiplying Monomials by Polynomials

The simplest application of the distributive property of multiplication over addition is that of the monomial by the polynomial. Consider the examples that follow.

EXAMPLE 13.1A: ***Expanding an Expression***

Expand the expression $7(-x + 3y - 6)$.

Solution:

Applying the distributive property to each term in the parentheses gives

$$\begin{aligned} 7(-x+3y-6) &= 7(-x)+7(3y)+7(-6) \\ &= (-7x)+(21y)+(-42) \\ &= -7x+21y-42 \end{aligned}$$

Notice that the operation before each term to be multiplied is treated as a sign.

EXAMPLE 13.1A: *Expanding an Expression (Continued)*

Thus, the number 7 is multiplied by each of the terms, $-x$, $+3y$, and -6, as shown. Each product is separated by addition. Then the operation and sign are again combined using rules of signed number multiplication to give the final answer.

In this example, no combining of like terms was possible because all of the terms were unlike.

EXAMPLE 13.1B: *Expanding an Expression*

The negative in front of parentheses is distributed as a change of sign of each term inside the parentheses.

Simplify the expression $\frac{1}{2}t^2 + t + 24 - (\frac{2}{5}t^2 + 3t - 14)$.

Solution:

The minus sign before the parentheses is interpreted as the coefficient -1 and so is multiplied by each term inside the parentheses. We sometimes refer to this as “multiplying through by -1.” Thus,

Apply the rules for fraction addition when coefficients are fractions.

$$\frac{1}{2}t^2 + t + 24 - \left(\frac{2}{5}t^2 + 3t - 14\right) = \frac{1}{2}t^2 + t + 24 - \frac{2}{5}t^2 - 3t + 14$$

Finally, combining like terms gives the simplified expression:

$$\frac{1}{10}t^2 - 2t + 38$$

EXAMPLE 13.1C: *Expanding an Expression*

Simplify $x^2 - 6x(7 - 2x) + 9x^2\left(x - 8\right)$.

Solution:

$$\begin{aligned} x^2 - 6x(7 - 2x) + 9x^2(x - 8) &= x^2 - 42x + 12x^2 + 9x^3 - 72x^2 \\ &= -59x^2 - 42x + 9x^3 \end{aligned}$$

The final answer shown is correct but is not in *standard form*. The standard form of a polynomial is one in which terms are listed in “descending order of the variable.” That is, the term with the highest exponent is written first, followed by the term with the next highest exponent and so on. Thus, in standard form, the solution becomes

$$9x^3 - 59x^2 - 42x$$

Notice how the coefficient of a term is connected to its sign, which moves with it when terms are rearranged.

Binominal Expansion Using the FOIL Method

Consider the expression $(x+2)^2$. The x + 2 term is a polynomial called a *binomial*.

When presented with an expression like this, the temptation is to write

$$\left(x+2\right)^2 = x^2 + 2^2$$ *. However, this is incorrect!*

$(x+2)^2 \neq x^2 + 2^2$

To avoid making this common error, recall that $(x + 2)^2 = (x + 2)(x + 2)$. Every term in the leftmost parentheses is multiplied by every term in the rightmost parentheses.

The correct expansion is:

$$\left(x+2\right)^2 = \left(x+2\right)\left(x+2\right) = x\cdot x + 2\cdot x + 2\cdot x + 2\cdot 2 = x^2 + 4x + 4$$

In general, we write $\left(a+b\right)^2 = \left(a+b\right)\left(a+b\right)$, as in

$$\left(x-6\right)^2 = \left(x-6\right)\left(x-6\right)$$
$$\left(m+\tfrac{1}{8}\right)^2 = \left(m+\tfrac{1}{8}\right)\left(m+\tfrac{1}{8}\right)$$
$$\left(3y-4\right)^2 = \left(3y-4\right)\left(3y-4\right)$$

The ultimate expanded form of $(a + b)^2$ is arrived at using a method known by the acronym **FOIL**,which stands for **F**irst **O**uter **I**nner **L**ast. The FOIL method ensures that all terms inside the first set of parentheses are multiplied by all terms in the second set of parentheses as shown in Example 13.2A.

EXAMPLE 13.2A: ***FOIL Method***

Carry out the polynomial expansion of $(x + 4)(x + 2)$ using the FOIL method.

Solution:

Using the FOIL method:

STEP 1: Multiply the **F**irst terms in each set of parentheses: $x\cdot x = x^2$

STEP 2: Multiply the **O**uter terms in each set of parentheses: $x\cdot 2 = 2x$

STEP 3: Multiply the **I**nner terms in each set of parentheses: $4\cdot x = 4x$

STEP 4: Multiply the **L**ast terms in each set of parentheses: $4\cdot 2 = 8$

STEP 5: Putting all of the terms together gives $x^2 + 2x + 4x + 8$

Finally, combing like terms and writing in standard form gives $x^2 + 6x + 8$

Example 13.2A is just a special case of binomial multiplication. Any two binomials can be multiplied through the FOIL method, as illustrated in the next several examples.

EXAMPLE 13.2B: *FOIL Method*

Carry out the indicated binomial expansion. Note how the FOIL method is used to multiply the first terms, outer terms, inner terms, and last terms in the intermediate step.

One way to avoid confusion on the signs is to show every operation in the intermediate step as addition and rewrite the minus signs as negative signs in the parentheses.

Each of the four binomial expansions shown differ only with respect to operation within parentheses:

$$\begin{aligned}(x+3)(x+2) &= x^2+2x+3x+(3)(2) = x^2+5x+6,\\ (x+3)(x-2) &= x^2-2x+3x+(3)(-2) = x^2+x-6,\\ (x-3)(x+2) &= x^2+2x-3x+(-3)(2) = x^2-x-6,\\ (x-3)(x-2) &= x^2-2x-3x+(-3)(-2) = x^2-5x+6.\end{aligned}$$

Notice the capture of the sign of the operation as terms are multiplied. For example, when multiplying the **O**uter terms in $(x + 3)(x – 2)$, the subtracted 2 is regarded as *negative* 2 to give $– 2x$. Likewise when multiplying the **L**ast terms, we have $(– 2)(3)$ to give $– 6$.

EXAMPLE 13.2C: *FOIL Method*

Carry out the polynomial expansion of $(2x – 5)(x – 3)$ using the FOIL method.

Solution:

$$\begin{aligned}(2x-5)(x-3) &= 2x(x)+2x(-3)+(-5)(x)+(-5)(-3)\\ &= 2x^2-6x-5x+15\\ &= 2x^2-11x+15\end{aligned}$$

The rules of multiplication of monomials are applied to each pair of monomials present. In an orderly way we multiply by the FOIL method. The first terms are $2x$ and x. The outer terms are $2x$ and -3, the inner terms are -5 and x, and the last terms are -5 and -3.

EXAMPLE 13.2D: *FOIL Method*

Carry out the polynomial expansion of $(x + 1)(x – 1)$ using the FOIL method.

Solution:

$$(x+1)(x-1) = x^2-1x+1x-1 = x^2-1$$

This kind of product is marked by the fact that the middle terms cancel upon multiplication. That is, the outer and inner products of FOIL cancel. In general,

$$(a+b)(a-b) = a^2+\cancel{a}b-\cancel{a}b-b^2 = a^2-b^2$$

This binomial has a special name: the *difference of squares*. The “difference” is the subtraction and “squares” refers to the fact that the two terms are perfect squares. Many equations contain such binomials; the ability to recognize and factor them leads to the solution of such equations.

The difference of squares is an important binomial with many physical applications.

General Polynomial Expansion

The FOIL method of polynomial expansion works well for multiplying pairs of binomials. In general, when multiplying two polynomials such as a binomial multiplied by a trinomial, each term of the first polynomial must be multiplied by each term of the second polynomial, and then the like terms combined. This process is illustrated in the next two examples.

EXAMPLE 13.3A: ***A Binomial Multiplied by a Trinomial***

Expand the polynomial $(a + b)(a + b + c)$.

Solution:

STEP 1: Multiply the first term of the binomial by all the terms of the trinomial:

$$(a)(a + b + c) = a^2 + ab + ac$$

STEP 2: Multiply the second term of the binomial by all the terms of the trinomial:

$$(b)(a + b + c) = ab + b^2 + bc$$

STEP 3: Combine like terms:

$$\begin{aligned}(a+b)(a+b+c) &= a^2 + ab + ac + ab + b^2 + bc\\ &= a^2 + 2ab + b^2 + ac + bc\end{aligned}$$

EXAMPLE 13.3B: ***A Binomial Multiplied by a Trinomial***

Expand the polynomial $(3x - 2)(x^2 + x + 1)$.

Solution:

STEP 1: Multiply the first term of the binomial by all the terms of the trinomial:

$$(3x)(x^2 + 2x + 1) = 3x^3 + 6x^2 + 3x$$

STEP 2: Multiply the second term of the binomial by all the terms of the trinomial. Notice that the minus sign goes with the 2:

$$(-2)(x^2 + x + 1) = -2x^2 - 2x - 2$$

STEP 3: Combine like terms:

$$\begin{aligned}(3x-2)(x^2+2x+1) &= 3x^3 + 6x^2 + 3x - 2x^2 - 2x - 2\\ &= 3x^3 + 4x^2 - x - 2\end{aligned}$$

EXERCISES

13.1 Expand each polynomial.

a) $(-3uy^2)(-4u^2w)(uw)(-4yw)$

b) $(6pqr)(2qrs)(-3)(-2prs)(q^2s^2)$

c) $(-3ab)(-3ab)(-3ab)$

d) $(3a^3b)(-4a^2b^2)(-2ab^3)(6bc)$

e) $(-5x^2y^3)(2y^2z^3)(-3z)$

f) $(22acd)(2bce)(ade)(3de)$

g) $6(a+b)$

h) $4x(3y-2z)$

i) $3bc(2e-3g+2)$

j) $5(3ab-6bd)$

k) $12z(xy-3y+2x)$

l) $(2b-7)(b+2)$

m) $(7p+3q)(3p-2q)$

n) $(x+3)(5x-2)$

13.2 Expand each polynomial.

a) $2x^2(x-1)$

b) $4a(a^2+5a+4)$

c) $6b^2(7b+2)$

d) $(4a+3)(6a^2+5a+4)$

e) $(2x-2)(7x^2+3x+2)$

f) $(a^2-ab+b^2)(a^2+ab+b^2)$

g) $(9+z^3)(z^2-5)$

h) $3ab(c+d+e)$

i) $(a+2)(a^2-3a+4)$

j) $(-3x+1)(2x^2-x+3)$

k) $(-4x+1)(x^2-3x+2)$

l) $(x^2+y)(2x^3+3y^2+x-2)$

m) $5x(x^2+3x)$

n) $(w+5)(w^2+5w-3)$

o) $(3x-3)(-2x^2+3x+1)$

p) $(2x+2)(3x-3)$

13.3 Expand each polynomial.

a) $4(a+b)^2$

b) $(z-3)^3$

c) $-2(3+x)^2$

d) $(3-z)^3$

e) $(x+1)(x+3)$

f) $(x+1)(x-3)$

g) $(x-1)(x+3)$

h) $(x-1)(x-3)$

13.2 Factoring Polynomials

In polynomial expansion we carried out multiplication of polynomials with polynomials. Factoring is the reverse of polynomial expansion, in which a polynomial is "broken down" to its constituent factors.

We begin with finding the greatest common monomial factors among terms in a polynomial. This can be thought of as the inverse of the distributive property, in which a monomial is multiplied through parentheses by each term in the polynomial. Thus, if each term in a polynomial contains a common factor, we *divide* each term in the polynomial by this factor and place it outside parentheses. Consider the following three polynomials:

$$4x^2+6x-12 \qquad\qquad 3a^2b^3-9a^2b^2+21a^3b \qquad\qquad -5xy+15xy^2-10y$$

Notice that the GCF of each term in the first polynomial is the factor 2. Therefore, we divide each term by 2 and place the 2 in front of parentheses containing the remaining polynomial factor:

$$4x^2+6x-12=2(2x^2+3x-6)$$

Greatest Common Factor = 2

We can now see that the polynomial $4x^2+6x-12$ has two factors:

"2" and $(2x^2+3x-6)$.

In the second polynomial, we seek the greatest common factor among coefficients and variables. The monomial 3*ab* divides each term as follows:

$$\frac{3a^2b^3-9a^2b^2+21a^3b}{3ab}=\frac{\cancel{3}a^2b^3}{\cancel{3}ab}-\frac{\overset{3}{\cancel{9}}\,a^2b^2}{\cancel{3}ab}+\frac{\overset{7}{\cancel{21}}\,a^3b}{\cancel{3}ab}$$

$$=a^{2-1}b^{3-1}-3a^{2-1}b^{2-1}+7a^{3-1}b^{1-1}$$

$$=ab^2-3ab+7a^2$$

The "GCF" of given terms is the term of highest degree that divides each given term.

Placing this result in parentheses with the removed factor in front, the final result of factoring is:

$$3a^2b^3-9a^2b^2+21a^3b=3ab(ab^2-3ab+7a^2)$$

Greatest Common Factor = 3ab

The third expression contains a GCF of –5*y* and so it factors as:

$$-5xy+15xy^2-10y=-5y(x-3xy+2)$$

Greatest Common Factor = –5y

Notice that since we factored out a negative sign, we had to change the sign of each term in the parentheses. Standard practice in factoring is to remove the negative sign from a leading coefficient by factoring it out.

The leading coefficient is the coefficient of the first term of a polynomial.

EXAMPLE 13.4A: ***Polynomial Factoring***

Factor the polynomial x^4+7x^3+14x.

Solution:

$$x^4+7x^3+14x=x\left(x^3+7x^2+14\right)$$

The only common factor is *x*. Although the last two terms have a common factor of 7, the first term does not, so 7 cannot be factored out.

EXAMPLE 13.4B: *Polynomial Factoring*

Factor the polynomial $18m^4n^2 - 3m^2n^2$.

Solution:

The GCF is $3m^2n^2$ and divides into each term as follows:

$$\frac{18m^4n^2 - 3m^2n^2}{3m^2n^2} = \frac{18m^4n^2}{3m^2n^2} - \frac{3m^2n^2}{3m^2n^2} = 6m^2 - 1$$

Putting all of the factors together gives the result:

$$18m^4n^2 - 3m^2n^2 = 3m^2n^2(6m^2 - 1)$$

EXERCISES

13.4 Fill in the parentheses to make each statement true.

a) $36x = 9(\ \ )$

b) $51t = 3(\ \ )$

c) $45u = (\ \ )(9u)$

d) $x^6y^2 = x^2y(\ \ )$

e) $42p^2 = 2p(\ \ )$

f) $-21w^3 = (\ \ )w^2$

g) $15w^3h^3 = 5w(\ \ )$

h) $30y = 15y^{\frac{1}{2}}(\ \ )$

i) $-12z^5 = -4z(\ \ )$

j) $96x^2yz^3 = -12yz^2(\ \ )$

13.5 Remove the greatest common factor from each expression.

a) $12x - 12y$

b) $3x - 15y$

c) $-5a^2 + 20b^2 - 15c^2$

d) $-4x + 12x^3$

e) $z^5 - z^3$

f) $6x^4 - 12x^3 + 6x^2 - 18x - 30$

g) $3(y+3) + 5(y+3)$

h) $3a(x-1)^3 + 9a^2(x-1)^5 - 6b(x-1)^4$

i) $-4(x-1)^3 + 12(x-1)^3$

j) $-18a^3b^5 - 36a^2b^4 + 27a^3b^{10}$

k) $3(a+1)^3 - 2(a+1)^5$

l) $-20c^2 + 42c$

m) $21x^6y^4z^8 + 63x^3y^2z^4 - 7x^3y^5z$

n) $-2x^3 + 14x^2 - 32x + 8$

o) $-56b^2c + 112bc$

13.3 Binomial Factors of a Trinomial

We are now ready to consider factoring polynomials into smaller polynomials. In particular, we will look at factoring *quadratic* polynomials (polynomials of degree 2) because of their importance to the solution of many practical technical problems.

The question becomes how do we find the binomial factors of a polynomial $ax^2 + bx + c$, where a, b, and c are integers? The answer to this question is not always simple or straightforward. The ease with which factors are determined depends on the signs and relative magnitudes of the coefficients a, b, and c.

We begin by looking at quadratic polynomials that are relatively easy to factor. These break down roughly into two classes:

- A trinomial with leading coefficient of 1. These can be factored by inspection.
- A binomial difference of squares. These can be factored according to a formula.

Factoring Trinomials with Leading Coefficient Equal to 1

Look again at one of the polynomial expansions we considered in Example 13.2B:

$$(x-3)(x+2) = x^2 - x - 6$$

If we divide the right side of this equation by either of the binomial factors on the left side of the equation, we get the other binomial factor:

$$\frac{x^2 - x - 6}{x-3} = x + 2 \qquad \text{and} \qquad \frac{x^2 - x - 6}{x+2} = x - 3$$

Now consider any trinomial of the form $x^2 + bx + c$, that is, a trinomial whose leading coefficient is 1. It can be factored into two binomials $(x+\square_1)(x+\square_2)$ whose **FOIL** product is $x^2 + bx + c$. That is,

$$(x+\square_1)(x+\square_2) = x^2 + bx + c$$

where $\square_1$ and $\square_2$ are *integers*, either positive or negative. The plus sign in the model does not mean that the factorization can only contain sums of positive terms; a negative value is likely to turn up in the factorization and it will affect the final operations in the binomial factors. Through **FOIL** multiplication of the left hand side of the equation, we have

$$x^2 + \square_1 x + \square_2 x + \square_1\square_2 = x^2 + \left(\square_1 + \square_2\right)x + \square_1\square_2$$

By symmetry, each term on the left side of the equation is equal to its like, or corresponding, term on the right. So it must be true that

$$x^2 = x^2$$

$$\left(\square_1 + \square_2\right)x = bx$$

$$\square_1\square_2 = c$$

Equate the coefficients of like terms on left and right hand sides. The x^2 terms already have an identical coefficient, namely, 1. The other terms are in equality relationships,

$$\square_1 + \square_2 = b \qquad \text{and} \qquad \square_1\square_2 = c$$

That is, the values of $\square_1$ and $\square_2$ in any factorization of $x^2 + bx + c$ are the two numbers whose sum is b and whose product is c. Generally, we do not need to perform this coefficient match-up every time we factor a trinomial; we need only look at the coefficients of the trinomial and ask:

What two values multiply to give c and add to give b?

EXAMPLE 13.5A: *Factoring a Trinomial with Leading Coefficient Equal to 1*

Factoring $x^2 + bx + c$.

Factor the quadratic polynomial $x^2 + 3x + 2$.

Solution:

For this problem we need values in the boxes that make the following equation true:

$$x^2 + 3x + 2 = (x + \square_1)(x + \square_2)$$

That is, $x^2 + 3x + 2 = x^2 + (\square_1 + \square_2)x + \square_1\square_2$. The coefficients match up as follows:

$$(\square_1 + \square_2)x = 3x$$

$$\square_1\square_2 = 2$$

The only values that work here are 1 and 2, since $1 + 2 = 3$ and $(1)(2) = 2$. Hence,

The order of the binomial factors does not matter, since multiplication is commutative.

$$(x+1)(x+2) = x^2 + 3x + 2$$

Multiply the binomial factors using **FOIL** to confirm that this is the correct factorization of the polynomial:

$$(x+1)(x+2) = x^2 + 2x + 1x + 2 = x^2 + 3x + 2$$

Example 13.5b: *Factoring a Trinomial with Leading Coefficient Equal to 1*

a) Factor $x^2 + 7x - 8$ into its binomial factors.

b) Factor $x^2 - 7x - 8$ into its binomial factors.

Note that these trinomials differ only in the sign of the x coefficient.

Recall that the "constant coefficient" is the constant term in a polynomial.

Solution:

a) In the factorization of $x^2 + 7x - 8$, what two values multiply to give the constant coefficient *and* add to give the coefficient of the middle term? Writing this as

$$\square_1\square_2 = -8 \qquad \text{and} \qquad \square_1 + \square_2 = 7$$

tells us the values are 8 and −1, since $(8)(-1) = -8$ and $8 + (-1) = 7$. The factorization is therefore

$$x^2 + 7x - 8 = (x+8)(x-1)$$

Factor order does not matter. It could be written as either $(x + 8)(x - 1)$ or $(x-1)(x+8)$

b) The factorization of the second polynomial $x^2 - 7x - 8$ requires that

$$\square_1\square_2 = -8 \qquad \text{and} \qquad \square_1 + \square_2 = -7.$$

The values that work are −8 and 1, since $(-8)(1) = -8$, and $-8 + 1 = -7$. The factorization is therefore

$$x^2 - 7x - 8 = (x-8)(x+1)$$

Notice that the minus sign has to go with the 1 and the plus sign with the 8.

Example 13.5c: *Factoring a Trinomial with Leading Coefficient Equal to 1*

Observe the factorization of each of the following polynomials:

$$x^2 + 4x + 3 = (x+3)(x+1)$$

$$x^2 - x - 12 = (x+3)(x-4)$$

$$x^2 - 9x + 8 = (x-8)(x-1)$$

$$x^2 + 20x + 100 = (x+10)(x+10) = (x+10)^2$$

Use **FOIL** to verify that the right hand side of each of these factorizations yield the polynomial on the left side.

Factoring Trinomials with Leading Coefficient Not Equal to 1

Factoring the trinomial $ax^2 + bx + c$, $a \neq 1$

In the last section, we considered the factorization of polynomials whose leading coefficient was 1. We now consider how to determine the binomial factors of quadratic polynomials whose leading coefficient is not 1.

The leading coefficient throws a wrench into the process and the method involves a bit more trial-and-error. For example, the form of the factorization of $2x^2 - 9x + 4$ becomes: $2x^2 - 9x + 4 = (2x + r_1)(x + r_2) = 2x^2 + 2xr_2 + xr_1 + r_1r_2$ where the values of r_1 and r_2 have to be determined.

This is because the product of the first terms is $2x^2$, so one of the two leading coefficients of the binomial factors has to be 2 and the other has to be 1. A certain indication of what the r values should be is that r_1r_2 must equal the constant coefficient, 4. Four possibilities exist:

$$r_1 = 2, r_2 = 2 \qquad r_1 = -2, r_2 = -2 \qquad r_1 = 4, r_2 = 1 \qquad r_1 = -4, r_2 = -1$$

The choices can be narrowed by virtue of the sign of the middle coefficient in $2x^2 - 9x + 4$, which is negative. A combination of two positive r values will not yield a negative coefficient in the sum. For example, $r_1 = 2$ and $r_2 = 2$ gives $(2x + 2)(x + 2) = 2x^2 + 8x + 4$, which is not the desired outcome.

Thus, we will now try only the negative r values.

Using $r_1 = -2$ and $r_2 = -2$ gives $(2x - 2)(x - 2) = 2x^2 - 8x + 4$, which is also incorrect.

Using $r_1 = -1$ and $r_2 = -4$ gives $(2x - 1)(x - 4) = 2x^2 - 9x + 4$, which is the correct solution.

Notice that in the case of polynomials with a leading coefficient other than 1, the position of the r values *does* affect the outcome. Switching the position of -4 and -1 in the binomial factors gives: $(2x - 4)(x - 1) = 2x^2 - 6x + 4$, which is not the correct solution! That is why trial-and-error is needed.

In the following examples, observe the procedure for finding the factors of trinomials with leading coefficient not equal to 1.

Example 13.6a: ***Factoring a Trinomial with Leading Coefficient Not Equal to 1***

Factor $3x^2 + 2x - 1$.

Solution:

$$3x^2 + 2x - 1 = (3x - 1)(x + 1)$$

Since $r_1r_2 = -1$, we need $r_1 = 1$, $r_2 = -1$ or $r_1 = -1$, $r_2 = 1$. The position of these numbers affects the product. Notice that $(3x + 1)(x - 1) = 3x^2 - 2x - 1$, which would not be correct.

EXAMPLE 13.6B: *Factoring a Trinomial with Leading Coefficient Not Equal to 1*

Factor $6x^2+x-2$.

Solution:

$$6x^2+x-2=(3x+2)(2x-1)$$

This problem is more complicated because the leading coefficient can be factored as either (3)(2) or (6)(1). Thus, we try all *r* value combinations and leading coefficient factor pairs—and hope we hit on the correct factorization sooner than later! A good method for investigating factoring possibilities is to write out a few binomial pairs with blank spaces for the *r* values. For example,

$$(6x+\quad)(x-\quad) \qquad (6x-\quad)(x+\quad)$$

$$(3x+\quad)(2x-\quad) \qquad (3x-\quad)(2x+\quad)$$

Since $r_1r_2=-2$, we have either $r_1=2$, $r_2=-1$ or $r_1=-2$, $r_2=1$. Placing these in each open position would look like this:

$$(6x+2)(x-1) \qquad (6x-1)(x+2) \qquad (6x-2)(6x+1) \qquad (6x+1)(6x-2)$$

$$(3x+2)(2x-1) \qquad (3x-1)(2x+2) \qquad (3x-2)(2x+1) \qquad (3x+1)(2x-2)$$

Only one of the combinations will lead to the correct solution. The trick is to find the combination that works sooner than later.

Hint: Start with the 2, 3 combinations since these yield smaller middle term coefficients.

EXERCISES

13.6 Factor each of the polynomials.

a) x^2-6x+8
b) $x^2+12+7x$
c) x^2+x-6
d) $x^2+x+\frac{1}{4}$
e) x^2-9x+7
f) $16x^2-16x+3$
g) $2x^2-18$
h) x^2+4x+3
i) $5x^2-1-4x$
j) $3x^2+5x+2$
k) $3x^2-x-2$
l) x^2+2x-8
m) $6x^2+\frac{17}{12}x+\frac{1}{12}$
n) $5x^2-3x-5$
o) $144x^2+24x+1$
p) $x^2+12x+36$
q) $3x^2+14x-5$
r) x^2+x-12
s) $3x^2+4x-4$
t) $x^2-15x+54$
u) $6x^2-4x-5$
v) $x^2+13x+42$
w) $25x^2-10x+1$
x) x^2-x-56
y) $2x^2+x-3$
z) $4x^2+4x+1$

13.4 Special Products

A *difference of squares* is a special class of quadratic polynomial to which binomials like $x^2 - 1$, $4x^2 - 9$, and $x^2 - 9y$ belong. Consider such binomials as if they were trinomials whose x term has a coefficient of zero. For example,

$$x^2 - 1 = x^2 + 0x - 1$$

As was the case for $x^2 + bx + c$, this factors as $(x + r_1)(x + r_2)$, where $r_1 r_2 = -1$ and $r_1 + r_2 = 0$. The r values that give this result are 1 and −1.

Checking:

$$(x+1)(x-1) = x^2 - 1x + 1x - 1 = x^2 - 1$$

Upon performing **FOIL** multiplication, the products of the outer and inner terms cancel, leaving no middle term in the resulting binomial. The general formula shows this is the case for factoring any difference of squares:

$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

In general,

Difference of squares

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE 13.7A: ***Factoring a Difference of Squares***

Observe how each difference of squares factors:

$$\begin{aligned} 1 - x^2 &= (1 + x)(1 - x) \\ 4x^2 - 9 &= (2x + 3)(2x - 3) \\ x^2 - \frac{1}{4} &= \left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right) \\ a^2t^2 - 49s^2 &= (at + 7s)(at - 7s) \end{aligned}$$

Sometimes a common factor can be removed to change a polynomial into a difference of squares, as shown in the next example.

EXAMPLE 13.7B: ***Factoring a Difference of Squares***

Factor the following polynomials:

$$5x^2 - 20$$

$$3ab^3 - 12a^3b$$

Solutions:

$$5x^2 - 20 = 5(x^2 - 4) = 5(x + 2)(x - 2)$$

$$3ab^3 - 12a^3b = 3ab(b^2 - 4a^2) = 3ab(b - a)(b + a)$$

EXAMPLE 13.7B: ***Factoring a Difference of Squares (Continued)***

Factoring out the 5 in $5x^2 - 20$ turns the problem into a difference of squares in the first polynomial; factoring out $3ab$ turns the second polynomial into a difference of squares.

Each of the preceding examples is a difference of *perfect* squares. In fact, we can take the square root of *any* constant, although the root may not be *rational*. Allowing this, we now see how to easily factor the difference of x^2 and any number. For instance, consider the following two $a^2 - b^2$ type factorizations:

$$x^2 - 7 = (x+\sqrt{7})(x-\sqrt{7}) \qquad b^2 = 7, \text{ so } b = \sqrt{7}$$

$$y^2 - 20 = (y+\sqrt{20})(y-\sqrt{20}) = (y+2\sqrt{5})(y-2\sqrt{5}) \qquad b^2 = 20, \text{ so } b = 2\sqrt{5}$$

Recall that

$\sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

Factors with Imaginary Numbers

While factoring a difference of squares is fairly straightforward, factoring a *sum of squares* is more challenging. These factorizations entail *imaginary numbers*.

Factoring the sum of squares ***$a^2 + b^2$***

A binomial sum of squares is a quadratic expression of the form $a^2 + b^2$. This binomial does not factor for any choice of b in the set of real numbers. Notice that $a^2 + b^2 \neq (a + b)(a + b)$, as the expression on the right expands to $a^2 + 2ab + b^2$.

In the factorization of the sum of squares, the roots of the constant term come from the set of *imaginary numbers.* These numbers have a component i defined as:

When in doubt about a solution, simply multiply the factors in question and see if the original polynomial results.

$$i \equiv \sqrt{-1}$$

The unique property of the imaginary number i is that since $i = \sqrt{-1}$, then $i^2 = -1$.

Consider the **FOIL** multiplication of the binomials $(x + i)$ and $(x - i)$:

$$(x+i)(x-i) = x^2 - ix + ix - i^2 = x^2 - i^2 = x^2 - (-1) = x^2 + 1$$

That is, $x^2 + 1 = (x+i)(x-i)$.

Powers of i:

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

The use of imaginary numbers allows us to factor a sum of squares, as shown in Example 13.8A.

EXAMPLE 13.8A: ***Factoring a Sum of Squares***

Factor $x^2 + 100$.

Solution:

$$x^2 + 100 = (x+10i)(x-10i)$$

Checking the factorization using FOIL gives:

$$x^2 - 10ix + 10ix - 100i^2 = x^2 - 100i^2 = x^2 - 100(-1) = x^2 + 100$$

EXAMPLE 13.8B: *Factoring a Sum of Squares*

Factor $8y^2 + 50$.

Solution:

$$8y^2 + 50 = 2(4y^2 + 25) = 2(2y + 5i)(2y - 5i)$$

The common factor of 2 is factored out of the polynomial first, revealing that this is a sum of squares. Then the binomial factors are determined.

EXAMPLE 13.8C: *Factoring a Sum of Squares*

Factor $a^2 + 5$.

Solution:

$$a^2 + 5 = \left(a + i\sqrt{5}\right)\left(a - i\sqrt{5}\right)$$

As can happen with the difference of squares, the sum of squares can admit a factorization in which the roots of the constant term are irrational. In the case of the sum of squares, the imaginary number can have an irrational coefficient.

EXERCISES

13.7 Factor each difference of squares.

a) $x^2 - 64$
b) $x^2 - 19$
c) $x^2 - 32$
d) $y^2 - 9x^2$
e) $b^2 - 4$
f) $16x^2 - y^2$
g) $25x^2 - 36y^2$
h) $50x^2 - 49$
i) $121z^2 - 1$
j) $5x^2 - 1$
k) $12x^2 - 75$
l) $20x^2 - 15y^2$
m) $24x^2 - 32$
n) $z^2 - 48$
o) $27x^2 - 32$

13.8 Factor each sum of squares.

a) $x^2 + 7$
b) $x^2 + 40$
c) $x^2 + 4y^2$
d) $20x^2 + 1$
e) $45x^2 + 64y^2$
f) $3x^2 + 1$
g) $x^2 + 36y^2$
h) $15x^2 + 2$
i) $49a^2 + 36b^2$
j) $4x^2 + 162$
k) $w^2 + 25$
l) $2x^2 + 288$
m) $a^2 + 32b^2$
n) $4x^2 + 16$
o) $6x^2 + 72y^2$

13.5 Algebraic Fractions

The topic of algebraic fractions could not have been addressed before factoring had been covered in depth. The reason for this will soon be obvious, but let us say here that performing operations on algebraic fractions entails almost all the arithmetic and algebra skills previously covered.

Unlike a purely numerical fraction, an algebraic fraction contains variables. Such fractions are also called *rational expressions*. We have already seen the simplest of these fractions, ones consisting only of monomials. For instance,

$$\frac{3x^4}{21x^2}=\frac{x^2}{7}$$

Now consider algebraic fractions that include addition and subtraction. For example,

$$\frac{2x-4}{10} \qquad \frac{4y+6y^2}{8y} \qquad \frac{x-3}{8x-24} \qquad \frac{a^2-25}{a+5} \qquad \frac{x^2+x-20}{5+x}$$

Simplifying Algebraic Fractions

The first step in simplifying an algebraic fraction is to look for ways to factor the numerator and denominator in each fraction. We find the following factorizations:

$$\frac{2x-4}{10}=\frac{2(x-2)}{10} \qquad \frac{4y+6y^2}{8y}=\frac{2y(2+3y)}{8y} \qquad \frac{x-3}{8x-24}=\frac{x-3}{8(x-3)}$$

$$\frac{a^2-25}{a+5}=\frac{(a+5)(a-5)}{a+5} \qquad \frac{x^2+x-20}{5+x}=\frac{(x+5)(x-4)}{5+x}$$

Notice that the numerator and denominator in each fraction have common factors. In some cases these are just constants or variables. In others, entire binomials are identical in the parts of the fraction. The expressions are simplified by canceling common factors as follows:

$$\frac{2(x-2)}{10}=\frac{x-2}{5} \qquad \frac{2y(2+3y)}{8y}=\frac{2+3y}{4} \qquad \frac{(x-3)}{8(x-3)}=\frac{1}{8}$$

$$\frac{(a+5)(a-5)}{(a+5)}=a-5 \qquad \frac{(x+5)(x-4)}{(5+x)}=x-4$$

The major points to consider when simplifying algebraic fractions are these:

1) In the first example, 2 is common to all three terms in the original fraction. That is, 2 divides both $2x$ and 4 in the numerator, and 2 divides 10 in the denominator. In the second example, $2y$ is common to all the terms.

When in doubt as to whether a factor can be canceled, check to see if it divides every monomial term in the expression.

Only factors appearing in *each* term may be cancelled. Consider the fraction $\frac{6b+c}{3}$. While 3 is common to 6 in the numerator and 3 in the denominator, it cannot be canceled, because 3 is not a factor of c. In other words, $\frac{6b+c}{3} \neq 2b+c$.

To help solidify this idea, notice that $\frac{6b+c}{3}$ can be broken down as $\frac{6b+c}{3} = \frac{6b}{3} + \frac{c}{3}$.

Now we can plainly see that 3 can only be factored out of the $\frac{6b}{3}$ term and not the $\frac{c}{3}$ term, to give a result of $2b + \frac{c}{3}$.

2) In each of the last three examples, the numerator and denominator have common binomial factors. The key to this kind of problem is to see if any of the polynomials involved can be factored into smaller polynomials. If so, it is quite possible that common binomial—or even trinomial—factors will then appear and the factorization can proceed with cancellations.

Subtraction is not commutative, so we may not reverse a and b in order to cancel the binomials in full. However, factoring –1 out of either numerator or denominator gives this: $\frac{a-b}{b-a} = 1$.

3) In the final example, the $x + 5$ and $5 + x$ cancel. This is true, of course, because of the commutative property of addition.

In general, $\frac{a+b}{b+a} = 1$, since $a+b = b+a$.

4) Keep in mind that $\frac{a-b}{b-a} \neq 1$. This is because subtraction is not commutative. Instead $\frac{a-b}{b-a} = -1$, as shown by the following factorization:

$$\frac{a-b}{b-a} = \frac{-1(-a+b)}{b-a} = \frac{-1(b-a)}{(b-a)} = -1.$$

Memorizing this fractional equivalent for −1 can be very helpful in simplifying a formula or equation.

Important Observation

$$\frac{a+b}{b+a} = 1 \qquad \frac{a-b}{b-a} = -1$$

EXAMPLE 13.9: ***Simplifying Algebraic Fractions***

Simplify each of the following algebraic fractions

a) $\frac{5x-20}{10}$ **b)** $\frac{x^2-36}{x+6}$

EXAMPLE 13.9: ***Simplifying Algebraic Fractions (Continued)***

c) $\dfrac{y^2-6y}{3y}$ **d)** $\dfrac{b^2-8b+7}{b-1}$

e) $\dfrac{3-m}{9m-27}$ **f)** $\dfrac{x^2+2x+1}{x^2-1}$

Solutions:

a) $\dfrac{5x-20}{10}=\dfrac{\overset{1}{\cancel{5}}(x-4)}{\underset{2}{\cancel{10}}}=\dfrac{x-4}{2}$ **b)** $\dfrac{x^2-36}{x+6}=\dfrac{\cancel{(x+6)}(x-6)}{\cancel{(x+6)}}=x-6$

c) $\dfrac{y^2-6y}{3y}=\dfrac{\cancel{y}(y-6)}{3\cancel{y}}=\dfrac{y-6}{3}$ **d)** $\dfrac{b^2-8b+7}{b-1}=\dfrac{(b-7)\cancel{(b-1)}}{\cancel{(b-1)}}=b-7$

e) $\dfrac{3-m}{9m-27}=\dfrac{(3-m)}{9(m-3)}=\dfrac{-1\cancel{(m-3)}}{9\cancel{(m-3)}}=\dfrac{-1}{9}$

f) $\dfrac{x^2+2x+1}{x^2-1}=\dfrac{\cancel{(x+1)}(x+1)}{\cancel{(x+1)}(x-1)}=\dfrac{x+1}{x-1}$

Multiplication and Division of Algebraic Fractions

Recall that when numerical fractions are multiplied or divided, we can sometimes cancel common factors before performing the operation. For example,

$$\frac{4}{7}\cdot\frac{3}{16}=\frac{\overset{1}{\cancel{4}}}{7}\cdot\frac{3}{\underset{4}{\cancel{16}}}=\frac{3}{28}.$$

The same idea applies to algebraic expressions under multiplication and division.

We begin with multiplication and division of monomials. The basic rules of fraction multiplication, as well as rules of exponents, apply here. For example,

$$\frac{x^2}{2}\cdot\frac{4y}{3xy^3}=\frac{4x^2y}{6xy^3}\text{, which reduces to }\frac{2x}{3y^2}.$$

The x variables are cross cancelled, as are the coefficients. Reduction of the y variables *within* the second fraction also occurs. Whenever possible, cancellation should be completed before performing a multiplication.

When polynomials are involved, the common factors across the fractions or within the fractions may not be obvious. As in the previous section, it is often necessary to factor a polynomial before cancellation can take place.

EXAMPLE 13.10A: ***Simplifying Algebraic Fractions***

CAUTION: Do not cancel terms within a binomial! The x terms do not cancel. Nor do the 1s.

Simplify $\frac{x^2-81}{x+1}\cdot\frac{x^2-1}{x-9}$

Solution:

$$\frac{x^2-81}{x+1}\cdot\frac{x^2-1}{x-9}=\frac{(x+9)(x-9)}{(x+1)}\cdot\frac{(x+1)(x-1)}{(x-9)}=(x+9)(x-1)$$

EXAMPLE 13.10B: ***Simplifying Algebraic Fractions***

Simplify $\frac{a^2+4a-5}{2a+4}\cdot\frac{6a^2-24}{a+5}$

Solution:

$$\begin{aligned}6a^2-24&=6(a^2-4)\\&=6(a+2)(a-2)\end{aligned}$$

$$\frac{a^2+4a-5}{2a+4}\cdot\frac{6a^2-24}{a+5}=\frac{(a+5)(a-1)}{2(a+2)}\cdot\frac{6(a+2)(a-2)}{(a+5)}$$

$$=\frac{6(a-1)(a-2)}{2}=3(a-1)(a-2)$$

The same principle applies to division of algebraic fractions. After the inversion and change of operation to multiplication, we seek common factors to cross cancel or to cancel within a fraction.

EXAMPLE 13.10C: ***Simplifying Algebraic Fractions***

Simplify $\frac{y^2-25}{y^2-9}\div\frac{y+5}{y^2+6y+9}$

Solution:

$$\frac{y^2-25}{y^2-9}\div\frac{y+5}{y^2+6y+9}=\frac{y^2-25}{y^2-9}\cdot\frac{y^2+6y+9}{y+5}$$

$$=\frac{(y+5)(y-5)}{(y+3)(y-3)}\cdot\frac{(y+3)(y+3)}{(y+5)}$$

$$=\frac{(y-5)(y+3)}{(y-3)}$$

The operation is changed to multiplication upon inversion of the second fraction. Then the various polynomials are factored and cancellation is performed.

EXAMPLE 13.10D: ***Simplifying Algebraic Fractions***

Simplify $\frac{2x^2+15x-77}{5}\cdot\frac{20}{x^2-121}$

Solution:

$$\frac{2x^2+15x-77}{5}\cdot\frac{20}{x^2-121}=\frac{2x^2+15x-77}{1}\cdot\frac{4}{(x+11)(x-11)}$$

Before proceeding, look at $2x^2+15x-77$ and determine if either of the factors already in the denominator is a factor of $2x^2+15x-77$.

By observation we can see that $(x+11)$ is a factor:

$$(2x-7)(x+11)=2x^2+15x-77$$

Thus,

$$\frac{2x^2+15x-77}{5}\cdot\frac{20}{x^2-121}=\frac{(2x-7)(x+11)}{1}\cdot\frac{4}{(x+11)(x-11)}=\frac{4(2x-7)}{(x-11)}$$

Addition and Subtraction of Algebraic Fractions

Now we look at addition and subtraction of algebraic fractions. Algebraic fractions, like purely numerical fractions, cannot be added if they have unlike denominators. A least common denominator (LCD) must be found first. Recall that an LCD is a least common multiple (LCM) among denominators.

Finding an LCM among algebraic expressions is more involved than doing so among integers. The exponents attached to variables as well as monomial and binomial factors of a polynomial must be taken into account. Table 13.1 offers a variety of typical situations encountered when seeking an LCM for algebraic expressions.

TABLE 13.1: Finding an LCM for Algebraic Fractions

Expressions	LCM
$5, x, 2x$	$10x$ It is the smallest expression that 5, x, and $2x$ divide.
$x+1, x$	$x(x+1)$ The caution here is that x does not divide $x+1$.
$x-1, x+4$	$(x-1)(x+4)$ Since neither expression is a factor of the other, the LCM is their product.
$x+6, x^2-36$	x^2-36 $x+6$ is a factor of x^2-36, which factors as $(x+6)(x-6)$.
$x^2-2x-3, x+1$	$x+1$ $x+1$ is a factor of x^2-2x-3, which factors as $(x+1)(x-3)$.

EXAMPLE 13.11: ***Combining Algebraic Fractions***

Simplify $\frac{x+1}{3x}-\frac{x-3}{2}+\frac{x}{4}$

Solution:

The LCM of $3x$, 2, and 4 is $12x$. This is the smallest monomial that $3x$, 2, and 4 divide. Proceeding as we would with numerical fractions, we adjust all fractions so that each has LCD $12x$ for it denominator.

$$\frac{x+1}{3x}\cdot\frac{4}{4}-\frac{x-3}{2}\cdot\frac{6x}{6x}+\frac{x}{4}\cdot\frac{3x}{3x}=\frac{4x+4}{12x}-\frac{6x^2-18x}{12x}+\frac{3x^2}{12x}$$

Multiplication of numerator and denominator by the same expression is the same as multiplying by 1: the form—not the value—of the original fraction is changed.

Proceed with the subtraction:

$$\begin{aligned}\frac{4x+4}{12x}-\frac{6x^2-18x}{12x}+\frac{3x^2}{12x}&=\frac{4x+4-6x^2-(-18x)+3x^2}{12x} &&\text{Distribute minus sign}\\ &=\frac{22x-3x^2+4}{12x} &&\text{Combine like terms}\\ &=\frac{-3x^2+22x+4}{12x} &&\text{Rearrange by descending exponent of variable}\end{aligned}$$

Since no common factors can be cancelled between the numerator and denominator, this is the final answer — no further simplification is possible.

EXAMPLE 13.12: ***Addition of Algebraic Fractions***

Simplify $\frac{x-1}{x+2}+\frac{x+3}{4x}$.

Solution:

The LCD is the product of $x + 2$ and $4x$ since the denominators do not have any common factors. So, to get a common denominator we multiply each fraction, top and bottom, by the other fraction's denominator:

$$\begin{aligned}\frac{x-1}{x+2}\cdot\frac{4x}{4x}+\frac{x+3}{4x}\cdot\frac{x+2}{x+2}&=\frac{(x-1)(4x)}{(x+2)(4x)}+\frac{(x+3)(x+2)}{(x+2)(4x)}\\ &=\frac{4x^2-4x}{(x+2)(4x)}+\frac{x^2+5x+6}{(x+2)(4x)}\\ &=\frac{5x^2+x+6}{(x+2)(4x)}\end{aligned}$$

Notice that the numerator in the final answer cannot be factored and therefore no further simplification can be done.

We could then write our final answer as $\frac{5x^2+x+6}{4x^2+8x}$.

EXAMPLE 13.13A: *Subtraction of Algebraic Fractions*

Simplify $\frac{7}{x-1}-\frac{3}{x+2}$

Solution:

As in the previous example, the LCD is the product of the denominators, $(x-1)(x+2)$.

$$\frac{7}{x-1}\cdot\frac{x+2}{x+2}-\frac{3}{x+2}\cdot\frac{x-1}{x-1}=\frac{7(x+2)}{(x-1)(x+2)}-\frac{3(x-1)}{(x+2)(x-1)}$$

$$=\frac{7x+14-3x-(-3)}{(x+2)(x-1)}$$

$$=\frac{4x+17}{(x+2)(x-1)}$$

Since no common factors can be cancelled between the numerator and denominator, this is the final answer — no further simplification is possible.

EXAMPLE 13.13B: *Subtraction of Algebraic Fractions*

Simplify $\frac{6x}{x-3}-\frac{2x}{x^2-9}$

Solution:

The first thing to look for when adding or subtracting algebraic fractions with quadratic denominators is whether either denominator can be factored. x^2-9 factors to $(x+3)(x-3)$. Since the other denominator, $x-3$, is one of these factors, the LCD is x^2-9. We need only adjust the first fraction to combine the fractions.

$$\frac{6x}{x-3}\cdot\frac{x+3}{x+3}-\frac{2x}{(x-3)(x+3)}=\frac{6x^2+18x}{(x-3)(x+3)}-\frac{2x}{(x-3)(x+3)}$$

$$=\frac{6x^2+16x}{(x-3)(x+3)}$$

$$=\frac{2x(3x+8)}{(x-3)(x+3)}$$

EXAMPLE 13.14: ***Addition and Subtraction of Algebraic Fractions***

Simplify $\frac{7x}{x^2-2x-24}+\frac{1}{x+4}-\frac{2x+3}{x^2-36}$

Solution:

This is a somewhat more challenging problem than the previous ones. We proceed as follows:

STEP 1: Factor all denominators fully to see which individual factors are present.

$$\frac{7x}{x^2-2x-24}+\frac{1}{x+4}-\frac{2x+3}{x^2-36}=\frac{7x}{(x-6)(x+4)}+\frac{1}{(x+4)}-\frac{2x+3}{(x+6)(x-6)}$$

STEP 2: List the distinct factors. They are $(x + 6)$, $(x - 6)$, and $(x + 4)$. Thus the LCD is the product of these,

$$(x + 6)(x - 6)(x + 4).$$

STEP 3: *Do not expand this expression*! Instead, refer to it in its factored form to decide which factor or factors each denominator needs in order to be converted to the LCD. Then multiply each fraction, top and bottom, by the necessary factor or factors.

$$\frac{7x}{(x-6)(x+4)}\cdot\frac{(x+6)}{(x+6)}+\frac{1}{(x+4)}\cdot\frac{(x+6)(x-6)}{(x+6)(x-6)}-\frac{(2x+3)}{(x+6)(x-6)}\cdot\frac{(x+4)}{(x+4)}$$

STEP 4: Expand each numerator.

$$\frac{7x^2+42x}{(x-6)(x+4)(x+6)}+\frac{x^2-36}{(x+4)(x+6)(x-6)}-\frac{(2x^2+11x+12)}{(x+6)(x-6)(x+4)}$$

STEP 5: Combine numerators over the LCD.

$$\frac{7x^2+42x+x^2-36-2x^2-11x-12}{(x-6)(x+4)(x+6)}=\frac{6x^2+31x-48}{(x-6)(x+4)(x+6)}$$

STEP 6: See if the final numerator factors and if any of its factors match those of the denominator. We find it does not factor, so we have a final answer.

Long Multiplication of Polynomials

Here is another way to multiply polynomials where more than two are involved. Consider

$$(x - 6)(x + 4)(x + 6)$$

Multiplying the first two binomials using **FOIL** leaves us with a trinomial multiplied by a binomial. We showed how to do this in Examples 13.3A and 13.3B.

Another method is to set up the multiplication as if it were a numerical long multiplication, with the longer polynomial on top:

$$\begin{array}{r} x^2-2x-24 \\ \underline{x+6} \end{array}$$

Unlike numerical long multiplication, begin by multiplying the leftmost expressions. First multiply x by x^2, then multiply x by $-2x$, then multiply x by -24. Write each product in turn in the first row under the problem, in descending order of the variables.

$$\begin{array}{r} x^2-2x-\ \ 24 \\ \underline{x+\ \ \ 6} \\ x^3-2x^2-24x \end{array}$$

Continue with the next term in $(x+6)$, the number 6, multiplying again with each term in the quadratic from left to right. First multiply 6 by x^2, then multiply 6 by $-2x$, then multiply 6 by -24. Each of these products is written beneath the first row, again in descending order of the variables, lining up the x^2 terms in one column, the x terms in the next column, and the constants in the last column.

$$\begin{array}{l} x^2-2x-24 \\ \underline{\quad\ \ x+\ 6} \\ x^3-2x^2-24x \\ \underline{\quad\ \ 6x^2-12x-144} \end{array}$$

Add the two rows:

$$\begin{array}{rrrr} & x^2 & -2x & -24 \\ & & x & +6 \\ \hline x^3 & -2x^2 & -24x & \\ & 6x^2 & -12x & -144 \\ \hline x^3 & +4x^2 & -36x & -144 \end{array}$$

Thus, the final answer is $x^3+4x^2-36x-144$.

Polynomials can also be multiplied on one line. This is not unlike FOIL, and in fact, is an extension of the idea behind FOIL.

For example, in the product $(3x^2+10x-5)(x-1)$, multiply each term in the trinomial by each term in the binomial and combine terms:

$$\begin{aligned} &(3x^2+10x-5)(x-1)= \\ &\qquad =3x^2(x-1)+10x(x-1)-5(x-1) \\ &\qquad =3x^3-3x^2+10x^2-10x-5x+5 \\ &\qquad =3x^3+7x^2-15x+5 \end{aligned}$$

Long Division of Polynomials

Any rational expression like $\frac{x^2+2x-3}{x+3}$ implies polynomial division. Factoring reveals if any common factors between the numerator and denominator can be canceled. After cancellation a polynomial may be left in place of the rational expression:

$$\frac{x^2+2x-3}{x+3}=\frac{(x+3)(x-1)}{(x+3)}=x-1$$

Here, the binomial that remains is the *quotient* of two other polynomials. However, this process does not always yield a polynomial quotient. As an analogy, consider the fraction $\frac{24}{10}$, which reduces to $\frac{12}{5}$. This also implies a division, namely 12 ÷ 5. Completion of this division yields the answer $2\frac{2}{5}$. Similarly, division of polynomials results in polynomial remainders when the denominator expression is not a factor of the numerator expression.

We will not explore the process here, but refer the reader to texts in college algebra. We also note that long division of polynomials is helpful in situations where (1) the factorization of a polynomial is not obvious; (2) the polynomial to be divided is of a degree greater than 2; (3) a polynomial is not factorable but an answer is desired in a form that is close to a polynomial form.

Preparing to Solve Quadratic Equations

The topic of factoring polynomials is crucial to *solving quadratic equations*. Though we presented many types of factoring situations, we will see when we work with solving quadratic equations many instances where factorization is not as straightforward as the examples given here. In those cases we will rely on application of the quadratic formula, which is presented in the next chapter.

EXERCISES

13.9 Perform the indicated operations and simplify.

a) $\frac{20a}{-4}$ **b)** $\frac{35cd}{7cd}$ **c)** $\frac{-32rst^2}{4rs}$

d) $\frac{128a^3bc^2}{-8abc}$ **e)** $\frac{6(-4a^3b^2)}{-12a}$ **f)** $\frac{2a(-3ab^2)}{ab}$

g) $\frac{21y^4}{y}$ **h)** $\frac{x^5}{x^3}$ **i)** $\frac{32a^4}{-8a^3}$

j) $\frac{(5a+15c)}{5}$ **k)** $\frac{(P+Prt)}{P}$ **l)** $\frac{(45x^2-15)}{15}$

m) $\dfrac{-56x^3y^4}{-7xy^2}$ **n)** $\dfrac{(x^3-7x^2+10x)}{-x}$ **o)** $\dfrac{(d^2+9d+14)}{(d+7)}$

p) $\dfrac{(x^2-3x-10)}{(x-5)}$ **q)** $\dfrac{(15a^2b^2-8ab+1)}{(3ab-1)}$ **r)** $\dfrac{(9y^2+24y+16)}{(3y+4)}$

13.10 Perform the indicated operations and simplify.

a) $\dfrac{3x^2}{4y}\cdot\dfrac{7y^3}{6x^2}$ **b)** $\dfrac{15xyz^2}{12yw}\cdot\dfrac{10yw^2}{25xz}$

c) $\dfrac{x}{x^2-9}\cdot\dfrac{x^2-3x}{x}$ **d)** $\dfrac{16x^2-4}{8x^2-6x+1}\cdot\dfrac{6x^2-5x+1}{16x^2+4x}$

e) $\dfrac{2x}{x^2-4}\cdot\dfrac{x^2-4x+4}{5x}$ **f)** $\dfrac{5}{3x^2}-\dfrac{3}{4x}+\dfrac{6}{x}$

g) $\dfrac{6}{x^2y}+\dfrac{8}{xy^2}$ **h)** $\dfrac{5x}{x^2-4}+\dfrac{3x}{x^2+6x+9}$

i) $\dfrac{x^2+5x+6}{x+3}\cdot\dfrac{x^2+6x+9}{x+2}$ **j)** $\dfrac{4}{x-3}+\dfrac{3}{x+2}$

k) $\dfrac{3}{x^2}+\dfrac{2}{3x}+\dfrac{5}{2x}$ **l)** $\dfrac{3a}{4xy}+\dfrac{4b}{3xy^2}-\dfrac{2a}{2x^2y}$

m) $\dfrac{x^3-4x}{x^3-9x^2+14x}\cdot\dfrac{5x^2-37x+14}{5x^3-2x}$ **n)** $\dfrac{3x^3+15x}{x^2-25}\cdot\dfrac{3x^2-17x+10}{15x^2-10x}$

o) $\dfrac{2x}{x^2-9}+\dfrac{4x}{x^2-3x}$ **p)** $\dfrac{x-2}{x^2+5x+6}-\dfrac{x-3}{x^2+6x+9}$

q) $\dfrac{2x-1}{x^2-3x-4}+\dfrac{x+2}{x^2-1}$ **r)** $\dfrac{2}{3x}+\dfrac{3}{2x}-\dfrac{2}{3x^2}$

13.11 Perform the indicated division and simplify.

a) $\dfrac{10w^2}{x}\div\dfrac{5w}{x}$ **b)** $\dfrac{36xy}{2w}\div\dfrac{6xy^2}{xw}$

c) $\dfrac{x^3-2x^2}{x^2-x-2}\div\dfrac{x^2-4}{x^2-4x+4}$ **d)** $\dfrac{5x^2-2x}{5x^2-37x+14}\div\dfrac{x^3-4x}{x^2-5x-14}$

e) $\dfrac{x^2-3x}{x^2+x-6}\div\dfrac{x^2-9}{x^2-x-2}$ **f)** $\dfrac{x^2-25}{3x^2+15x}\div\dfrac{3x^2-17x+10}{15x^2-10x}$

14

SOLVING QUADRATIC EQUATIONS

We have seen how to solve linear equations, or equations of first degree, or degree 1, in which the highest exponent on the variable x is 1. Now we turn our attention to second degree equations, in which the highest exponent on the variable is 2, such as x^2. These equations of degree 2 are called *quadratic equations*.

The word "quadratic" contains the root word "quadras," the Latin word for "square."

14.1 Solving Quadratic Equations of Form x^2 = Constant

Consider the following quadratic equations:

$$x^2 = 9 \qquad \text{and} \qquad x^2 - 1 = 0$$

The first two equations can be easily solved by inspection. In the first equation, the obvious solution is $x = 3$, since $3^2 = 9$. However, this is not a complete solution because $x = -3$ also satisfies this equation since $(-3)^2 = 9$. In the second equation we can also find the solutions by simply examining the equation. We find that $x = 1$ and $x = -1$ both satisfy the equation, because $1^2 - 1 = 0$, and $(-1)^2 - 1 = 0$.

Solving $x^2 = c$, c Positive or Zero

As was true in equations of degree 1, to solve a quadratic, we must isolate x. Again consider $x^2 = 9$. The x^2 stands alone; so to isolate x we need only take the square root of both sides of the equation:

$$\sqrt{x^2} = \sqrt{9}$$

Recall that in Chapter 6 we said that a radical standing alone denotes only *positive* roots, as in $\sqrt{9} = 3.$ However, when a radical appears in an *equation*, as one does here, the correct interpretation is that both positive and negative roots are implied. We take *both* roots because a quadradic equation represents a parabola whose graph may intersect the x-axis twice. Thus,

$$x = \pm\sqrt{9} = \pm 3$$

Equations in x^2 have, at most, two solutions; equations in x have, at most, one solution.

Solving $x^2 - c = 0$, c Positive or Zero

Consider the equation $x^2 - 1 = 0$ discussed earlier. In this equation, x is isolated in the series of steps shown:

$$x^2 - 1 = 0$$
$$x^2 - 1 + 1 = 0 + 1$$
$$x^2 = 1$$
$$\sqrt{x^2} = \pm\sqrt{1}$$
$$x = \pm 1$$

As before, the constant and variable terms are separated on opposite sides of the equation.

EXAMPLE 14.1A: ***Solving a Quadratic of Form $x^2 = c$***

Solve for x in the equation $x^2 - 7 = 1$.

Solution:

Combine constants on the right hand side of the equation:

$$x^2 - 7 = 1$$
$$x^2 - 7 + 7 = 1 + 7$$
$$x^2 = 8$$

Isolate x by taking the square root of both sides:

$$\sqrt{x^2} = \pm\sqrt{8} = \pm\sqrt{4 \cdot 2} = \pm\sqrt{4}\sqrt{2} = \pm 2\sqrt{2}$$

Thus, the solution consists of two values, $x = 2\sqrt{2}$ and $-2\sqrt{2}$.

EXAMPLE 14.1B: ***Solving a Quadratic of Form $(x - a)^2 = c$***

Solve for x in the equation $(x - 5)^2 = 1$.

Solution:

Extract the root of $(x - 5)^2$ first, in order to get to the variable x. We do this the same way that we extract the root of x^2: by taking the square root of both sides of the equation.

$$(x - 5)^2 = 1$$
$$\sqrt{(x-5)^2} = \pm\sqrt{1}$$
$$x - 5 = \pm 1$$

When taking roots to solve equations, the radical must be applied to the whole side of the equation – not to individual terms.

EXAMPLE 14.1B: ***Solving a Quadratic of Form $(x - a)^2 = c$ (Continued)***

We need to apply the additive inverse of −5 to both sides of the equation. Before doing so, we separate the intermediate solution into two parts, $x - 5 = 1$ and $x - 5 = -1$. Solving the equations separately,

$$\begin{array}{cc} x-5=1 & x-5=-1 \\ x-5+5=1+5 & x-5+5=-1+5 \\ x=6 & x=4 \end{array}$$

Thus, the solution consists of two values, $x = 4$ and $x = 6$.

These values can be checked by substituting them into the original equation and verifying that the resulting expression is true.

Solving $x^2 = c$, c Negative

The solutions to the quadratic equations we have shown so far have been real numbers, both rational and irrational.

Solutions to quadratic equations can also have imaginary numbers. Consider the equation $x^2 + 100 = 0$. To solve this, we proceed as always, isolating x:

$$x^2 + 100 - 100 = 0 - 100 \quad \rightarrow \quad x^2 = -100 \quad \rightarrow \quad x = \pm\sqrt{-100}$$

Recall that square roots of *negative* real numbers are imaginary. By definition,

$$i = \sqrt{-1}$$

Thus, $\pm\sqrt{-100} = \pm\sqrt{(100)(-1)} = \pm\sqrt{100}\sqrt{-1} = \pm 10i$.

Imaginary numbers are necessary to solve certain quadratic equations, like this one, which is of the form $x^2 = c$ with $c < 0$.

Solving Quadric Equations of Form $x^2 + bx + c = 0$

We now proceed to the general quadratic equation, whose form we encountered in Chapter 13 where we were concerned with quadratic *expressions*. We begin by noting that a quadratic equation such as $x^2 - 1 = 0$, which we already solved by isolating x, may also be solved by *factoring*. Observe that $x^2 - 1$ is a difference of squares, which factors to $(x + 1)(x - 1)$. Thus, we can restate the problem as:

$$(x + 1)(x - 1) = 0$$

By the zero principle of multiplication, the product of two numbers is zero if either number is zero. Likewise the product of the two binomial factors is zero if either binomial has a value of zero. Consequently, $(x + 1)(x - 1) = 0$ is solved if either $(x + 1) = 0$ or $(x - 1) = 0$.

The zero principle of multiplication states that if $AB = 0$, then $A = 0$ or $B = 0$.

$$(x+1)(x-1)=0 \qquad \Rightarrow \qquad x+1=0 \text{ or } x-1=0$$

Therefore, $x = -1$ or $x = 1$.

The method for factoring trinomials of this type is explained in Chapter 13.

Employing factorization and the multiplication principle $AB = 0 \Rightarrow A = 0$ or $B = 0$ can solve many quadratic equations. For instance, the values of x that satisfy $x^2 + 3x - 4 = 0$ can be found by factoring the polynomial:

$$x^2 + 3x - 4 = 0 \Rightarrow (x+4)(x-1) = 0$$

Setting each factor equal to zero and solving:

$$\begin{aligned} x+4 &= 0 \qquad \Rightarrow \qquad x = -4 \\ x-1 &= 0 \qquad \Rightarrow \qquad x = 1 \end{aligned}$$

To use this method, the constant on the right of the equation must be zero. This sometimes means putting the quadratic equation in general form, $ax^2 + bx + c = 0$ by moving a non-zero constant to the left side of the equation, as the next example illustrates.

EXAMPLE 14.2: ***Solving a Quadratic Equation by Factoring***

Solve $x^2 + 9x + 8 = -6$ using factorization.

Solution: This quadratic equation is not in the general form $ax^2 + bx + c = 0$. The constant on the right side of the equation is –6, rather than zero. Adding 6 to both sides puts the equation in the correct form:

$$x^2 + 9x + 8 = -6$$

$$x^2 + 9x + 8 + 6 = -6 + 6$$

$$x^2 + 9x + 14 = 0$$

$$(x+2)(x+7) = 0$$

The equation is solved if either $(x+2) = 0$ or $(x+7) = 0$. Accordingly, the solutions to this quadratic equation are $x = -2$ and $x = -7$.

Check the solution by substituting the values found for x into the original equation, first one, then the other.

For $x = -2$:

$$x^2 + 9x + 8 = -6$$

$$(-2)^2 + 9(-2) + 8 = 4 - 18 + 8 = -6$$

$$-6 = -6 \qquad \text{Checks}$$

and for $x = -7$:

$$(-7)^2 + 9(-7) + 8 = 49 - 63 + 8 = -6$$

$$-6 = -6 \qquad \text{Checks}$$

Summary of Steps for Solving a Single-Variable Quadratic Equation

If the equation is a sum or difference of squares $(x^2 + c = 0$ or $x^2 - c = 0)$:

STEP 1: Isolate x^2 on the left and the constant on the right: $x^2 = c$, with c either positive or negative.

STEP 2: Take the square root of both sides of the equation: $\sqrt{x^2} = \sqrt{c}$

STEP 3: Solve for x: $x = \pm\sqrt{c}$

If the equation is a trinomial that can be factored using integer values $(ax^2 + bx + c = 0)$:

STEP 1: If the equation is not set equal to 0, rearrange terms so that all variables and constants other than 0 are on the left hand side of the equation.

STEP 2: Factor the quadratic expression on the left into two binomials.

STEP 3: Set each binomial factor equal to zero.

STEP 4: Solve for x.

EXERCISES

14.1 Solve the quadratic equations by isolating the variable.

a) $x^2 = 99$
b) $3x^2 = 192$
c) $-x^2 = -32$
d) $x^2 + 45 = 0$
e) $12x^2 - x - 1 = 0$
f) $x^2 - 50 = 24$
g) $2x^2 + 38 = 70$
h) $x^2 = -16$
i) $2x^2 = -162$
j) $x^2 + x - 12 = 0$
k) $x^2 + 4x + 3 = 0$
l) $x^2 + x - 56 = 0$

14.2 Solve the quadratic equations by isolating the variable.

a) $3x^2 + x - 2 = 0$
b) $x^2 - 13x + 42 = 0$
c) $6x^2 + 8x - 8 = 0$
d) $10x^2 - 17x + 3 = 0$
e) $x^2 + 15x - 54 = 0$
f) $x^2 - 4x + 3 = 0$
g) $x^2 + x = 12$
h) $3x^2 + 4x = 4$
i) $2x^2 - 8x = -6$
j) $9x^2 + 3x = 6$
k) $x^2 - 10x = 24$

14.2 The Quadratic Formula

The *quadratic formula* provides another way to find the values of x that satisfy $ax^2 + bx + c = 0$. We use the quadratic formula when we can not easily factor a quadratic equation to set it equal to zero. It yields the roots of $ax^2 + bx + c = 0$ directly when the coefficients a, b, and c are substituted into the formula:

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As we show in the examples that follow, we can solve any quadratic equation using this formula. First, however, we have to put the equation into its general form, $ax^2 + bx + c = 0$, if the problem is not in this form to begin with.

EXAMPLE 14.3A: ***Using the Quadratic Formula***

Use the quadratic formula to solve the equation $2x^2 + 5x = 3$.

Solution: The equation must first be put in general form, so 3 is subtracted from both sides to yield the equivalent form:

$$2x^2 + 5x = 3$$

$$2x^2 + 5x - 3 = 3 - 3$$

$$2x^2 + 5x - 3 = 0$$

The coefficients in the equation are $a = 2$, $b = 5$, and $c = -3$. Substituting these into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{-5 \pm \sqrt{25 + 24}}{4} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4}$$

Thus, $x = \frac{-5+7}{4} = \frac{1}{2}$ and $x = \frac{-5-7}{4} = -3$

Substituting each value, one at a time, into the original equation, we find these are the correct solutions.

Note: Although many equations with leading coefficients not equal to 1 are difficult to factor, this one is not. We could have solved the problem by factoring the left side and setting each factor equal to zero. That is,

$$2x^2 + 5x - 3 = (2x - 1)(x + 3) = 0$$

$$2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$

$$x + 3 = 0 \quad \Rightarrow \quad x = -3$$

The point to be taken is that the quadratic formula can be used in all situations, whether factoring is straightforward or not.

The development and proof of the quadratic formula can be found in many college algebra books. The proof requires a technique called "completing the square," which is not covered here.

EXAMPLE 14.3B: *Using the Quadratic Formula*

Use the quadratic formula to solve the equation $4x^2-3=0$.

Solution:

In general form, this equation becomes $4x^2+0x-3=0$.
The coefficients are $a = 4$, $b = 0$, and $c = -3$. Putting these values into the quadratic formula gives the following:

$$x=\frac{-0\pm\sqrt{0^2-4(4)(-3)}}{2(4)}=\frac{\pm\sqrt{48}}{8}=\frac{\pm\sqrt{16}\sqrt{3}}{8}=\frac{\pm4\sqrt{3}}{8}=\frac{\pm\sqrt{3}}{2}$$

The solution to this quadratic equation could also be found fairly easily using the factorization technique. The square root of $4x^2$ is $2x$, and the square root of 3 is $\sqrt{3}$. We can readily see that:

$$4x^2-3=(2x+\sqrt{3})(2x-\sqrt{3})=0$$

$$2x+\sqrt{3}=0 \quad\Rightarrow\quad x=\frac{-\sqrt{3}}{2}$$

$$\text{or}\quad 2x-\sqrt{3}=0 \quad\Rightarrow\quad x=\frac{\sqrt{3}}{2}$$

EXAMPLE 14.3C: *Using the Quadratic Formula*

Use the quadratic formula to solve the equation $x^2+x+2=0$.

Solution:

The coefficients of $x^2 + x + 2 = 0$ are $a = 1$, $b = 1$, $c = 2$. Substituting these into the quadratic formula:

$$\begin{aligned}x&=\frac{-1\pm\sqrt{(1)^2-4(1)(2)}}{2(1)}\\&=\frac{-1\pm\sqrt{-7}}{2}=\frac{-1}{2}\pm\frac{\sqrt{-7}}{2}=\frac{-1}{2}\pm\frac{i\sqrt{7}}{2}\end{aligned}$$

Hence, $$x=\frac{-1}{2}+\frac{i\sqrt{7}}{2}\ \text{or}\ \frac{-1}{2}-\frac{i\sqrt{7}}{2}.$$

A number with a real and an imaginary part is called a complex number.

The solution to Example 14.3C is *complex*. A complex number has a real part and an imaginary part. The factorization of $x^2 + x + 2 = 0$ would be very difficult to discern by inspection. It is for this kind of quadratic equation that the formula is indispensable.

EXERCISES

14.3 Solve each equation using the quadratic formula.

a) $x^2+6x-1=0$ **b)** $x^2+2x-4=0$ **c)** $5x^2-20x-5=0$

d) $x^2-12x=-8$ **e)** $2x^2-6x-2=0$ **f)** $x^2+5x+5=0$

g) $-x^2-x+4=0$ **h)** $4x^2+8x-5=0$ **i)** $2x^2-14x+1=0$

j) $9x^2+9x-2=0$ **k)** $x^2-50=24$ **l)** $x^2+16x-28=0$

m) $2x^2+38=70$ **n)** $2x^2+5x-3=0$ **o)** $x^2+3x+1=0$

14.4 Solve each equation using the quadratic formula.

a) $x^2-4x+2=0$ **b)** $x^2+15x-54=0$ **c)** $x^2-6x+9=0$

d) $x^2+x+3=0$ **e)** $2x^2+3x-4=0$ **f)** $x^2-4x+3=0$

g) $2x^2-8x=-6$ **h)** $10x^2-17x+3=0$ **i)** $4x^2+25=-20x$

j) $x^2+16x=-64$ **k)** $3x^2-4x-2=0$ **l)** $5x^2-21x+4=0$

m) $3x^2-x-4=0$ **n)** $x^2-13x+42=0$ **o)** $3x^2-9x+3=0$

14.5 Solve each equation using the quadratic formula.

a) $x^2+\frac{x}{2}-1=0$ **b)** $2x^2+6x=-\frac{9}{2}$ **c)** $6x^2+8x-8=0$

d) $x^2-4x-4=0$ **e)** $9x^2-6x+1=0$ **f)** $x^2+2x-\frac{1}{4}=0$

g) $6x^2+\frac{x}{12}+\frac{1}{12}=0$ **h)** $9x^2+3x=6$ **i)** $x^2=-16$

j) $2x^2=-162$ **k)** $-2x^2+\frac{x}{2}+\frac{3}{2}=0$ **l)** $x^2-\frac{x}{12}-\frac{1}{12}=0$

m) $x^2+x-12=0$ **n)** $x^2+4x+3=0$ **o)** $x^2+x-56=0$

14.6 Solve each equation using the quadratic formula.

a) $x^2-10x-24=0$ **b)** $3x^2+x-2=0$ **c)** $x(4x-1)=2+x$

d) $x^2+x=12$ **e)** $3x^2+4x=4$ **f)** $x^2-10x=24$

g) $-2x+6(x^2-5)+6=14x$

15

LINES, ANGLES, POLYGONS, AND SOLIDS

Technical shop mathematics relies on *plane geometry*, the study of the properties and relationships of figures including points, lines, and polygons, which lie in the two-dimensional plane, and polyhedrons, which exist in three-dimensional space.

The word "geometry" is from the Greek words meaning "the measure of the earth."

15.1 Points, Lines, and Planes

Points

We begin with *points*, the most fundamental of geometric figures. A point is *zero-dimensional*—it has no length, width, or thickness. However, to draw a picture of a point, we usually draw a dot, which necessarily has dimension—a pencil mark, for example, or a pixel on a computer screen. Figure 15.1 shows a typical illustration of a point.

Points as locations are studied in analytic geometry, which provides ways to graph plane figures in the Cartesian plane and describe them with algebraic methods.

FIGURE 15.1 ***Point A is zero-dimensional***

A single point can denote a location in space. When connected, points form one-, two-, and three-dimensional geometric figures.

Lines and Line Segments

We intuitively understand what fundamental geometric figures look like. Defining them is not so easy. Let us agree that a *line* is a set of points existing in one dimension: length. A line's length is infinite. Lines may be either straight or curved. A curved line continually changes direction. A straight line does not change direction. In this discussion, we always mean a straight line when we use the word "line."

Although an illustration of a line has finite length and obvious width, a true line has infinite length and no width. A line can be drawn through any two points. A drawing of a line is shown in Figure 15.2. Two letters name two arbitrary points on the line. Arrows at either end imply that the points making up the line extend infinitely in either direction. We write the name of this line as $\overleftrightarrow{AB}$ or "line *AB*," but not "line $\overleftrightarrow{AB}$."

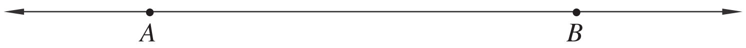


FIGURE 15.2 ***Line AB***

This line may also be named $\overleftrightarrow{BA}$. The order of the letters does not matter. Sometimes single lowercase letters are used in drawings to label lines, as in "line *m*" shown in Figure 15.3.

FIGURE 15.3 ***Line m***

Skew lines are non-intersecting lines lying in different planes.

Parallel lines are lines in the same plane that have no points in common and are everywhere equidistant from each other. In other words, parallel lines are lines that are oriented in the same direction so they never intersect or touch one another. The symbol for parallel lines is ||. Figure 15.4 shows parallel lines, *m* and *n*.

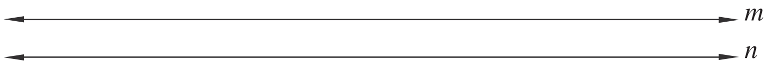


FIGURE 15.4 ***Parallel lines***

Perpendicular lines intersect in a plane to form 90° angles, or right angles, as defined later in this chapter. A plumb-bob on the end of a length of cord is an ancient method of erecting perpendiculars in building many kinds of structures. The symbol for perpendicular lines is ⊥. In diagrams, perpendicular lines are often indicated by a square box. Figure 15.5 shows perpendicular lines *m* and *n*.

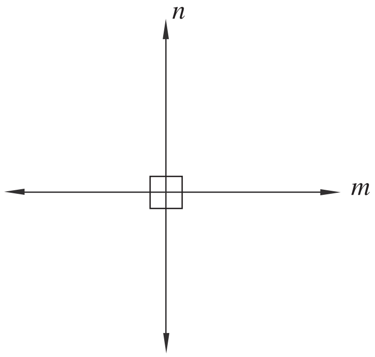


FIGURE 15.5 ***Perpendicular lines***

Lines relate in one of three ways:

1) they intersect at one point,

2) they do not intersect,

3) they intersect at every point (such lines are said to be *collinear*).

A *line segment* is a portion of a line. It is named by its endpoints. Figure 15.6 depicts line segment *AB* (also written $\overline{AB}$). Any line segment has finite length.

The shortest path between two points is a line segment. Used informally, the word "line" often refers to a "line segment."

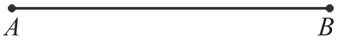


FIGURE 15.6 ***Line segment AB***

Rays

A *ray* is a portion of a line extending in one direction from a point. A ray is sometimes called a "half-line." A ray that begins at point *A* and progresses through point *B*, which can be any point on the ray, is designated as $\overrightarrow{AB}$ or "ray *AB*," but not "ray $\overrightarrow{AB}$." Figure 15.7 gives an example of a ray.

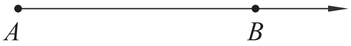


FIGURE 15.7 ***Ray AB***

Like a line, a ray is infinite in length. It is always named with its starting point first.

Angles

An *angle* is formed by two rays meeting at their endpoints, as in Figure 15.8. The angle shown is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The common point A is the vertex of the angle, and the rays are the angle's sides. When lines or line segments intersect, more than one angle is formed.

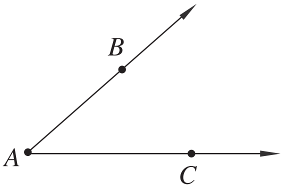


FIGURE 15.8 ***∠A, or∠BAC, or ∠CAB***

An angle is identified by its vertex or by one point on each side of the angle and the vertex. The order of the points does not matter as long as the vertex is named between the other points. Hence, in Figure 15.8, ∠*A* is also called ∠*BAC* or ∠*CAB*.

If two or more angles have vertices at a common point, each angle must be identified with both side points and vertex so that it is clear which angle is being cited, as shown in Figure 15.9.

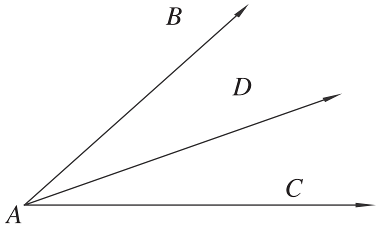


FIGURE 15.9 ***Both ∠BAD and ∠DAC have vertex at A***

Become familiar with commonly used Greek letters, lower and upper cases, to become proficient in recognizing and pronouncing them.

Angles are sometimes identified by a number or a Greek letter placed within or near the angle, for example $\angle\beta$ or ∠1 and ∠2 in Figure 15.10.

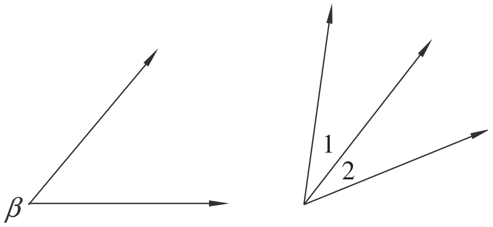
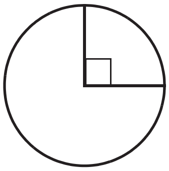


FIGURE 15.10 ***∠β, ∠1 and ∠2***

Adjacent angles are formed when, for example, three or more rays meet at a point. In Figure 15.10 ∠1 and ∠2 are adjacent angles.

A right angle is one quarter of a full revolution.



A *right angle* is an angle measuring 90° as shown in Figure 15.11. A right angle is indicated symbolically by a □ at the intersection of the perpendicular lines.

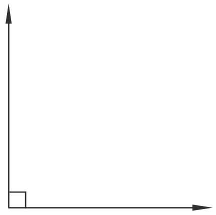


FIGURE 15.11 ***A right angle measures 90°***

A *straight angle* is the angle of a straight line measuring 180°, named for a point on the line that serves as the angle vertex, as in straight angle *A* shown in Figure 15.12.

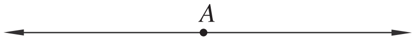


FIGURE 15.12 ***Straight angle A***

An *acute angle*, such as shown in Figure 15.13, is an angle of measure less than 90°.

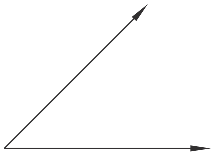


FIGURE 15.13 ***An acute angle***

An *obtuse angle*, such as shown in Figure 15.14, has measure greater than 90° but less than 180°.

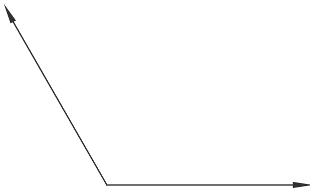
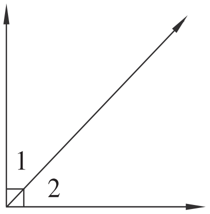


FIGURE 15.14 ***An obtuse angle***

Complementary angles are two angles whose measures sum to 90, the measure of a right angle. In Figure 15.15 $\angle 1$ and $\angle 2$ are complementary angles.

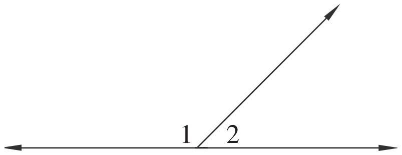


$\angle 1$ is the complement of $\angle 2$;

$\angle 2$ is the complement to $\angle 1$.

FIGURE 15.15 ***Complementary angles***

Supplementary angles are two angles whose measures sum to 180. When placed adjacent to each other, supplementary angles form a line. In Figure 15.16, $\angle 1$ and $\angle 2$ are supplementary.

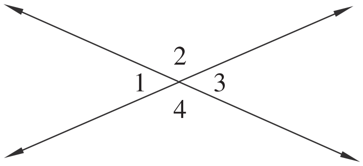


$\angle 1$ is the supplement of $\angle 2$;

$\angle 2$ is the supplement to $\angle 1$.

FIGURE 15.16 ***Supplementary angles***

Vertical angles are the opposite angles formed by intersecting lines. In Figure 15.17, $\angle 1$ and $\angle 3$ are vertical angles, as are $\angle 2$ and $\angle 4$.



$\angle 1$ and $\angle 3$ are vertical angles;

$\angle 2$ and $\angle 4$ are vertical angles.

FIGURE 15.17 ***Vertical angles***

Consider Figure 15.18, in which parallel lines *m* and *n* are cut by *transversal line t*. A transversal line intersects two or more other lines at points where the other lines are not intersecting.

Angles formed by parallel lines and a transversal.

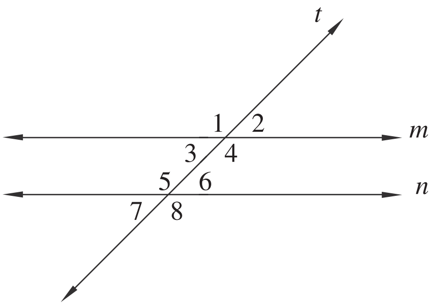


FIGURE 15.18 ***Angles formed when two lines are cut by a transversal***

There are many pairs of vertical angles in Figure 15.18. In addition, each angle is characterized with respect to its position in the figure relative to other angles and the lines themselves:

Name all pairs of corresponding angles in Figure 15.18 for practice.

$\angle 1$, $\angle 2$, $\angle 7$, $\angle 8$ are *exterior angles*.
$\angle 3$, $\angle 4$, $\angle 5$, $\angle 6$ are *interior angles*.
$\angle 1$ and $\angle 8$ are *alternate exterior angles*, as are $\angle 2$ and $\angle 7$.
$\angle 3$ and $\angle 6$ are *alternate interior angles*, as are $\angle 4$ and $\angle 5$.
$\angle 1$ and $\angle 5$ are *corresponding angles*, as are $\angle 4$ and $\angle 8$.

Planes

A handy rule to remember in the shop is that three points define a plane. The three points on which a tripod sits makes it very stable.

A *plane* is a two-dimensional figure. It extends infinitely in dimensions of length and width. It has no edges or thickness. A plane contains an infinite number of points and an infinite number of lines. In reality, objects modeled by flat planes are finite in these dimensions. For example, floors, tabletops, and panes of glass have finite dimensions.

Three points define a plane—as long as they do not all lie on one line. We name a plane by a single arbitrary point in the plane, for example, "plane *A*." Planes are usually represented in drawings by four-sided figures, as in Figure 15.19.

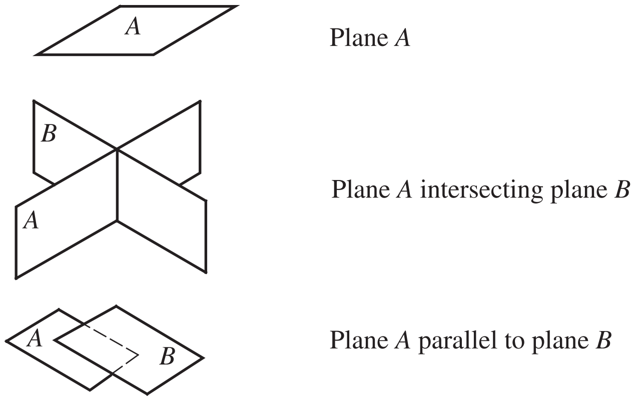


FIGURE 15.19 ***Planes***

Notice that parallel lines lie in the same plane.

EXERCISES

15.1 Draw a sketch for each of the following geometrical situations:

a) Line *AB* and line *CD* intersect at point *P*.
b) *E* is not a point on line *FG*, but point *Q* is on that line.
c) *Z* is a point of ray *MN*.
d) Angle *PQR*.
e) Segment *HJ*.

15.2 Can perpendicular lines intersect one another at more than one point?

15.3 Two points determine a line. How many points determine a plane?

15.4 State the three ways in which two lines in a plane can relate to one another.

15.5 Which two angles are identical? $\angle ABC$, $\angle BCA$, $\angle BAC$, $\angle CAB$

15.6 State whether the indicated angle is acute, obtuse, or neither.

a) 34° **b)** 185° **c)** 84° **d)** 150°
e) 90° **f)** 135° **g)** 91° **h)** 21°

15.7 Calculate the complementary angle measure for each angle:

a) $\angle A = 42°$ **b)** $\angle B = 61°$ **c)** $\angle C = 22°$ **d)** $\angle D = 19°$

15.8 Calculate the supplementary angle measure for each angle:

a) $\angle\alpha = 35°$ **b)** $\angle\beta = 110°$ **c)** $\angle\gamma = 89°$ **d)** $\angle\delta = 33°$

15.9 Draw two complementary adjacent angles.

15.10 Draw two supplementary adjacent angles.

15.11 Name the indicated angles from Figure 15.20.

a) Exterior angles
b) Interior angles
c) Alternate exterior angles
d) Alternate interior angles
e) Corresponding angles

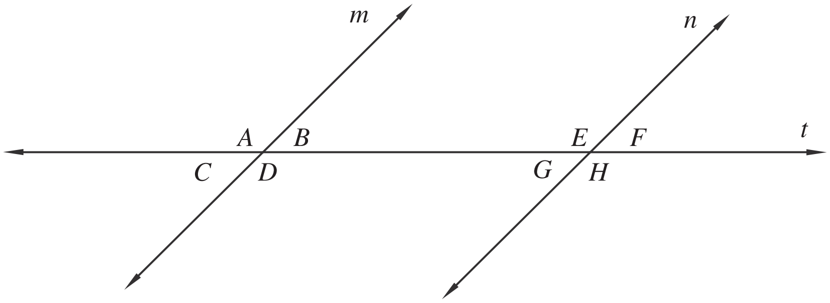


FIGURE 15.20

15.2 Polygons

An enclosed figure formed by three of more line segments is called a *polygon*, from the Greek words poly meaning “many” and gonia meaning “angle.” Figures with many angles, of course, have many sides, as well. Polygons are encountered in many areas of practical shop mathematics. Examples of convex polygons are pictured in Figure 15.21. Examples of concave polygons are shown in Figure 15.22.

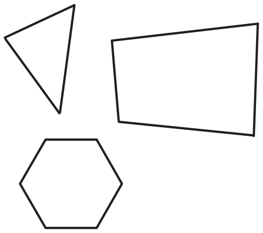


FIGURE 15.21 ***Convex polygons***

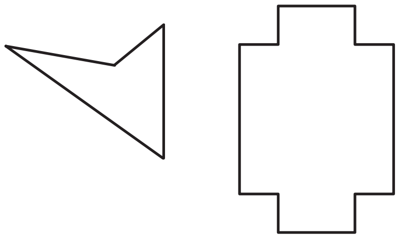


FIGURE 15.22 ***Concave polygons***

Polygons are named according to number of sides, such as triangle (three sides), quadrilateral (four sides), and hexagon (six sides).

A diagonal of a polygon is a line segment joining two nonadjacent vertices of the polygon. In other words, a diagonal is a line which joins two vertices that do not lie one next to the other. Any polygon of four or more sides has diagonals. In a convex polygon, all diagonals lie inside the figure as illustrated in Figure 15.23A. In a concave polygon, at least one diagonal, or a portion of a diagonal, lies outside the figure as shown in Figure 15.23B.

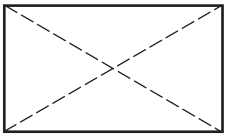


FIGURE 15.23A ***All diagonals lie inside a convex polygon***

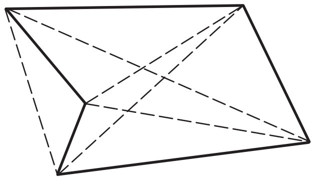


FIGURE 15.23B ***A diagonal lies outside this concave polygon***

A pentagon has five diagonals as shown in Figure 15.24.

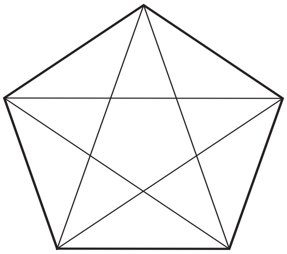


FIGURE 15.24 ***The five diagonals of a pentagon are shown in blue***

Line segments of equal length are said to be *congruent*. The same is said of angles of equal measure—they are congruent angles. Two polygons are congruent to each other if the dimensions of the sides and interior angles of one polygon are equal to the dimensions of the corresponding sides and interior angles of the other polygon.

The symbol for congruence is $\cong$. In Figure 15.25, $\overline{AB} \cong \overline{CD}$ and $\angle\alpha \cong \angle\beta$.

Figure 15.25 ***Congruent line segments and congruent angles***

The phrase "the sides (or angles) are equal" is an informal way of saying "the sides (or angles) are congruent."

When we discuss the angles of a polygon, we generally mean its *interior angles*. All interior angles in an *equiangular* polygon are congruent: they have equal measure. However, the sides of an equiangular polygon are not necessarily of equal measure, that is the sides are not necessarily congruent. All sides in an *equilateral* polygon are congruent, but this is not necessarily the case for its angles. *Regular* polygons are those having all sides congruent *and* all angles congruent. All rectangles are equiangular polygons because the angles in a rectangle measure 90°. Nonetheless, a rectangle is not necessarily equilateral, unless it is a square. All sides of a square are congruent, hence it is a regular polygon.

Angles of equal measure are called congruent angles. Line segments of equal length are called congruent line segments.

We will examine congruence in further detail in this and the following geometry chapters.

Triangles

The polygon with the least number of sides is the *triangle*. A triangle, of course, has three sides, and hence, three angles. The point at which any two sides of a polygon meet is called a *vertex.* By custom, a triangle's vertices are labeled A, B, and C using uppercase letters. The sides opposite the angles are designated by lowercase letters corresponding to the angles, that is, a, b, and c, as shown in Figure 15.26.

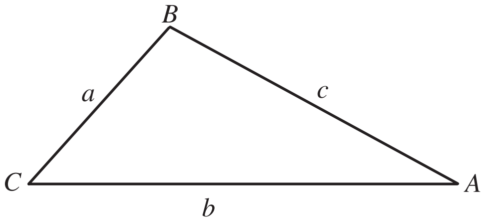


Figure 15.26 ***Triangle ABC***

The sum of the measures of the interior angles in any triangle is 180. That is,

$$m\angle A + m\angle B + m\angle C = 180$$

The sum of interior angle measures in any triangle is 180.

The letter "*m*" before the angle symbols is read "measure". When angles are used in equations, we refer to the measure of the angle without naming units (degrees).

The sum of the lengths of any two sides of a triangle is greater than the length of the third side. That is,

$$a + b > c$$

This fact is referred to as the *triangle inequality*.

A formula that gives the sum of the interior angles for any polygon is given in Chapter 17.

As indicated in Figure 15.27A, all sides of an *equilateral triangle* are congruent. Hence, all angles are also congruent, and the triangle is equiangular. Note that since the sum of angle measure in any triangle is 180°, each angle in an equiangular triangle is 60°. Identical letters are used to mark congruent sides and angles. Another way to show congruence is to use "score" marks on the sides having equal measure. The triangle is the only polygon for which "equilateral" implies "equiangular."

Equilateral Triangles

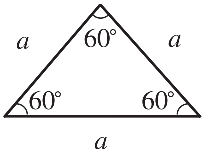


FIGURE 15.27A ***Sides labeled to show congruence***

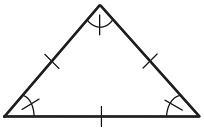


FIGURE 15.27B ***Sides and angles scored to show congruence***

In an *isosceles* triangle only two sides and two angles are congruent as shown in Figure 15.28A. The congruent angles in an isosceles triangle are called the *base angles*.

Isosceles Triangles

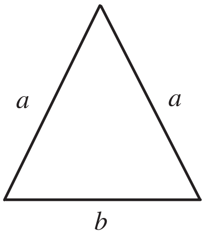


FIGURE 15.28A ***Sides labeled to show congruence***

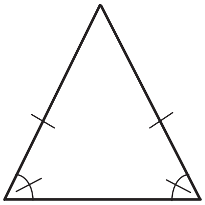


FIGURE 15.28B ***Sides and angles scored to show congruence***

In a scalene triangle, as illustrated in Figure 15.29, no two sides and hence no two angles have the same measure.

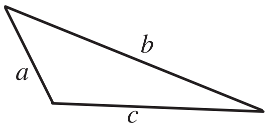


FIGURE 15.29 ***A scalene triangle***

Triangles are also classified with regard to their angular conditions. If one of its angles measures 90°, as in Figure 15.30, it is a *right* triangle. Right triangles are usually drawn with a small square inside the 90° angle, the same symbol used to show when intersecting lines are perpendicular. Triangles with no right angle are *oblique triangles*. Among oblique triangles are *obtuse triangles*, in which one angle is greater than 90° as shown in Figure 15.31, and *acute* triangles in which each angle is less than 90° as shown in Figure 15.32.

Acute and obtuse triangles are oblique triangles.

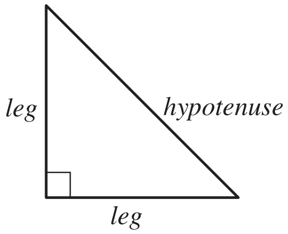


FIGURE 15.30 ***A right triangle***

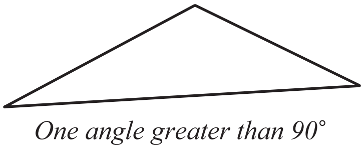


FIGURE 15.31 ***An obtuse triangle***

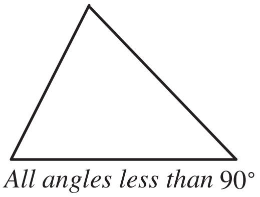


FIGURE 15.32 ***An acute triangle***

Theorems concerning triangles are discussed fully in Chapter 18. The study of the ratios of the lengths of the sides of right triangles is the subject of trigonometry, covered in Chapter 20.

Quadrilaterals

A four-sided polygon is a *quadrilateral*. Like a triangle, a quadrilateral is named by its vertices as illustrated in Figure 15.33.

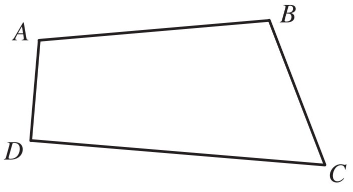


FIGURE 15.33 ***Quadrilateral ABCD***

The word "quadrilateral" is the most general name for four-sided polygons, implying no special angle or side relationships or measures of parts. Specific quadrilaterals—the parallelogram, the rectangle, the rhombus, and the square—are defined by their special properties.

Parallelograms

A *parallelogram* is a quadrilateral in which opposite sides are parallel. In Figure 15.34, the sides labeled *a* are parallel to one another, as are the sides labeled *b*. As the identical letter implies, opposite sides are also congruent in a parallelogram. The same is true of opposite angles.

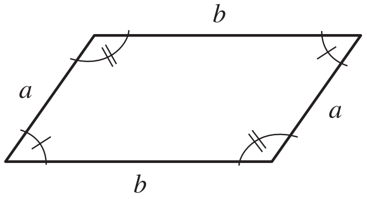


FIGURE 15.34 ***A parallelogram***

Rectangles

Rect is the Greek word for "right."

A parallelogram with all right angles as shown in Figure 15.35 is a *rectangle*. Besides having the all the properties of a parallelogram, angles of a rectangle are all right angles. Hence, a rectangle is equiangular.

Special properties of rectangles are related to its diagonals. In a rectangle:

1) diagonals are of equal length,
2) diagonals bisect each other, and
3) diagonals form pairs of congruent, isosceles triangles.

All rectangles are parallelograms, but not all parallelograms are rectangles.

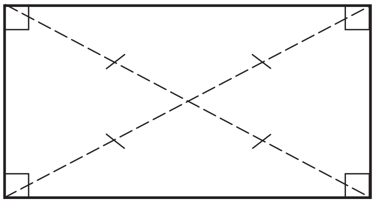


FIGURE 15.35 ***A rectangle***

Rhombi

The plural form of rhombus is rhombi. We sometimes call a rhombus a "diamond."

An equilateral parallelogram is a *rhombus*. A typical rhombus is shown in Figure 15.36. Its sides are equal in measure, but its angles are not necessarily equal. Diagonals of a rhombus bisect each other, but they are unequal in length. They bisect the interior angles and form four congruent triangles.

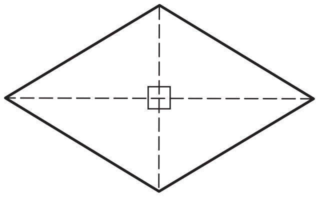


FIGURE 15.36 ***A rhombus***

Squares

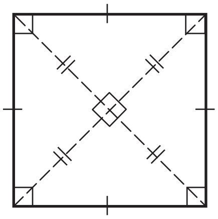
Although we typically think of rhombi as having two obtuse and two acute angles (like a square that has been stretched), a square is a rhombus, too, in that it is an equilateral parallelogram, but all its angles are 90° as shown in Figure 15.37.

Because a square is both equilateral and equiangular, it is a regular quadrilateral. A square is always a rectangle, so it has all the properties of a rectangle. Its additional properties are related to its diagonals. In a square:

All squares are rectangles, but not all rectangles are squares.

All squares are rhombi, but not all rhombi are squares.

1) diagonals are angle bisectors, dividing the angles into pairs of 45° angles,
2) diagonals are perpendicular bisectors, and
3) diagonals divide the square into four congruent, right, isosceles triangles.



Congruent sides and congruent angles

FIGURE 15.37 ***A square***

Trapezoids

All the quadrilaterals we have considered so far are parallelograms. *Trapezoids* have one—and only one—pair of parallel sides, called the *bases* of the trapezoid. The nonparallel sides are called the *legs*. Figure 15.38A shows a typical trapezoid.

An *isosceles trapezoid* has congruent legs and, hence, congruent base angles. The diagonals of an isosceles trapezoid do not bisect the interior angles, nor do they bisect each other. However, the diagonals of an isosceles trapezoid do form pairs of *similar triangles*, which are covered in Chapter 18. Furthermore, the triangles whose sides include the parallel sides are isosceles.

In a non-isosceles trapezoid, the diagonals do NOT form similar triangles.

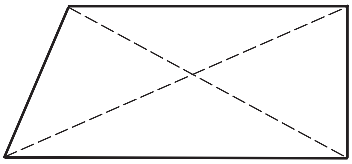


FIGURE 15.38A
Non-isosceles trapezoid

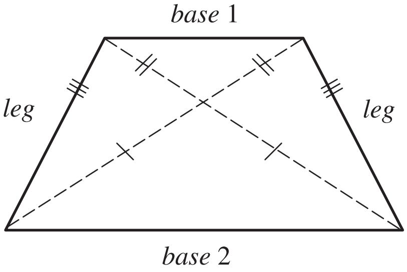


FIGURE 15.38B
Isosceles trapezoid

Pentagons

A regular pentagon has interior angles measuring 540°

A five-sided polygon is called a *pentagon*. This polygon is familiar to the general public because of the building in Washington, D.C., housing the United States Department of Defense, whose popular name is "The Pentagon." This building is a mammoth, five-sided structure. Another familiar pentagon is the home plate in baseball. Figure 15.39A shows two kinds of pentagons, regular and nonregular.

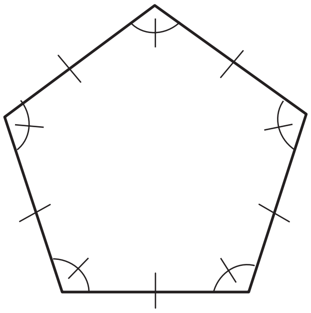


FIGURE 15.39A
Regular pentagon, congruent sides, congruent angles

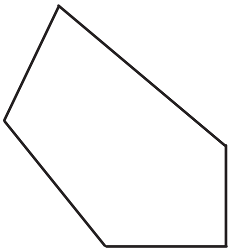


FIGURE 15.39B
Nonregular pentagon

Hexagons

Tessellations are interlocking geometric patterns. La Alhambra in Granada, Spain, is a Muslim fortress with mosaic art where hexagons can be seen. Tile floor patterns are often tessellations. The popular artist M.C. Escher produced many popular and intriguing tessellations.

The six-sided polygon, or *hexagon*, as shown in Figure 15.40A, has been present in art and architecture since ancient times. Long before humans began constructing buildings, honey bees stumbled across the inherent structural strength and efficiency of the hexagon and adopted it for the shape of honeycomb. Humans use the hexagonal shape for rigid structures that do not waste space, such as in aircraft with structural units made of a hexagonal core covered with sheet metal. Bolt heads are commonly hexagonal.

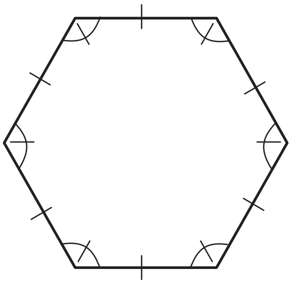


FIGURE 15.40A
Regular hexagon, congruent sides, congruent angles

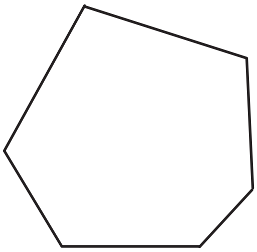


FIGURE 15.40B
Nonregular hexagon

Octagons

An *octagon* as shown in Figure 15.41 is an eight-sided polygon. Stop signs are octagonal; bar steel is available in octagonal cross-sections.

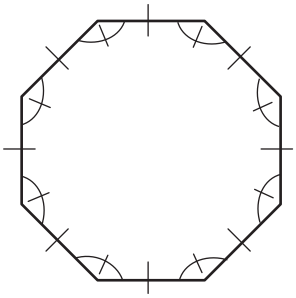
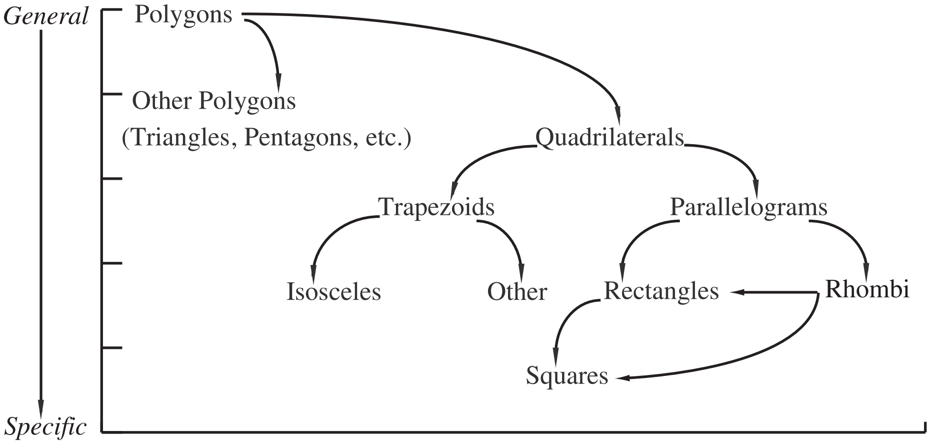


FIGURE 15.41 ***Regular octagon, congruent sides, congruent angles***

Polygon Hierarchy

In our discussion of polygons, we have shown that there is a hierarchy, or ordering, of polygon types, especially among triangles and quadrilaterals. Figure 15.42 gives the hierarchical ordering of several polygons from general to specific.



A rhombus could be a rectangle if it is a square.

FIGURE 15.42 ***A partial hierarchy of polygons***

EXERCISES

15.12 Using a protractor and a ruler, draw triangle ABC with side $AB = 3$ inches, $\angle A = 100°$ and $\angle B = 40°$.

15.13 Identify triangle ABC in Exercise 15.12 using both the descriptive names that apply to it.

15.14 A six-sided figure has all sides of equal length. What is its name?

15.15 Draw a triangle with all sides equal. What is the name of this triangle?

15.16 Draw a triangle with three different length sides. What is the name of this triangle?

15.17 A four-sided figure is drawn in which all the sides are of equal length and the interior angles each measure 90. How many names can this figure be called? What are the names?

15.18 Can all rectangles, squares, and rhombi be classified as parallelograms?

15.19 Can a rectangle be a rhombus?

15.20 Are scalene triangles always obtuse?

15.21 Why does a triangle never have more than one angle measuring 90?

15.3 Polyhedrons and Other Solid Figures

Geometric Solids

Polyhedron is made up of the Greek words poly, meaning "many," and hedron, meaning "face."

A *polyhedron* is a three-dimensional geometric solid with flat surfaces, or faces, and straight edges. The point where edges meet is called a *vertex*, as it is for polygons. The polyhedrons of most interest in technical fields are *regular polyhedrons*, *prisms*, and *pyramids*.

Regular Polyhedrons

The faces of a regular polyhedron are regular, congruent polygons. As shown in Figure 15.43, there are five regular polyhedrons:

1) the *tetrahedron* has 4 equilateral triangle faces,
2) the *cube* has 6 square faces,
3) the *octahedron* has 8 equilateral triangle faces,
4) the *dodecahedron* has 12 faces, all regular pentagons, and
5) the *icosahedron* has 20 equilateral triangle faces.

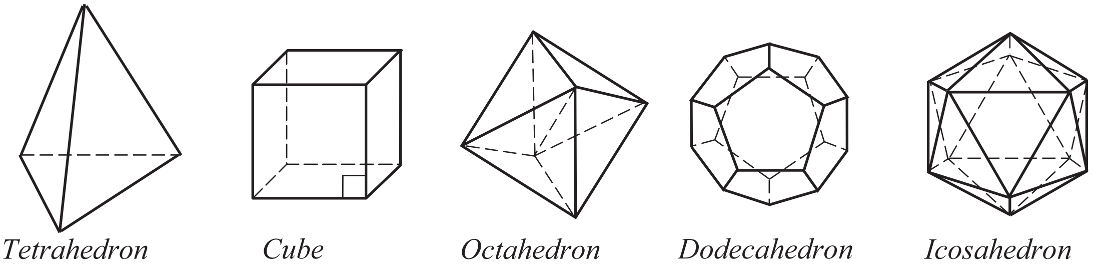


FIGURE 15.43 ***The five regular polyhedrons***

Prisms

A *prism* is a geometric solid whose bases are congruent polygons lying in parallel planes. The bases of a prism are the top and bottom polygons. Note that the sides of a prism are parallelograms. A cube, which is a regular polyhedron, is sometimes called a *right prism* because its sides are perpendicular to the bases. All the prisms we will consider will be right prisms. Most are named after their base shape—for example, the *triangular* and *rectangular prisms* shown in Figures 15.45A and 15.45B.

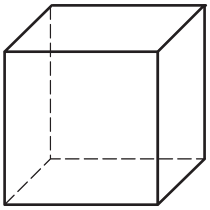


FIGURE 15.44 ***A cube is a regular polyhedron and a right prism***

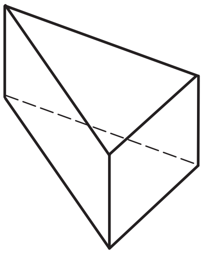


FIGURE 15.45A
Triangular prism

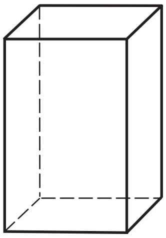


FIGURE 15.45B
Rectangular prism

Pyramids

If a prism were to have one of its bases removed and top vertices of the edges brought to one point, the result would be a *pyramid.* A pyramid is named according to the shape of its base. Figures 15.46A and 15.46B show the two most common pyramids, the *triangular pyramid* and the *rectangular pyramid.*

Any polygon can be the base of a prism or pyramid, giving rise to prisms and pyramids that are pentagonal, hexagonal, and so on.

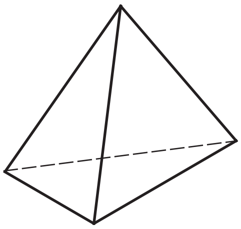


FIGURE 15.46A
Triangular pyramid

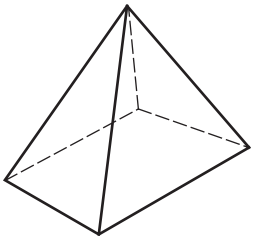


FIGURE 15.46B
Rectangular pyramid

Three-dimensional shapes can be solid or not solid, but polyhedrons are assumed hollow unless otherwise stated.

Other three-dimensional solid shapes to recognize are shown in Figure 15.47. They are the cone, the cylinder, and the sphere.

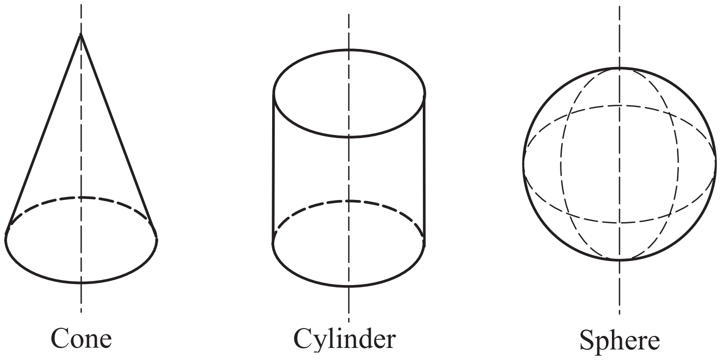


Figure 15.47 ***Other three-dimensional shapes***

EXERCISES

15.22 Name all the possible regular polyhedrons and state the number of sides for each one.

15.23 Name two types of prisms.

15.24 Name two types of pyramids.

16

PERIMETER, AREA, AND VOLUME

In technical shop and engineering problems we often need to the know length, area, or volume (capacity) of an object. Instances of such problems include calculating the number of feet of fence necessary to enclose a construction project; determining the number of gallons of paint to cover a house exterior; calculating how much of a bolt's thread surface will support the weight of a bridge; and deciding whether a tank of a certain size will hold enough coolant to meet the requirements of a cutting machine.

This chapter gives the important formulas for calculating perimeter, area, and volume of two- and three-dimensional figures.

16.1 Perimeter

Perimeter refers to the combined lengths of the sides of a closed figure, such as a polygon. It is a linear measure, given in inches, feet, miles, meters, kilometers, rods, and so on.

A rod is a unit of linear measure equal to 16.5 feet.

Perimeter of a Polygon

The perimeter P of a polygon with n sides is found by adding the lengths of individual sides:

$$P_{\text{polygon}} = s_1 + s_2 + s_3 + \ldots + s_n$$

where s_1, s_2,..., s_n are the individual lengths of each of the sides of the polygon. A square is an equilateral polygon; its sides are of equal length. Hence, a square's perimeter P is found by multiplying the length s of a side by 4, or $P = 4s$. In fact, the perimeter P of *any equilateral polygon* is

$$P_{\substack{\text{regular}\\ \text{polygon}}} = ns$$

Regular means "equilateral."

where n is the number of sides of the polygon and s is the length of any one side.

In the manufacture of metal stampings, consideration of perimeter is a very important part of the overall design. It is part of the calculation of the force required to blank a part from a sheet or coil of metal. In turn, the force is used to estimate the size of the press required for the blanking operation.

EXAMPLE 16.1: ***Perimeter of a Polygon***

Calculate the perimeter of the machine part shown. Dimensions are in millimeters.

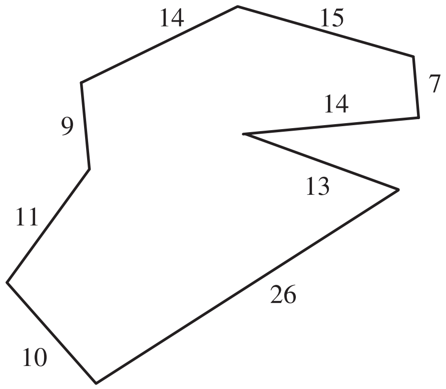


FIGURE 16.1

Solution:

Let P represent perimeter. Using $P_{\text{polygon}} = s_1 + s_2 + s_3 + ... + s_n$ gives

$$P = (14 + 7 + 15 + 14 + 9 + 11 + 10 + 26 + 13) \text{ mm} = 119 \text{ mm}$$

Finding Perimeter of a Square from its Area

Many situations arise in which the length of a side of a polygon is related to a quantity other than the polygon's perimeter. The square provides a ready example. The area of a square with side length s is given by $A = s^2$. Taking the square root of both sides and reversing the order of the equation gives:

$$s = \sqrt{A}$$

Thus, the perimeter is $P_{\text{square}} = 4s = 4\sqrt{A}$.

EXAMPLE 16.2: ***Perimeter of a Square of Known Area***

Example 16.2 is a good illustration of equation transformation, as learned in Chapter 11.

Find the perimeter of a square with area equal to 25 ft^2.

Solution:

$$P_{\text{square}} = 4s = 4\sqrt{A} = 4\sqrt{25 \text{ ft}^2} = 4(5 \text{ ft}) = 20 \text{ ft}$$

Finding Perimeter of a Square from Its Diagonal

The length s of a square's side is related to its diagonal length d by the formula

$$s = 0.7071d$$

Thus, perimeter $P = 4s = 4(0.7071d) = 2.828d$.

The coefficient in this formula is a result of applying the Pythagorean theorem to a right triangle in the square formed by the diagonal. The Pythagorean theorem is discussed in Chapter 18.

EXAMPLE 16.3: ***Perimeter of a Square of Known Diagonal***

Find the perimeter of a square with diagonal length $d = 4.5$ inches.

Solution: Using $P = 4s = 4(0.7071d) = 2.828d$, we substitute d into the formula:

$$P = 2.828d = 2.828(4.5 \text{ inches}) = 13 \text{ inches}$$

In the given information, the diameter has two significant digits, and, hence, so does the answer.

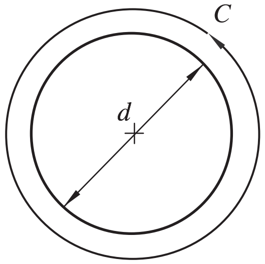
Circumference of a Circle

A *circle* is a set of points in the plane lying an equal distance from a fixed point, the center. The line segment from the center to any point on the circle is the *radius*. The *diameter* is the line segment from one point on the circle to another, passing through the center. Its length is twice that of the radius.

Chapter 19 discusses circles in more detail.

The *circumference* of a circle is the boundary of the circle. In any particular circle, the length of the circumference is about 3.14 diameter lengths, as shown in Figure 16.2. Algebraically, this means that the approximate ratio of circumference C to diameter d is 3.14.

Like other measurements in one dimension, circumference is expressed in units of length.



$$\frac{C}{d} \approx 3.14.$$

FIGURE 16.2 ***Circumference of a circle is equal to approximately 3.14 diameter lengths***

The number 3.14 (in fraction form, $3\frac{1}{7}$ or $\frac{22}{7}$, is an *approximation* of a physical constant known as π, after the Greek letter pi. The exact value of π cannot be written as a decimal fraction because it is an irrational number. This is why we use the symbol π to represent its exact value. Hence, $\frac{C}{d} = \pi$, which, solving for C, yields the formula for circumference:

$$C = \pi d$$

Circumference of a circle

Most calculators have a π key, which when pressed displays the decimal approximation 3.141592654 or an even longer number.

This formula holds for any circle of any given diameter. As circle size increases, diameter d increases by a factor of π, and hence, circumference C increases. A circle's circumference is directly proportional to its diameter.

Another version of the circumference formula can be derived as follows:

$$C = \pi d = \pi(2r) = 2\pi r$$

EXAMPLE 16.4: *Circumference of a Shaft*

Calculate the circumference of a 25-millimeter radius shaft.

The notation shown in Figure 16.3 is typical of blueprints. ***R25*** *means that radius r = 25. Measurement dimensions may be given in a blueprint legend.*

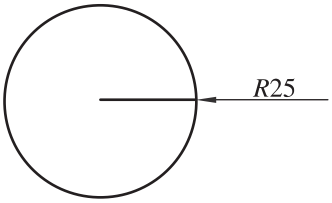


FIGURE 16.3

Solution: Substituting the value for the radius into the formula:

$$C = 2\pi r = (2\pi)(25) = 157 \text{ mm}$$

EXERCISES

16.1 Find the perimeter of a square whose sides measure

a) 1 foot **b)** $6\frac{1}{4}$ cm **c)** 2 miles

16.2 Find the perimeter of a square whose area measures

a) 1 square foot **b)** 10 square meters **c)** 100 square miles

16.3 Find the perimeter of a square whose diagonal measures

a) 1 foot **b)** 3.25 cm **c)** 0.5 mm

16.4 Find the perimeter of

a) a regular hexagon with sides measuring 9 inches;
b) a regular octagon with sides measuring 7.125 inches;
c) a rectangle 22 meters long by 12.375 meters wide.

16.5 What is the circumference of a shaft whose diameter measures

a) $\frac{3}{4}$ inch? **b)** 50 mm? **c)** 12 cm?

16.6 A rectangular fenced lot is 52 feet, 6 inches, by 28 yards. How many feet of fencing were used?

16.7 A swimming pool is 40 feet long and 20 feet wide with a depth from 3 feet at one end to 10 feet at the other. What is the perimeter at ground level?

16.8 Calculate the perimeter of a rectangular press blank 30 inches long by 14.25 inches wide.

16.9 Calculate the perimeter of the object shown in Figure 16.4 (a rectangle with a triangular part removed). All dimensions are in millimeters.

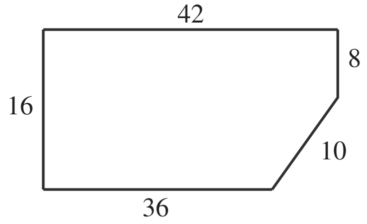


FIGURE 16.4

16.2 Area of a Polygon

Area is given in units of length squared, such as ft^2, m^2, or mile^2, as well as in units such as acres and hectares. The fundamental SI unit for area is the square meter (m^2), though for large land area the hectare may be used: 1 hectare = 10,000 m^2.

A plane figure of three or more sides bounds a surface whose area is calculated fairly easily. This is because the sides are *rectilinear*, that is the sides are straight line segments. It is from such figures that the phrase "square units" derives. Areas of figures with curved boundaries are also given in square units.

The formula to calculate the area of a square is: $A_{\text{square}} = s^2$ where s is the length of any side of the square. Figure 16.5 illustrates that a square with sides measuring 1 m has area equal to 1 m^2 and a square measuring 2 m on a side contains four 1-meter squares, hence, its area is 4 m^2.

\

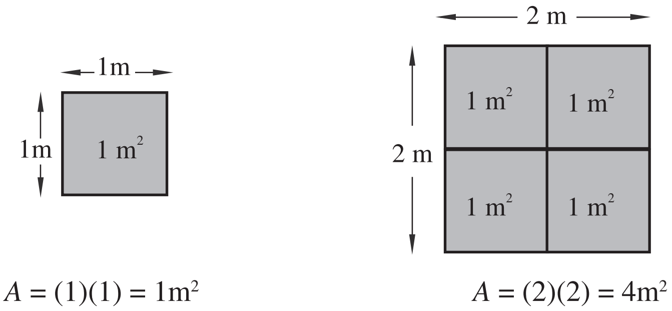


FIGURE 16.5 ***Area of a square***

Sometimes one dimension must be converted into another when calculating an area, as the next example demonstrates.

EXAMPLE 16.5: ***Square Feet in a Square Yard***

How many square feet are in one square yard?

Solution:

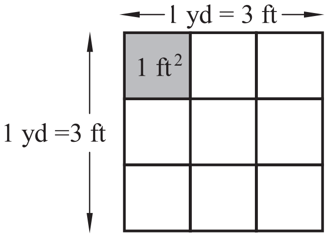


FIGURE 16.6

Since 1 yd = 3 ft, a square measuring 1 yd to a side has dimensions of 3 ft by 3 ft, so the area is (3 ft)(3 ft) = 9 ft^2.

Area of a Rectangle

The formula for rectangle area is derived in a way similar to that of the square. It is the product of the length l and the width w of the rectangle, or $A = lw$ as shown in Figure 16.7.

Area formulas for basic shapes should be committed to memory, as they come up frequently in technical shop situations.

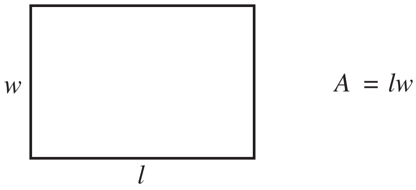


FIGURE 16.7 ***Rectangle area***

EXAMPLE 16.6: ***Rectangle Dimensions and Area***

What is the length of a rectangle whose area is 72 ft^2 and whose width is 9 ft?

Solution: Transform the formula $A = lw$ so that it can be solved for length l:

$$A \cdot \frac{1}{w} = l\cancel{w} \cdot \frac{1}{\cancel{w}}$$

$$\frac{A}{w} = l \quad \text{or} \quad l = \frac{A}{w}$$

Substituting the values of A and w gives

$$l = \frac{A}{w} = \frac{72\ \text{ft}^2}{9\ \text{ft}} = 8\ \text{ft}$$

Notice that the square is a rectangle for which l and w are both equal to s. Thus, $A = s \times s = s^2$ for a square becomes $A = l \times w$ for a rectangle.

Area of a Parallelogram

The parallelogram pictured in Figure 16.8 has base length b and a height h. Its area is simply $A = bh$.

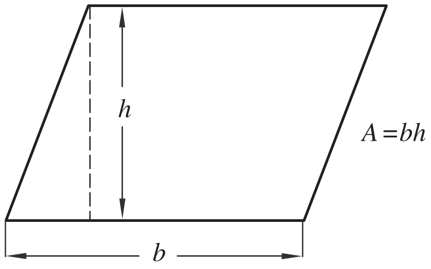


FIGURE 16.8 ***Parallelogram area***

This formula comes about from the way a parallelogram can be transformed into a rectangle by sliding the triangular portion marked off by the height over to the other side of the parallelogram as shown in Figure 16.9. Then dimensions b and h are the same as the rectangle's dimensions l and w, respectively.

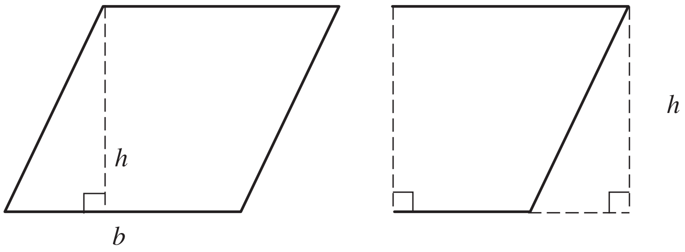


FIGURE 16.9

EXAMPLE 16.7: ***Parallelogram Area***

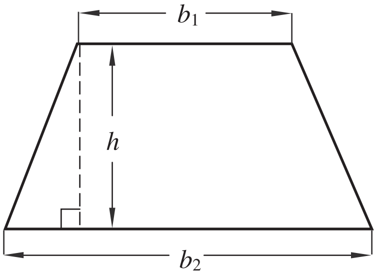
What is the area of the parallelogram whose base measures 18 cm and height measures 10 cm?

Solution:

$$\begin{aligned} A &= bh \\ &= (18 \text{ cm})(10 \text{ cm}) \\ &= 180 \text{ cm}^2 \end{aligned}$$

Area of a Trapezoid

Trapezoid area is related to area of a parallelogram, $A = bh$. The trapezoid, however, has not one base b, but two bases, b_1 and b_2, of different lengths. The mean of these bases, $\frac{b_1+b_2}{2}$, multiplied by the h, gives the trapezoid's area, as shown Figure 16.10.



$$A = \frac{(b_1 + b_2)}{2} h$$

FIGURE 16.10 ***Trapezoid area***

EXAMPLE 16.8: ***Area of a Trapezoid***

What is the area of a trapezoid with bases measuring 18 m and 27.5 m, and height measuring 16 m?

Solution:

$$A = \frac{(b_1 + b_2)}{2} h = \frac{(18\,\text{m} + 27.5\,\text{m})}{2}(16\,\text{m}) = 364\,\text{m}^2$$

Area of a Triangle

The area *A* of a triangle is calculated from the base *b* and height *h*, or altitude of the triangle according to the formula:

$$A = \frac{1}{2} bh$$

Consider Figure 16.11 in which two congruent triangles I and II are situated to form a parallelogram.

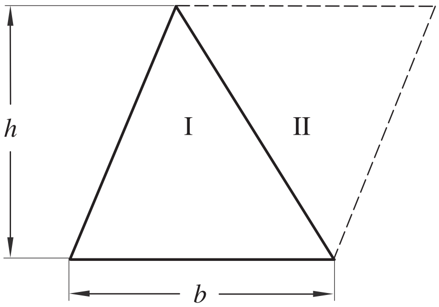
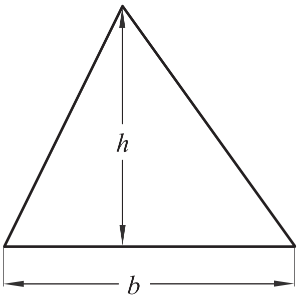


FIGURE 16.11 ***Congruent triangles I and II of a parallelogram***

Recall that the formula for the area of a parallelogram is $A = bh$. Since congruent triangles have equal area, each of the triangles in Figure 16.11 must have area $A = \frac{1}{2} bh$. The result is shown in Figure 16.12.



$$A = \frac{1}{2}bh$$

FIGURE 16.12 *Area of a triangle*

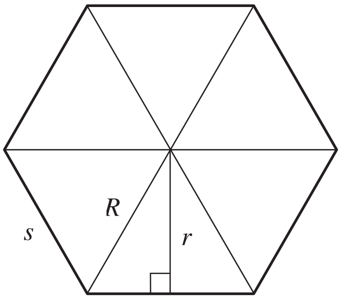
Area of a Regular Hexagon

As shown in Figure 16.13, for a regular hexagon, line segment R going from the hexagon's center to a vertex is equal in length to any side s. In turn, s and R relate to perpendicular bisector r, from hexagon center to a side, according to $s = R = 1.155r$. Dividing the hexagon into six congruent triangles each with base s and height r gives the total area of the hexagon to be:

$$A = (6)\left(\frac{1}{2}\right)(s)(r)$$

or, since $R = s$ and $r = \dfrac{R}{1.155}$, $A = (6)\left(\dfrac{1}{2}\right)(R)\left(\dfrac{R}{1.155}\right) = 2.597R^2$.

We can also write: $A = 2.597s^2$ or $A = 3.464r^2$.



$s = R = 1.155r$

$A = 2.597R^2$

$= 2.597s^2$

$= (2.597)(1.155r)^2$

so, $A = 3.464r^2$

FIGURE 16.13 *Area of a regular hexagon*

Notice that as the number of sides of a polygon increases, the side lengths decrease. Eventually the polygon turns into a circle. This is one way to arrive at the area formula of a circle, $A = \pi r^2$, in which r is radius length.

r is called the apothem. It is the perpendicular from the center of a regular polygon to any side.

EXAMPLE 16.9: ***Area of a Regular Hexagon***

Find the area of a regular hexagon with side length 3.3 cm.

Solution:

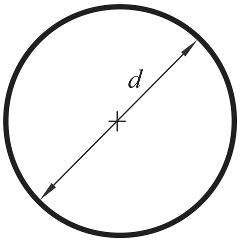
$$A = 2.597\,s^2 = 2.597(3.3\text{ cm})^2 = 28.3\text{ cm}^2$$

Area of a Circle

The formula for area A of a circle is $A = \pi r^2$ or $A = \dfrac{\pi d^2}{4}$ where r is the circle's radius and d is the circle's diameter as shown in Figure 16.14.

$$A = \pi r^2$$

$$\pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$$



$$r = \frac{d}{2}$$

$$A = \pi r^2 \text{ or } A = \frac{\pi d^2}{4}$$

FIGURE 16.14 ***Area of a circle***

A motivation for this formula is given in Figure 16.15. Eight wedges of a circle are placed side by side. They lie, more or less, in a rectangle. The length of this rectangle is equal to half the circumference of the circle, and of width equal to the circle's radius r. Hence, the area of this rectangle is $\frac{1}{2}(2\pi r)r = \pi r^2$.

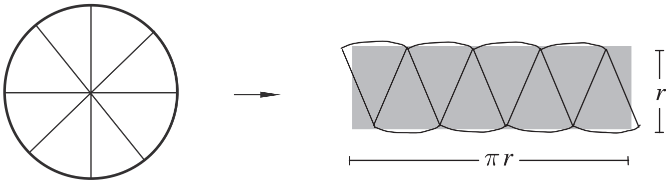


FIGURE 16.15 ***An approximate derivation of the area formula for circles***

If the number of circle wedges is increased by decreasing their size, the laid out wedges fill the rectangle better. Making the wedges infinitely small, that is, making infinitely many wedges, fills the open spaces, and makes the shape a rectangle whose area is precisely $A = \pi r^2$.

EXAMPLE 16.10: ***Area of a Circle***

What is the area of a circle whose diameter measures 19.7 inches?

Solution:

Take $r = \frac{d}{2}$ and substitute into $A = \pi r^2$, or use the diameter in $A = \frac{\pi d^2}{4}$. With either formula, the area is found to be 305 square inches.

Area of a Compound Shape

Other techniques for estimating area are given in Machinery's Handbook.

Most shapes common to industrial situations can be broken into simpler shapes such as rectangles, triangles, and circles, whose perimeter and area formulas are known. The individual areas are added to give the area of the original *compound shape*. Regions that cannot be broken down to familiar shapes can sometimes be approximated using known shapes.

EXAMPLE 16.11: *Area of a Compound Shape*

Calculate the area of the compound shape shown in Figure 16.16. Notice that it is composed of a semicircle (half-circle), a triangle, and a trapezoid.

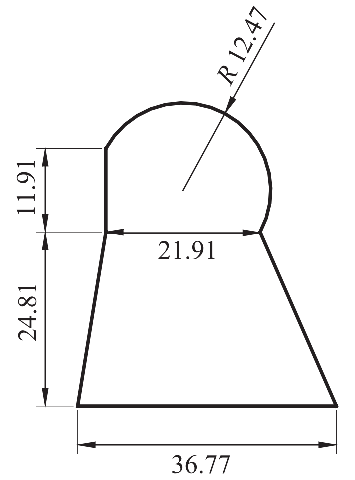


FIGURE 16.16 ***Area calculation of a compound shape, dimensions in mm***

Solution: Break the compound shape into the simpler shapes from which it is composed:

In Figure 16.17, the line that separates region I and region II is the diameter of the semicircle, and the line that separates region II and region III is parallel to the bottom line, since this shape is a trapezoid.

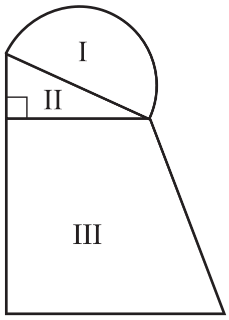


FIGURE 16.17 ***Splitting a compound shape into its components***

Region I is a semicircle. Its area is half the area of a circle with radius 12.47 mm.

$$A_{\mathrm{I}} = \frac{1}{2}\pi r^2 = \frac{1}{2}(\pi)(12.47\text{ mm})^2 = 244.3\text{ mm}^2$$

Region II is a triangle. Its area is equal to $\frac{1}{2}bh$ where height h is the top vertical dimension (11.91 mm) and base b is 21.91 mm. Substituting $b = 21.91$ mm and $h = 11.91$ mm into $\frac{1}{2}bh$ gives the area of region II:

$$A_{\mathrm{II}} = \frac{1}{2}(21.91\text{mm})(11.91\text{mm}) = 130.5\text{ mm}^2$$

EXAMPLE 16.11: *Area of a Compound Shape (Continued)*

Finally, region III is a trapezoid, for which $A = \frac{(b_1 + b_2)}{2}h$. Here, h is the other vertical dimension, 24.81 mm. Thus,

$$A_{\text{III}} = \frac{(21.91\text{ mm} + 36.77\text{ mm})}{2}(24.81\text{ mm}) = 727.9\text{ mm}^2$$

The total area of the compound shape is

$$A_{\text{total}} = A_{\text{I}} + A_{\text{II}} + A_{\text{III}} = 244.3\text{ mm}^2 + 130.5\text{ mm}^2 + 727.9\text{ mm}^2 = 1102.7\text{ mm}^2$$

EXAMPLE 16.12: *Cross-Sectional Area of a Bushing*

Calculate the cross-sectional area of the bushing shown. The dimensions have units of millimeters.

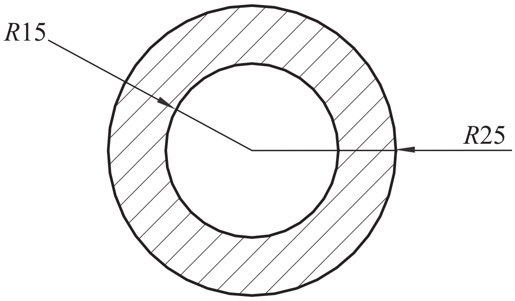


FIGURE 16.18

Solution:

The cross-section of the bushing looks like a ring. We subtract the area of the small hole from the area of the large circle. The diagram gives the radius of the large circle as 25 mm and the radius of the hole as 15 mm. Hence,

$$\begin{aligned} A_{\text{bushing}} &= A_{\text{large circle}} - A_{\text{small circle}} = \pi(r_{\text{large circle}})^2 - \pi(r_{\text{small circle}})^2 \\ &= \pi(25\text{ mm})^2 - \pi(15\text{ mm})^2 = 1963\text{ mm}^2 - 707\text{ mm}^2 = 1256\text{ mm}^2 \end{aligned}$$

EXERCISES

16.10 Find the area of a square whose sides measure

a) 1 foot **b)** $6\frac{1}{4}$ cm **c)** 2 miles **d)** 3 feet

16.11 Find the area of a rectangle whose length and width measure

a) 8.75 meters by 2.25 meters
b) 12 feet by 11.5 feet
c) 0.05 mm by 0.75 mm

16.12 Find the area of a parallelogram that has

a) 8.75-meter base and 2.25-meter height
b) 12-foot base and 11.5-foot height
c) 0.05-mm base and by 0.75-mm height

16.13 Find the area of a trapezoid that has

a) 8.75- and 3.25- meter bases and 2.25-meter height
b) 12- and 15-foot bases and 11.5-foot height
c) 5- and 2- mm bases and 0.75-mm height

16.14 Find the area of a triangle that has

a) 8.75-meter base and 2.25-meter height
b) 12-foot base and 11.5-foot height
c) 0.05-mm base and by 0.75-mm height

16.15 Find the area of a regular hexagon whose side length measures

a) 1 foot **b)** $6\frac{1}{4}$ cm **c)** 3 meters

16.16 Find the area of a circle whose radius measures

a) 1 foot **b)** $6\frac{1}{4}$ cm **c)** 3 meters

16.17 How many square inches are in a steel plate that is 54 inches wide and 13 feet 8 inches long?

16.18 The interior measurements of a factory building are 155 feet 8 inches by 368 feet, 5 inches. How many square yards of flooring are required?

16.19 A square field measuring 8 acres is to be fenced to store material. How many feet of fencing are required?

16.20 Floor tiles measure 8 inches square. How many will be required to tile an office floor containing 1600 square yards?

16.21 A room is 25 feet by 36 feet with an 8-foot ceiling height. How many square feet of wall space must be painted?

16.22 An exterior wall of a factory is to be painted. The wall is 355 feet, 8 inches long and $18\frac{1}{2}$ feet high. The paint will cover 60 square yards per gallon. How many gallons of paint will be required with 5% extra for loss and spillage?

16.23 If Douglas fir plywood sells at 32 cents per square foot, what will be the price for 23 pieces of regular 4-foot by 8-foot plywood panels?

16.24 A swimming pool is 40 feet long and 20 feet wide with a depth from 3 feet at one end to 10 feet at the other. The side walls are trapezoids.

a) What is the area of one 40-feet wall?
b) How much will it cost to paint the total interior surface with a waterproof paint at $0.875 per square foot?

16.25 Find the surface area of a rectangular prism 32 cm by 22 cm by 8.3 cm.

16.26 Find the surface area of a cube 26 cm on an edge.

16.27 The nonregular polygon shown in Figure 16.19 consists of a square, 20 cm on a side, plus an isosceles triangle with a height of 25 cm. What is the area of the polygon?

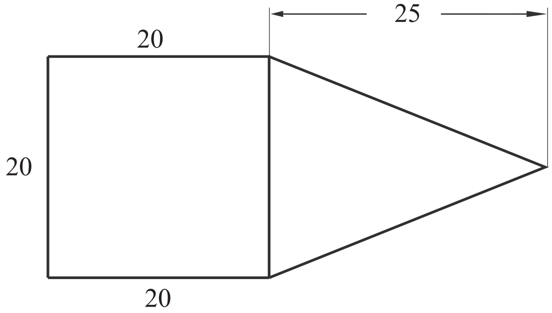


FIGURE 16.19

16.28 Find the area of the field shown in Figure 16.20. All dimensions are in meters.

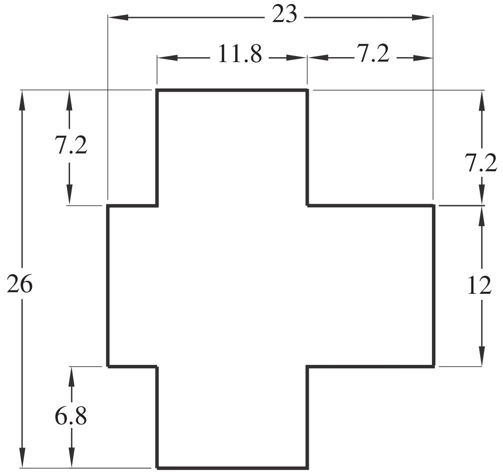
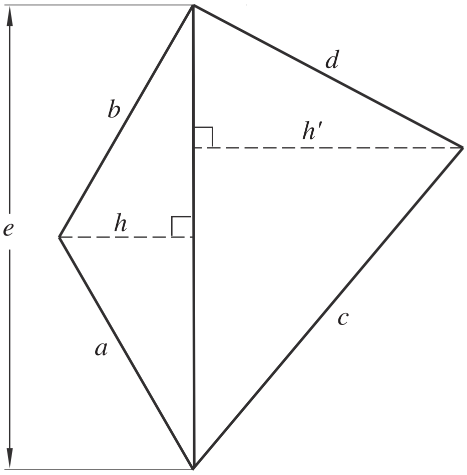


FIGURE 16.20

16.29 Write an expression for the area shown in Figure 16.21, using the letters to represent the lengths of the sides.



Hint: Notice that the shape can be divided into two triangles.

Figure 16.21

16.30 A screen has 96 openings with dimensions as shown in Figure 16.22. Find the total area of the openings in the screen.

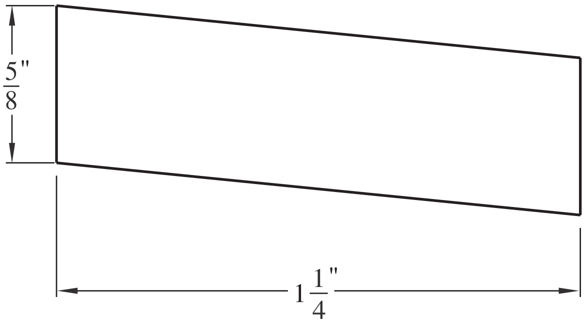


Figure 16.22

16.31 A steam engine cylinder is to have 9 port openings as shown in Figure 16.23. If the total area of the port openings is to be 10.125 square inches, find the width *X* of each port. Answer to the nearest 0.001 inch.

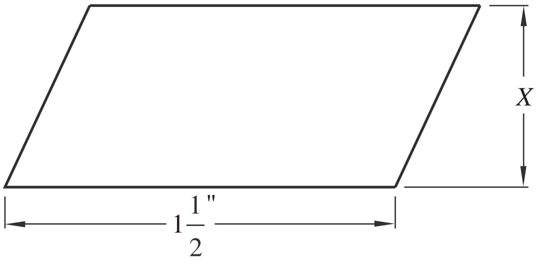


Figure 16.23

16.32 Calculate the cross-sectional area of the storm drain shown in Figure 16.24. Give answer in square feet and also in square meters.

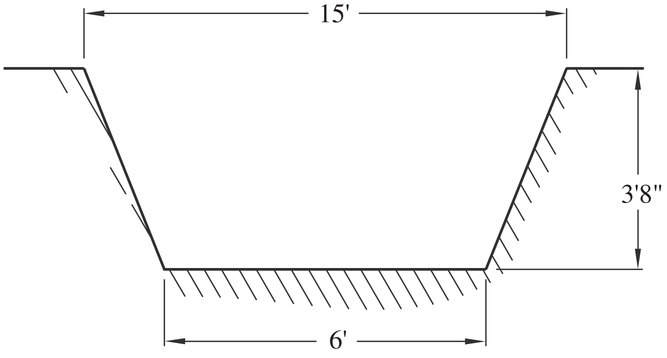


FIGURE 16.24

16.3 Surface Area and Volume of a Solid

Surface area is given in square units. Volume is given in cubic units, liters, or gallons.

Volume gives information about the capacity of a three-dimensional object, for example, the amount of fuel a tank can hold or fluid an actuator displaces. Volume is expressed as a length unit cubed, for example, ft^3, m^3, or cm^3. Volume is also reported in liters and gallons.

Comprehensive lists of formulas are available in references such as *Machinery's Handbook.*

Prisms

A prism's *surface area* is equal to the sum of the solid's face areas.

Surface Area and Volume of a Rectangular Prism

The two-dimensional surface area of three-dimensional figures is given in square units, just as plane area is, for example, ft^2, m^2, or cm^2.

The rectangular prism shown in Figure 16.25 has base dimensions l and w and height h. Hence, its base area is lw and its side areas are lh and wh, making its surface area:

$$SA_{\substack{\text{Rectangular}\\\text{prism}}} = 2lw + 2lh + 2wh = 2(lw + lh + wh)$$

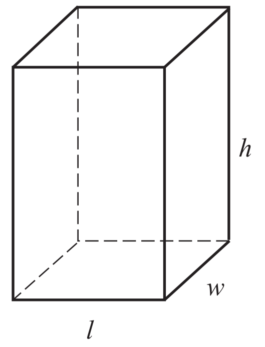
Notice that the rectangular prism comprises six rectangles — two of each dimension combination.

The fundamental unit of surface area in SI is the square meter.

The volume of a prism is found by multiplying its base area by its height. Thus, the volume of a rectangular prism is simply

$$V_{\substack{\text{Rectangular}\\\text{prism}}} = lwh$$

where l is length, w is width, and h is height.



$$SA_{\substack{\text{Rectangular}\\\text{prism}}} = 2(lw + lh + wh)$$

$$V_{\substack{\text{Rectangular}\\\text{prism}}} = lwh$$

FIGURE 16.25 ***A rectangular prism has six rectangular faces***

EXAMPLE 16.13: ***Rectangular Prism***

Find the surface area and volume of the rectangular prism whose base dimensions are 1.50 inches and 2.75 inches, and whose height is 3.10 inches.

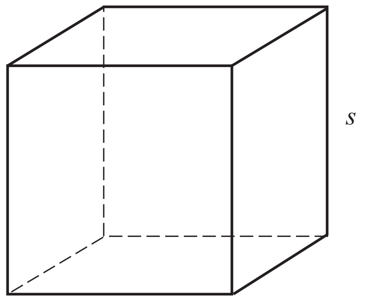
Solution:

$$\begin{aligned} SA_{\substack{\text{Rectangular}\\\text{prism}}} &= 2lw + 2lh + 2wh \\ &= 2(1.50)(2.75) + 2(1.50)(3.10) + 2(2.75)(3.10) \\ &= 34.6 \text{ square inches} \end{aligned}$$

$$V_{\substack{\text{Rectangular}\\\text{prism}}} = lwh = (1.50)(2.75)(3.10) = 12.8 \text{ cubic inches}$$

Cube—a Special Rectangular Prism

A cube, such as the one shown in Figure 16.26, is a rectangular prism whose length, width, and height dimensions are all the same, that is, they have *edge* length s. Since $l = w = h = s$, the formulas for surface area and volume reduce to equations in s alone.



$$SA_{cube} = 6s^2$$
$$V_{cube} = s^3$$

FIGURE 16.26 ***A cube has six congruent square faces***

To help visualize the meaning of cubic volume, consider the 10-cm cube shown in Figure 16.27. Since each edge of the cube measures 10 cm, the base of the cube is a square with area 10 cm × 10 cm = 100 cm^2. The bottom layer of the cube is a slab measuring 10 cm by 10 cm by 1 cm. Stacking up 10 such layers gives the volume: 10 cm × 10 cm × 10 cm = 1000 cm^3.

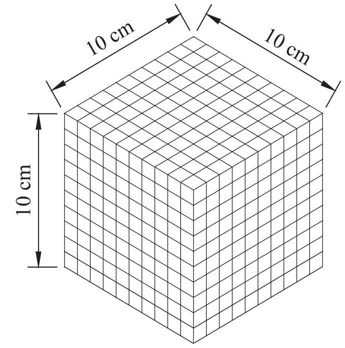


FIGURE 16.27 ***Volume of a 10-cm cube***

The fundamental unit of volume in SI is the cubic meter.

A common error is to state the volume of a cubic yard as 3 cubic feet!

The volume of a cube measuring 1 cm by 1 cm by 1 cm is 1 cubic centimeter, abbreviated by cm^3 or cc.

EXAMPLE 16.14: ***Surface Area and Volume of a Cube***

What is the surface area in square feet and volume in cubic feet of a cube whose edges each measure 1 yard?

Solution: The dimensions of the cube are 3 feet by 3 feet by 3 feet. Hence,

$$SA = 6s^2 = 6(3)^2 = 6(9) = 54 \text{ ft}^2$$

$$V = s^3 = (3)^3 = 27 \text{ ft}^3$$

Therefore, 1 cubic yard has surface area of 54 ft^2 and volume of 27 cubic feet.

Volume of a Triangular Prism

Confusion may occur in references to the dimensions in a triangular prism such as the one illustrated in Figure 16.28. The base is a triangle and the height is the length of the perpendicular from the base to the top. Be careful to note that base and height also refer to the *dimensions* of the triangular base itself, whose area is $\frac{1}{2}bh.$ We choose to label the prism's height as c, the bottom triangle height as h and its base length as b, to give $V_{\substack{\text{triangular}\\\text{prism}}} = \dfrac{1}{2}bhc.$

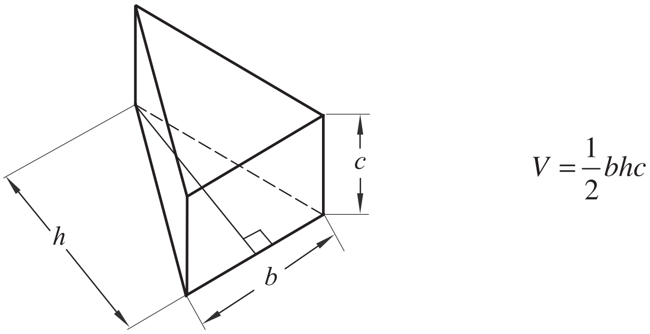


FIGURE 16.28 ***Triangular prism dimensions***

In other books, different lettering may be used for a solid's dimensions. Sometimes formulas are given with edge lengths of the base. Always use the formula that corresponds to the dimension information available.

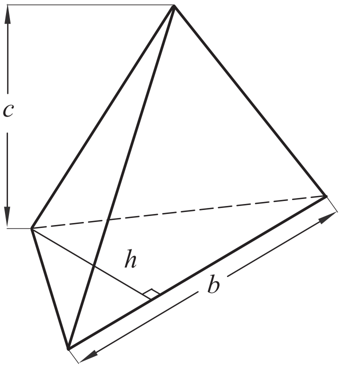
Be clear about each dimension's label before using a formula.

Volume of a Pyramid

The volume of a pyramid is found by multiplying base area by height by $\frac{1}{3}$:

$$V = \frac{1}{3} \times \text{height of pyramid} \times \text{area of base}$$

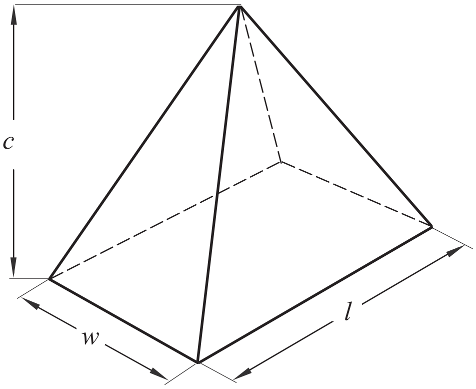
The volume V of a triangular pyramid is $V = \left(\frac{1}{3}\right)(c)\left(\frac{1}{2}\right)(bh) = \frac{1}{6}bhc$, as shown in Figure 16.29A, and the volume V of a rectangular pyramid as shown in Figure 16.29B, is $V = \frac{1}{3}lwc$. The height is labeled c. The base triangle dimensions of the triangular pyramid are b and h. The base rectangle dimensions of the rectangular pyramid are l and w.



$$V = \frac{1}{6}bhc$$

FIGURE 16.29A ***Triangular pyramid***

Notice that the factor of $\frac{1}{6}$ comes from the product of $\frac{1}{3}$ from the general pyramid volume formula and $\frac{1}{2}$ from the triangle area formula.



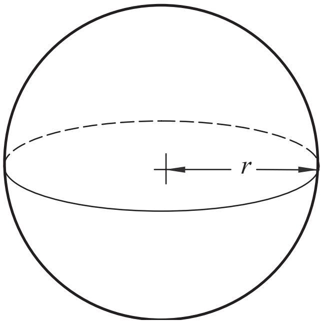
$$V = \frac{1}{3}lwc$$

FIGURE 16.29B ***Rectangular pyramid***

Sphere and Cylinder

Formulas to calculate the surface areas and volumes of the *sphere* and *right circular cylinder* are shown in Figures 16.30 and 16.31.

Surface area and volume of a sphere

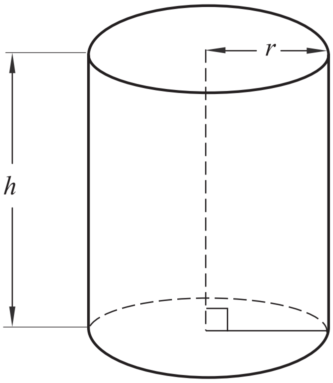


$$SA_{\text{Sphere}} = 4\pi r^2$$

$$V_{\text{Sphere}} = \frac{4}{3}\pi r^3$$

FIGURE 16.30 ***A sphere***

Surface area and volume of a right circular cylinder



$$SA_{\text{Cylinder}} = 2\pi r^2 + 2\pi rh$$

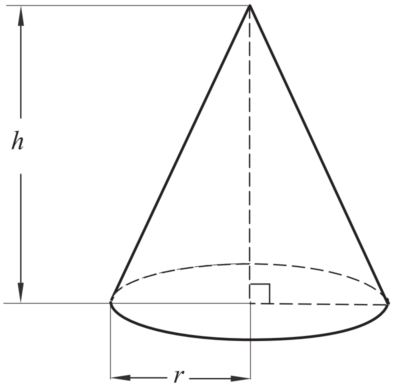
$$V_{\text{Cylinder}} = \pi r^2 h$$

FIGURE 16.31 ***Right circular cylinder***

Cone and Frustum

Formulas to calculate the volumes of a *right circular cone* and the *frustums* of a rectangular pyramid and right circular cone are shown in Figures 16.32, 16.33A and 16.33B.

Volume of a cone



$$V_{\text{cone}} = \tfrac{1}{3}\pi r^2 h$$

FIGURE 16.32 ***Right circular cone***

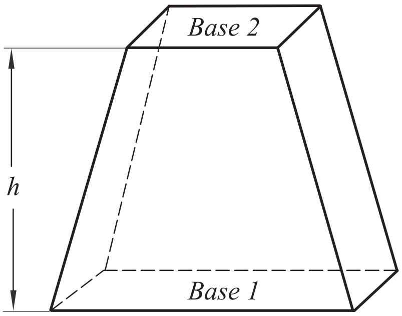


FIGURE 16.33A
Frustum of a rectangular pyramid

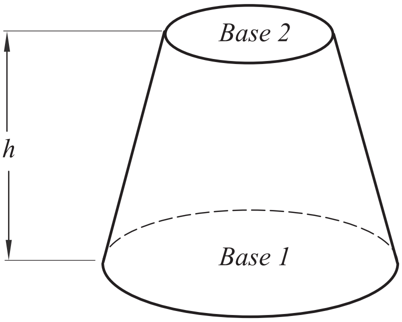


FIGURE 16.33B
Frustum of a cone

A frustum is the portion of a solid that lies between two parallel planes cutting the solid. Usually frustrums are in reference to cones or pyramids.

$$V_{\text{frustum}} = \frac{h}{3}\left(A_{\text{base 1}} + A_{\text{base 2}} + \sqrt{A_{\text{base 1}} \cdot A_{\text{base 2}}}\right)$$

Volume of a frustum

Volume of a Compound Shape

We have shown that breaking a compound shape into simpler shapes helps us find the area of a compound shape. Similarly, the volume of a compound solid is found by separating the shape into pieces whose volumes can be readily determined. The sum of these volumes gives the total volume, as demonstrated in the following example.

EXAMPLE 16.15: ***Volume of a Compound Shape***

The dimensions of the arch shown in Figure 16.34 are in meters. What volume of concrete is required to build the arch?

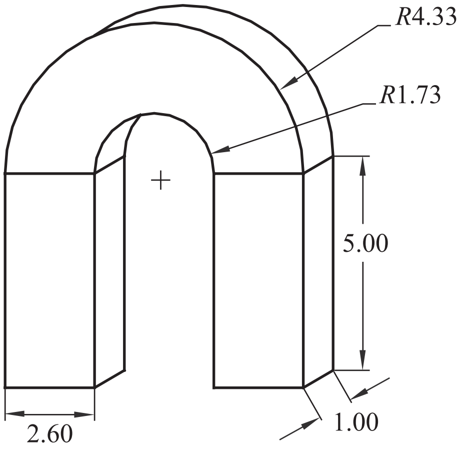


FIGURE 16.34

Solution: First, notice that the arch is made up of two identical rectangular prism legs and a semicircular arch with a smaller semicircular arch removed.

EXAMPLE 16.15: *Volume of a Compound Shape (Continued)*

The volume of one of the rectangular prism legs is:

$$V_{\text{leg}} = lwh = (2.60)(1.00)(5.00) = 13.00 \text{ m}^3$$

The volume of the semicircular portion of the arch is

(thickness of arch) × (area of larger semicircle – area of smaller semicircle)

Thus,

$$V_{\text{top}} = (1.00)\left[\frac{1}{2}\pi(4.33)^2 - \frac{1}{2}\pi(1.73)^2\right] = 24.75 \text{ m}^3$$

Since the arch has two legs, the total volume is:

$$V_{\text{total}} = (2)(13.00) + 24.75 = 50.75 \text{ m}^3$$

EXERCISES

16.33 Find the surface area and volume of a rectangular prism whose base dimensions are 423 mm and 534 mm, and whose height is 892 mm.

16.34 Find the volume of a triangular prism as shown in Figure 16.28 with dimensions b = 17 inches, c = 31 inches, and h = 4 inches.

16.35 Find the volume of a triangular pyramid as shown in Figure 16.29A with dimensions b = 8 feet, c = 18 feet, and h = 7 feet.

16.36 Find the volume of a rectangular pyramid as shown in Figure 16.29B with dimensions l = 8 feet, c = 18 feet, and w = 7 feet.

16.37 Find the surface area and volume of a sphere with the indicated radius.

a) r = 1 foot **b)** r = 1 meter **c)** r = 23 mm **d)** r = 4 inches

16.38 Find the surface area and volume of a right circular cylinder with the indicated height and radius.

a) h = 1 foot and r = 6 inches **b)** h = 543 mm and r = 1 meter
c) h = 91 mm and r = 23 mm **d)** h = 7 inches and r = 4 inches

16.39 Find the volume of a right circular cone with the indicated height and radius.

a) h = 1 foot and r = 6 inches **b)** h = 543 mm and r = 1 meter
c) h = 91 mm and r = 23 mm **d)** h = 7 inches and r = 4 inches

16.40 How many cubic inches are in one cubic foot?

16.41 A stock crib is 20 feet long, 30 feet wide, and 12 feet high. How many cubic feet does it contain? How many cubic meters is this?

16.42 A box is 3 feet long, 4 feet wide, and 2 feet high. What is its volume in cubic inches? In cubic centimeters?

16.43 A rainfall on a certain day was $3\frac{1}{2}$ inches. How many cubic inches fell on a $2\frac{1}{2}$ hectare field?

16.44 A storage tank is 8 feet by 20 feet by 6 feet. How many cubic feet are in the tank? How many cubic meters?

16.45 The oil reservoir of a machine that is operated by fluid power is 30 inches long, 14 inches wide, and $2\frac{1}{2}$ feet deep. How many gallons of oil are required to:

a) completely fill the reservoir? **b)** fill the reservoir to $\frac{7}{8}$ of its depth?

16.46 A room is 12 feet by 16 feet by 8 feet. How many cubic feet does it contain? How many square meters of wall does it contain? How many yards of carpet will be required to cover the floor (wall to wall)?

16.47 A basement for a house is to be 60 feet by 35 feet by 8 feet. After 525 cubic yards have been removed by the excavator, how many cubic yards remain to be dug?

16.48 A rectangular water storage pond is 345 feet long by 675 feet wide. After a heavy rain the water level rose 4 inches. How many cubic feet of water entered the pond?

16.49 A rectangular grain storage bin is 150 feet long, 45 feet wide, and 22 feet high. How many cubic yards of storage does it have? How many cubic meters?

16.50 An excavation is 35 feet long, 27 feet wide, and 9 feet deep. How many cubic meters of dirt have been removed?

16.51 How many liters of gasoline can be put into a service station storage tank in Madrid that measures 4 meters in diameter by 7 meters long?

16.52 How many meter sticks, laid end to end, would be needed to enclose a square hectare?

16.53 If a flat field measuring 2.5 hectares is evenly covered by water to a depth of 2.5 cm, how many gallons of water are in each hectare?

16.54 A machinist milled 0.025 inch from the side of a flat steel bar, 5 inches wide and 20 inches long. How many cubic centimeters of steel were removed from one side of the bar?

16.55 Calculate the volume of a rectangular tank that is 32 feet long, 40 feet wide, and 10 feet deep. If the tank is full of water, how many gallons does it hold?

16.56 How many square feet of steel plate are needed to build a rectangular tank that is $7\frac{1}{2}$ feet long, $4\frac{1}{2}$ feet wide, and $3\frac{1}{2}$ feet deep? Allow 5% for cutting and welding.

16.57 A concrete cistern is used to collect rainwater in rural France. It is 30 meters long, 22 meters wide, and $11\frac{1}{2}$ meters deep (inside dimensions). How many liters of water are required to fill up the cistern?

16.58 The cold air return duct in a heating system measured 8 inches by 14 inches in cross section. Fifty-five feet of duct was used.

a) How many cubic feet of air will it hold? **b)** How many cubic meters?

16.59 The hydraulic oil storage tank on a machine tool is 250 cm wide by 360 cm long by 150 cm deep.

a) How many liters of oil will it hold? **b)** How many gallons?

16.60 Calculate the volume of a rectangular prism oil reservoir of length 5 meters, width 2500 cm, and depth 1500 cm. Give the answer in kiloliters and in gallons.

16.61 A cylindrical tank 15 feet in diameter and 10 feet high is filled with oil.

a) How many gallons are required to fill it?

b) If the full tank is drained so that a depth of 6.25 feet of oil remains, how many gallons drained?

c) If the empty tank is filled to 8 feet depth, drained to 3.75 feet and then filled to capacity, how many gallons were drained? How many kiloliters were then needed to fill it to capacity?

16.62 Find the weight of a solid sphere 75 cm in diameter if the material weighs 450 pounds per cubic foot.

16.63 How many liters of oil will a circular tank hold if its diameter is 10 meters and its length is 35 meters?

16.64 A sphere is cut from a cube 20 inches on an edge. What is the minimum amount of waste in cubic inches?

16.65 Two spherical tanks, 30 inches in diameter, are packed in a box whose inside measurements are 36 inches by 36 inches by 70 inches. How much space (cubic inches) is left for packing material?

16.66 A round bar of steel 5.5 inches in diameter and 14 inches long is machined out to have a wall thickness of 0.625 inch. What is the volume of the resulting tube, and how many cubic inches of metal were removed?

17

AXIOMS, POSTULATES, AND THEOREMS

An *axiom* is a universally recognized statement whose truth is accepted without proof. For example, the claim that "Jennifer cannot be in Virginia at the same time she is in California" is self-evident and does not require proof. *Postulates* are also mathematical statements that are accepted without proof. Logic and pure mathematics begin with these unproved assumptions and build theorems from them.

The use of axioms in mathematics by the ancient Greeks represents the beginning of pure mathematics as it is known today.

17.1 Axioms and Postulates

Axioms

The axioms that form the basis of mathematical reasoning are summarized below. They are accepted without proof. These statements also appear in Chapter 1, where they are presented as the *real number properties*. In fact, Axioms I, II, and III are simply the *definitions* of reflexivity, symmetry, and transitivity, respectively. These definitions and properties are used more or less automatically in algebra exercises, but they are also needed to understand theorem proofs and to solve geometry problems.

Recall from Chapter 15 that m∠A is read, "The measure of angle A." When angles are used in equations, we refer to the measure of the angle and not to the units (degrees).

Axioms of Mathematical Reasoning

Reflexive property of equality: $a = a$ ***Axiom I***

Examples: $7 = 7$ and $m\angle A = m\angle A$

Symmetric property of equality: If $a = b$, then $b = a$ ***Axiom II***

Examples: If $2 = x$ then $x = 2$

If $m\angle A = 45$, then $45 = m\angle A$

Axiom III

Transitive property of equality: If $a = b$ and $b = c$, then $a = c$

Examples: If $x = y + 1$ and $y + 1 = 9$, then $x = 9$

If $m\angle A = m\angle B$ and $m\angle B = m\angle C$, then $m\angle A = m\angle C$

Axiom IV

Substitution property of equality: If $a + b = c$ and $b = b'$, then $a + b' = c$

Examples: If $x + c = 10$ and $c = 2$, then $x + 2 = 10$

If $m\angle A + m\angle B = 90$ and $m\angle B = 30$, then $m\angle A + 30 = 90$

Axiom V

Addition property of equality: If $x + a = x + b$, then $a = b$

Subtraction property of equality: If $x - a = x - b$, then $a = b$

Examples: If $x^2 + y^2 = x^2 + 25$, then $y^2 = 25$

If $m\angle A - m\angle B = m\angle A - 38$, then $m\angle B = 38$

Axiom VI

Multiplication and division properties of equality:

If $ab = ac$, $a \neq 0$, then $b = c$

If $a = b$ and $c = d$, then $\frac{a}{c} = \frac{b}{d}$ and $\frac{a}{d} = \frac{b}{c}$ for $c \neq 0$, $d \neq 0$

Examples: If $x^2y = x^2z$, $x \neq 0$, then $y = z$

If $x^2 = w$ and $y^2 = z$ then $\frac{x^2}{y^2} = \frac{w}{z}$ and $\frac{x^2}{z} = \frac{w}{y^2}$ for $z \neq 0$, $y \neq 0$

Axiom VII

The whole is greater than or equal to any of its individual parts:

If $c = a + b$, then $c \geq a$ and $c \geq b$

Examples: If $14 = x + 5$, then $14 \geq x$ or $14 \geq 5$

If $90 = m\angle A + m\angle B$, then $90 \geq m\angle A$ and $90 \geq m\angle B$.

In Axiom VII, notice that if $a = 0$, then $c = b$, while if $b = 0$, then $c = a$. Hence the need for "greater than or equal to" in the statement.

Euclid's Postulates

"Axiom" and "postulate" are often used interchangeably.

Euclid's axioms regarding plane figures are often called *postulates*. He considered these postulates to be self-evident, observed truths. Euclid used his postulates to construct proofs of hundreds of theorems.

The following postulates were given in Euclid's *Elements* about 2300 years ago.

Postulate 1: One and only one line can be drawn between any two points.

The phrase "one and only one" is a mathematically precise way to say, "no less and no more than one."

Postulate 2: Two noncollinear lines lying in the same plane can intersect at one and only one point.

Notice that Postulate 2 does not say the noncollinear lines *must* intersect at one and only one point. As we saw in Chapter 15, lines may be parallel, and thus, intersect nowhere.

Postulate 3: A straight line is the path of shortest distance between two points in one plane (see Figure 17.1).

FIGURE 17.1

Postulate 4: Any geometric figure can be moved without its size or shape being changed (see Figure 17.2).

We can slide a figure or rotate it or reflect it across a line without changing its shape.

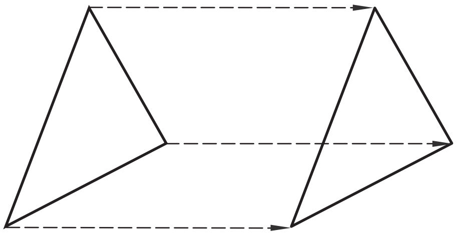


FIGURE 17.2

Postulate 5: A line segment has one and only one midpoint (see Figure 17.3).

The midpoint M is the point on the line segment AB that divides the segment into two segments of equal length.

FIGURE 17.3

Postulate 6: An angle has one and only one bisector (see Figure 17.4).

A bisector is a ray that divides an angle into two angles of equal measure.

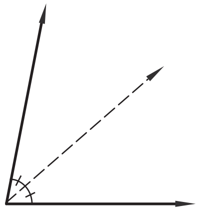


FIGURE 17.4

Postulate 7: Through a point external to a line, one and only one line parallel to the line can be drawn (see Figure 17.5).

Parallel lines are lines in the same plane that have no points in common.

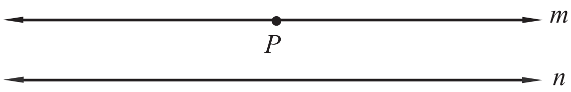


FIGURE 17.5

Perpendicular lines meet to form right angles.

Postulate 8: Through any point outside a line, one and only one perpendicular can be drawn to the line (see Figure 17.6).

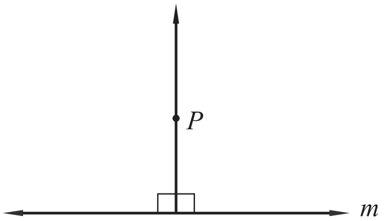


FIGURE 17.6

Postulate 9: Through any three points that are not on one line, there is exactly one plane (see Figure 17.7).

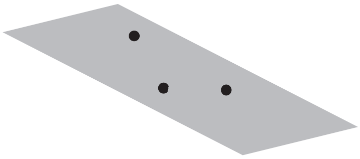


FIGURE 17.7

Postulate 10 is needed to prove the theorem that states a similar fact: A line perpendicular to one of two parallel lines is also perpendicular to the other line.

Postulate 10: Two lines in a given plane perpendicular to a third line are parallel to each other (see Figure 17.8).

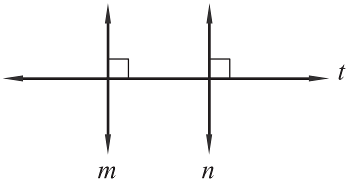


FIGURE 17.8

Postulate 11: In a given plane, one and only one circle can be drawn with any given point as center and line segment of a given length as radius (see Figure 17.9).

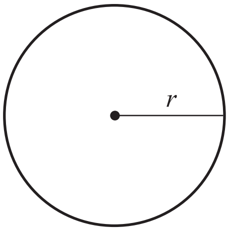


FIGURE 17.9

EXERCISES

17.1 State the axiom or postulate that justifies each statement:

a) $x + y = y + x$
b) $2^3 = 8$ and $6 + 2 = 8$, therefore $2^3 = 6 + 2$
c) *F* and *G* are points on one and only one line.
d) If points *J*, *K*, and *L* are not on the same line, they lie in one and only one plane.
e) If point *P* in plane *A* is the center of a circle with a given radius, only one circle is described.
f) Segment *AD* has only one midpoint.
g) If point *Q* does not lie on line *XY*, only one line parallel to line *XY* can be drawn through point *Q*.

17.2 How many perpendicular lines can be drawn to a given line from a point outside the given line? Which postulate justifies the conclusion?

17.3 What geometric figure describes the shortest distance between two points in one plane? Which postulate justifies the conclusion?

17.4 How many bisectors can angle *TUV* have? Which postulate justifies the conclusion?

17.5 At most, at how many points can two noncollinear lines lying in the same plane intersect? Which postulate justifies the conclusion?

17.6 Lines *EF* and *GH* are both in the same plane and are both perpendicular to line *QR*. Are lines *EF* and *GH* parallel? Which postulate justifies the conclusion?

17.2 Theorems About Lines and Angles in a Plane

Theorems are statements posed about geometric figures, specifically, about the interrelationships of points, lines, and angles in the plane. The truth of a theorem is established only after it is proved by means of definitions, axioms, and postulates. In turn, these theorems can be used to establish other theorems.

The following theorem statements should be memorized. Nonrigorous outlines of proofs are provided for some of the more straightforward theorems.

Vertical angles are congruent **_Theorem 17-1_**

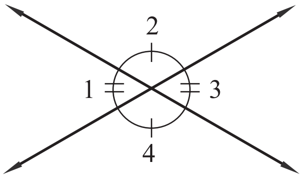


FIGURE 17.10 ***Vertical angles theorem***

Recall from Chapter 15 that vertical angles are formed by intersecting lines as shown in Figure 17.10. Referring to Figure 17.10, we will prove this theorem by showing that $\angle 1$ and $\angle 3$ have equal measure, as do $\angle 2$ and $\angle 4$. That is, that $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$.

Proof of Theorem 17-1

The method of proof for this theorem is to prove a series of statements based on the figure and on the relevant definitions and axioms. These statements must lead to the statement we wish to prove.

1) In Figure 17.10, $m\angle 1 + m\angle 2 = 180$, and $m\angle 2 + m\angle 3 = 180$.
2) By transitivity (Axiom III), $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$.
3) Subtracting $m\angle 2$ from both sides we have $m\angle 1 = m\angle 3$.
4) $\angle 1 \cong \angle 3$ by definition of congruence, which is what we needed to show.

The proof that $\angle 2 \cong \angle 4$ is carried out the same way.

Theorem 17-2

Supplements of the same angle or congruent angles are congruent to each other. Complements of the same angle or congruent angles are congruent to each other.

Recall that supplementary angles have measures whose sum is 180.

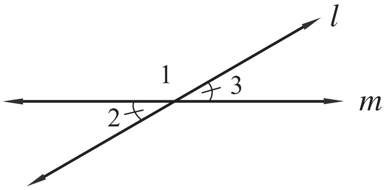


Figure 17.11 ***Supplements of the same angle are congruent to each other***

The first part of Theorem 17-2 follows directly from Theorem 17-1. Referring to Figure 17.11, $\angle 1$ and $\angle 2$ lie on line *l*, and $\angle 1$ and $\angle 3$ lie on line *m*. $\angle 2$ and $\angle 3$ are each supplementary to $\angle 1$. Since $\angle 2$ and $\angle 3$ are also vertical angles, they are congruent to each other by Theorem 17-1.

The proof of the second part of Theorem 17-2 is very similar to the proof of Theorem 17-1.

Recall that complementary angles have measures whose sum is 90.

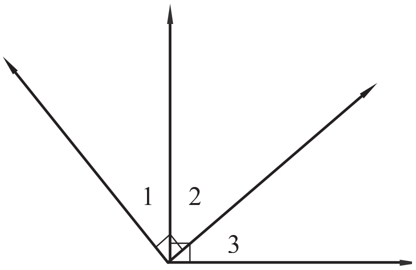


Figure 17.12 ***Complements of the same angle are congruent to each other***

1) In Figure 17.12, $\angle 1$ and $\angle 3$ are each complementary to $\angle 2$. That is,

$$m\angle 1 + m\angle 2 = 90$$
$$m\angle 2 + m\angle 3 = 90$$

2) By transitivity (Axiom III), $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$.

3) Subtracting $m\angle 2$ from both sides we have $m\angle 1 = m\angle 3$.

4) $\angle 1 \cong \angle 3$, by definition of congruence, which is what we needed to show.

The next three theorems are proved by means beyond the scope of this book. These theorem are, nevertheless, very important to know.

A line perpendicular to one of two parallel lines is perpendicular to the other.

Theorem 17-3

See Postulate 10 for justification of Theorem 17-3.

Two parallel lines cut by a transversal form alternate interior angles that are congruent.

Theorem 17-4

In Figure 17.13, $\angle 1 \cong \angle 7$ and $\angle 6 \cong \angle 4$.

Remember that parallel lines do not intersect. This is the same as saying that no matter how far they are extended, they never meet.

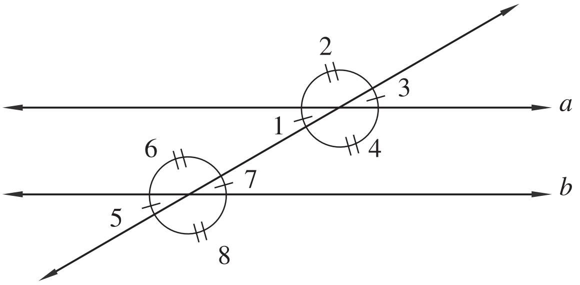


FIGURE 17.13 ***Parallel lines intersected by a transversal***

Two parallel lines cut by a transversal form congruent corresponding angles.

Theorem 17-5

In Figure 17.13, $\angle 2 \cong \angle 6$, $\angle 1 \cong \angle 5$, $\angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$.

The sum of the measures of the interior angles of a triangle is 180.

Theorem 17-6

Consider Figure 17.14 as proof that Theorem 17-6 is true.

Theorem 17-6 is the most fundamental rule of triangles.

In Figure 17.14, the congruent angles a and a′ are identified as being equal in measure by the single mark placed on each angle arc symbol. Angles c and c′ are likewise identified as being congruent by a double mark on the symbol.

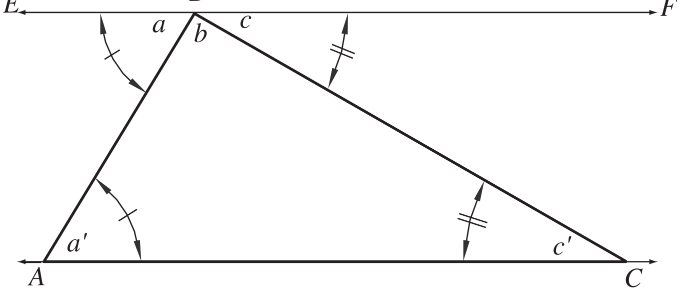


FIGURE 17.14 ***Sum of the interior angles of a triangle.***

In Figure 17.14, $\overleftrightarrow{EF}$ is drawn through point B, parallel to $\overleftrightarrow{AC}$.

Then, $m\angle a + m\angle b + m\angle c = 180$ Straight angle

$m\angle a = m\angle a'$; $m\angle c = m\angle c'$ Alternate interior angles

$m\angle a' + m\angle b + m\angle c' = 180$ Substitution

a', b, and c' are the three interior angles of the triangle given. Thus, we have shown that the sum of the measures of the interior angles of a triangle is 180.

EXAMPLE 17.1A: ***Interior Angle Rule for Triangles***

A triangle has angles measuring 42 and 56. What is the measure of the third angle?

Solution: Since the sum of the measures of the interior angles must equal 180, the third angle is found by subtracting the measures of the other two angles from 180:

$$180 - 42 - 56 = 82$$

Therefore, the third angle measures 82.

EXAMPLE 17.1B: ***Interior Angle Rule for Triangles***

In the figure shown, $m\angle CAB$ = 32 and $m\angle CBD$ = 58. What are the measures of $\angle ABC$ and $\angle BCA$?

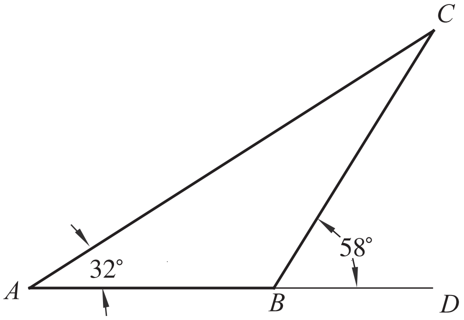


FIGURE 17.15

Solution: Since $\angle ABC$ and $\angle CBD$ are supplementary,

$$m\angle ABC + m\angle CBD = 180$$

$$\begin{aligned} m\angle ABC &= 180 - m\angle CBD \\ &= 180 - 58 \\ &= 122 \end{aligned}$$

Therefore, $m\angle ABC$ = 122.

Now, since we know that the sum of the measures of the interior angles of a triangle must equal 180,

$$m\angle ABC + m\angle BCA + m\angle CAB = 180.$$

EXAMPLE 17.1B: *Interior Angle Rule for Triangles (Continued)*

Solving for $m\angle BCA$,

$$\begin{aligned} m\angle BCA &= 180 - m\angle ABC - m\angle CAB \\ &= 180 - 122 - 32 = 26 \end{aligned}$$

Therefore, $m\angle BCA$ measures 26.

EXAMPLE 17.1C: *Interior Angle Rule for Triangles*

Solve for angles x and y in Figure 17.16 where $\overline{CD}$ bisects $\angle BCA$.

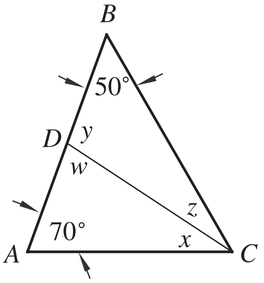


FIGURE 17.16

Solution: Neither of the smaller triangles have enough information to solve for x or y. Since $\angle BCA$ includes one of the desired angles, we first determine its measure as follows:

$$m\angle A + m\angle B + m\angle BCA = 180$$

So, $$m\angle C = 180 - m\angle A - m\angle B = 180 - 70 - 50 = 60$$

Since $\overline{CD}$ bisects $\angle BCA$,

$$m\angle x = \frac{1}{2} m\angle BCA = \frac{1}{2} \cdot 60 = 30$$

$$m\angle x = 30$$

Thus, we may now piece together the information we have to find the other unknown angles. Since we know the measure of $\angle x$ we can apply Theorem 17-6 again to find the measure of $\angle w$.

$$m\angle A + m\angle x + m\angle w = 180$$

So, $$m\angle w = 180 - m\angle A - m\angle x = 180 - 70 - 30 = 80$$

Since y and w form a straight angle, $m\angle y + m\angle w = 180$.

So, $$m\angle y = 180 - m\angle w = 180 - 80 = 100$$

Thus $$m\angle y = 100.$$

Summary: $\angle x$ must be 30°, and $\angle y$ must be 100°.

Theorem 17-6 can be generalized to polygons of any number of sides, as shown in Theorem 17-7.

The sum of the measures of the interior angles of a polygon of n sides is equal to $n - 2$ multiplied by 180.

Theorem 17-7

Theorem 17-7 is often referred to as the "n minus 2 rule." It applies to any polygon, not just regular ones.

Theorem 17-7 is exemplified by a pentagon as shown in Figure 17.17. The concept is extended to other polygons.

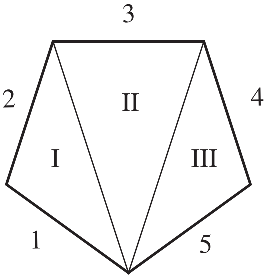


FIGURE 17.17 ***The sum of the interior angles of a polygon***

In the pentagon shown in Figure 17.17, diagonals are drawn from any one vertex to each of the other vertices, dividing the pentagon into three triangles. Triangle II includes one side of the original pentagon, triangles I and III include two sides of the original pentagon. The five-sided polygon is thus divided into (5 – 2) triangles. Since each triangle has interior angle measure of 180, we see that the total interior angle measure of the pentagon is

$$\frac{180}{\text{triangle}}(5-2\text{ triangles})=180\cdot 3=540$$

Extending this to any polygon, we would find that each triangle in such a division produces a number of middle triangles with just one side from the original polygon and two outer triangles with two such sides. The number of triangles that are produced is always two less than the number of sides of that polygon. This is expressed mathematically as is $n - 2$ where n is the number of sides. For the pentagon, $n = 5$, so $n - 2 = 5 - 2 = 3$. As the sum of the measure of the interior angles of the polygon equals the sum of the measure of the interior angles of the triangles constructed within it, we conclude that the sum of the measures of the interior angles of a polygon of n sides is determined by the formula:

Sum of the measures of the interior angles = $180(n - 2)$.

EXAMPLE 17.1D: *Sum of the Interior Angles of a Triangle*

Be careful: Theorem 17-7 applies to interior angles only.

Use the "*n* minus 2 rule" (Theorem 17-7) to show that the sum of the measures of the interior angles of a triangle add to 180.

Solution: In the case of a triangle, a polygon in which $n = 3$, the n minus 2 rule gives

$$180(n-2)=(3-2)(180)=(1)(180)=180.$$

EXAMPLE 17.2: ***Sum of Interior Angles of a Polygon***

Show that the sum of the measures of the interior angles of a quadrilateral add to 360. Confirm using the *n* minus 2 rule.

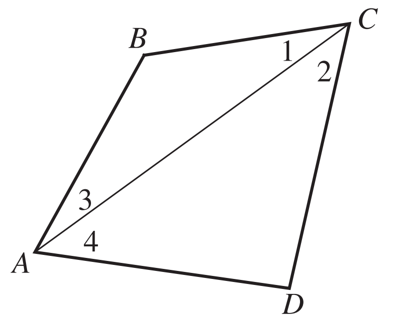


FIGURE 17.18

Solution : Solving without applying the *n* minus 2 rule directly to the quadrilateral:

STEP 1: Split quadrilateral $ABCD$ into two triangles by drawing $\overline{AC}$.

STEP 2: $m\angle 1 + m\angle 2 = m\angle BCD$ and $m\angle 3 + m\angle 4 = m\angle BAD$ (by construction)

STEP 3: $m\angle 1 + m\angle 3 + m\angle CBA = 180$ and $m\angle 2 + m\angle 4 + m\angle CDA = 180$ (Theorem 17-7)

STEP 4: $m\angle 1 + m\angle 3 + m\angle CBA + m\angle 2 + m\angle 4 + m\angle CDA = 180 + 180$ (transitivity)

STEP 5: For clarity in the next step, group like terms of the equation in Step 4 to get $m\angle 1 + m\angle 2 + m\angle CBA + m\angle 3 + m\angle 4 + m\angle CDA = 360$

STEP 6: $m\angle A + m\angle B + m\angle C + m\angle D = 360$ (substitution)

In Step 6 we name the angles by the vertex angles alone.

Confirm the solution by applying the *n* minus 2 rule directly to the quadrilateral. In the case of a quadrilateral, $n = 4$ and the *n* minus 2 rule gives

$$(180)(n-2) = (180)(4-2) = (180)(2) = 360$$

EXAMPLE 17.3: ***n Minus 2 Rule***

Find the sum of the measures of the interior angles of a hexagon.

Solution: A hexagon has six sides, so $n = 6$.

The calculation is:

$$\text{Sum of interior angles } = (n-2)(180) = (6-2)(180) = (4)(180) = 720$$

EXAMPLE 17.4A: *Interior Angle Rule for Polygons*

Consider the plot of land shown in Figure 17.19 and determine $m\angle\alpha$.

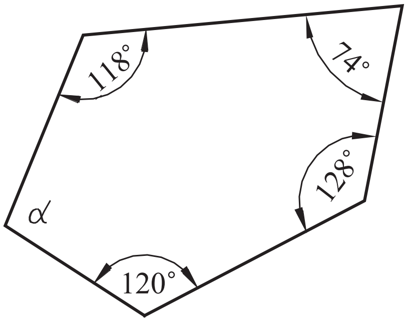


FIGURE 17.19

Solution:

The plot of land is in the shape of a pentagon. Therefore, from Theorem 17-7,

$$\text{Sum of interior angles} = (180)(5 - 2) = (180)(3) = 540$$

All angles except $\angle\alpha$ are given. Thus,

$$m\angle\alpha + 118 + 74 + 128 + 120 = 540$$

$$m\angle\alpha = 540 - 118 - 74 - 128 - 120$$

Therefore, $\angle\alpha$ must be 100°.

EXAMPLE 17.4B: *Interior Angle Rule for Polygons*

Determine $m\angle\alpha$ for the die shown in Figure 17.20.

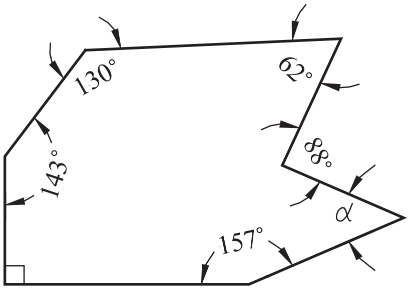


FIGURE 17.20

Solution:

The die is a seven-sided object. Therefore, from Theorem 17-7 the sum of the measures of the interior angles is (7 – 2)(180) = 900.

Notice that the angle with measure equal to 88 is an exterior angle. Theorem 17-7 holds only for interior angles. Therefore, the interior angle must be found first. The measure of the interior angle is simply 360 – 88 = 272. Then,

$$m\angle\alpha + 272 + 62 + 130 + 143 + 90 + 157 = 900$$

EXAMPLE 17.4B: ***Interior Angle Rule for Polygons***

So, $m\angle\alpha = 900 - 272 - 62 - 130 - 143 - 90 - 157 = 46$

Therefore, $\angle\alpha$ must be 46°.

The measure of an exterior angle of a triangle, formed by extending one side of a triangle, is equal to the sum of the measures of the two opposite interior angles.

Theorem 17-8

Theorem 17-8 can be easily demonstrated, as illustrated in Figure 17.21.

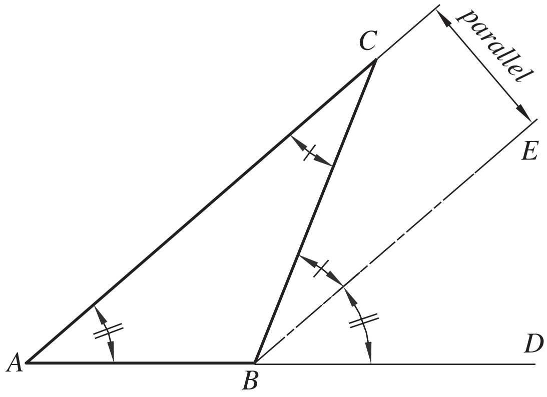


FIGURE 17.21 ***Exterior angle rule for triangles***

In Figure 17.21, $\overline{BE}$ is parallel to $\overline{AC}$. Also, $\overline{AD}$ and $\overline{BC}$ are transversals to the parallel lines. Then,

$m\angle BAC = m\angle DBE$ *by congruence of corresponding angles*

$m\angle BCA = m\angle EBC$ *by congruence of alternate interior angles*

$m\angle BAC + m\angle BCA = m\angle DBE + m\angle EBC$ *by congruence*

$m\angle DBC = m\angle DBE + m\angle EBC$ *by construction*

$m\angle DBC = m\angle BAC + m\angle BCA$ *by substitution*

Strategies for Solving Geometry Problems

Proving a theorem is a form of geometric problem solving. Solving geometry problems requires *logic*—that is, careful, correct reasoning. The information given in a *problem statement* is considered and various definitions, axioms, postulates, and theorems are logically applied in reasoned steps until the desired outcome is reached.

A good beginning in solving a problem involving geometry is to sketch a relevant figure or figures if one is not provided. Elaborate drawings with exact proportions are not necessary. However, a sketch should be large enough so that geometric relationships, such as those among angles, dimensions, and labels can be clearly seen.

This strategy is useful for other types of problems encountered in technical fields.

The charting technique is a traditional form of solving geometry problems or proving geometry theorems.

Next, verify that the information is reflected in the sketch.

Then state as simply as possible the question to be answered or the problem to be solved. Then, reasoning one step at a time, apply definitions, axioms, postulates, and theorems to the known information until the solution is found. This is called *charting* the data. A charted proof consists of two columns. *Statements* are written in the left column and the reason or justification for the statement is written in the right column, often by reference to a definition, axiom, postulate, or theorem.

Finally, after solving the problem, decide if the result is reasonable.

EXAMPLE 17.5: ***Solving a Geometry Problem by Charting***

Given the triangle shown in Figure 17.22, let $m\angle A = 70$, $m\angle B = 50$, and $\overline{CD}$ bisect $\angle ACB$. Solve for angles x and y using the charting method.

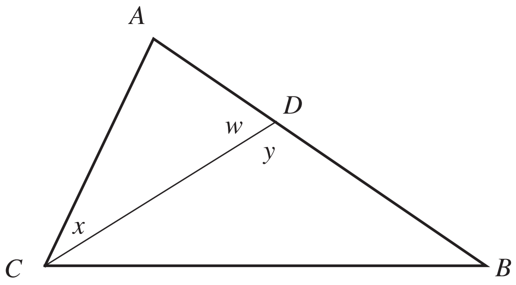


FIGURE 17.22

Solution:

Statement	*Reason*
STEP 1: $m\angle A = 70$ and $m\angle B = 50$	Given
STEP 2: $m\angle A + m\angle B + m\angle ACB = 180$	Theorem 17-7
STEP 3: $m\angle ACB = 180 - m\angle A - m\angle B$ $= 180 - 70 - 50$	Subtraction property
STEP 4: $m\angle x = \frac{1}{2} m\angle ACB$ $= \frac{1}{2}(60)$ $= 30$	$\overline{CD}$ bisects $\angle C$

EXAMPLE 17.5: ***Solving a Geometry Problem by Charting (Continued)***

STEP 5: In triangle *ADC*, $m\angle A + m\angle x + m\angle w = 180$ $m\angle w = 180 - m\angle A - m\angle x$ $= 180 - 70 - 30$ $= 80$	Theorem 17-7
STEP 6: $m\angle y + m\angle w = 180$ $m\angle y = 180 - m\angle w$ $= 180 - 80$ $= 100$	Definition of straight angle

EXERCISES

17.7 In the oblique triangle shown in Figure 17.23, solve for ∠*B*.

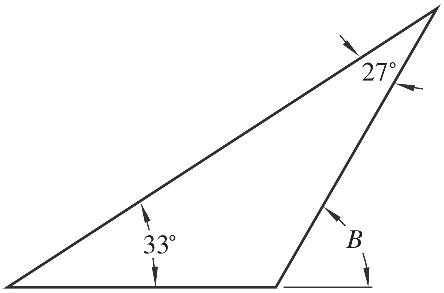


FIGURE 17.23

17.8 In the polygon shown in Figure 17.24, solve for angle *A* using the (*n* – 2) rule.

Hint: Be sure to use only interior angles.

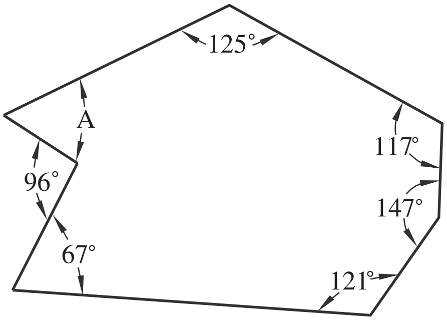


FIGURE 17.24

17.9 Determine $\angle B$ in Figure 17.25.

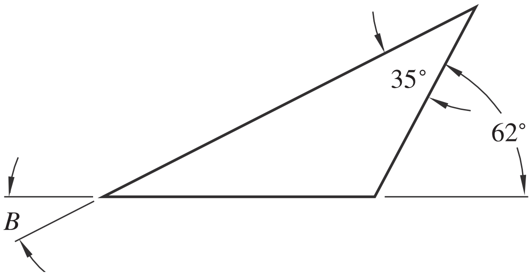


FIGURE 17.25

17.10 Solve for angle C in Figure 17.26.

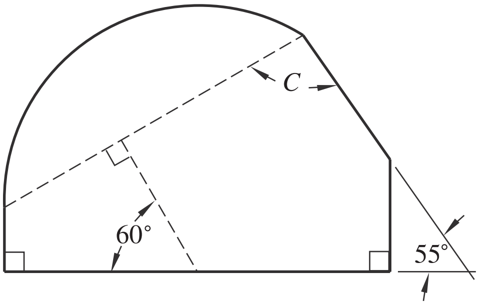


FIGURE 17.26

17.11 In the right triangle ACB shown in Figure 17.27, solve for angle ABC.

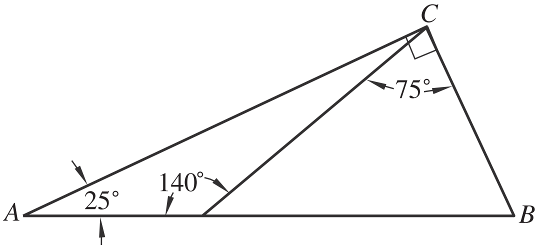


FIGURE 17.27

Hint: Find the supplement of the114° angle, then solve for β in the four-sided polygon pictured, using (n – 2)(180°).

17.12 A triangle is circumscribed around a circle in Figure 17.28. Solve for angle β.

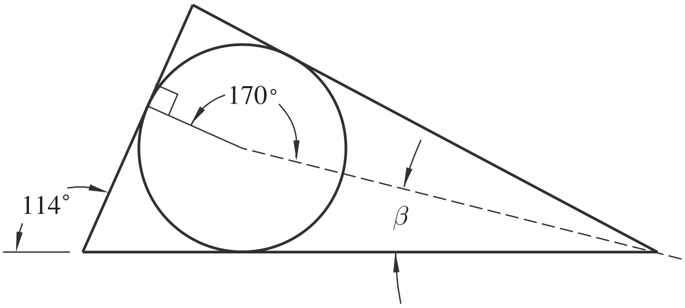


FIGURE 17.28

17.13 In the polygon shown in Figure 17.29, determine angle *A* by using the *n* – 2 rule. Be sure to use only interior angles.

Hint: Notice the parallel sides indicated by dimension C.

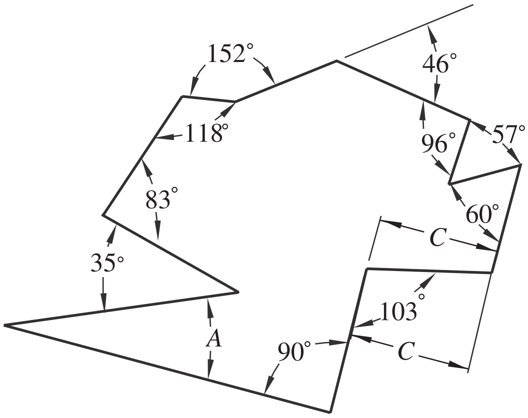


FIGURE 17.29

17.14 Solve for angle *B* in the polygon shown in Figure 17.30 by using the *n* – 2 rule.

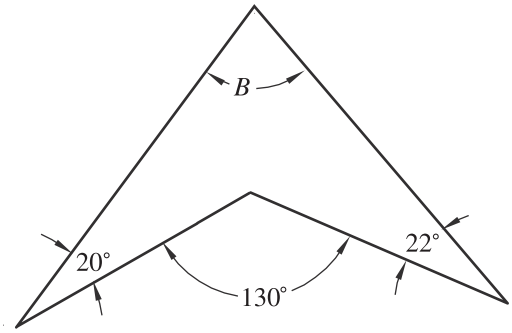


FIGURE 17.30

17.15 Determine angle *C* in Figure 17.31.

Hint: Consider opposite angles, supplements, and sum of interior angles in a triangle.

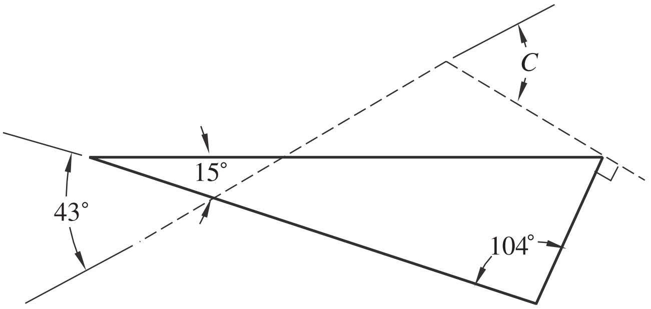


FIGURE 17.31

17.16 The dimensions *C* and *C* in Figure 17.32 indicate the lines are parallel. Solve for angle *A*.

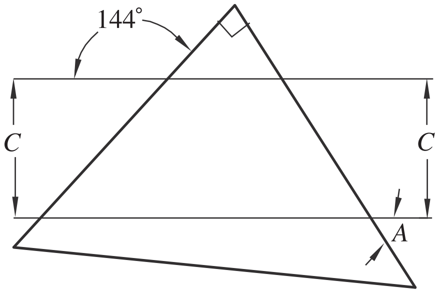


FIGURE 17.32

17.17 A molding gauge is shown in Figure 17.33. Solve for angle α.

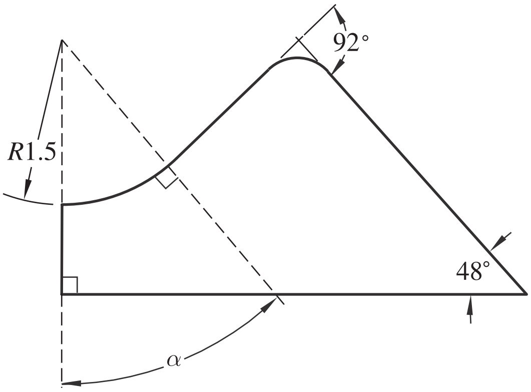


FIGURE 17.33

17.18 A lift-stop on a machine is shown in Figure 17.34. What is the angle of the lift wedge *β* when the ball stop angle is 117º?

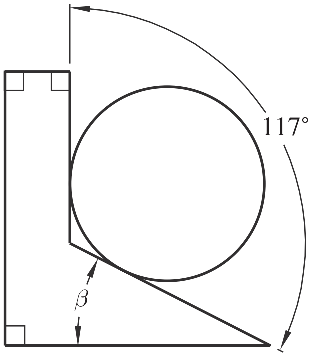


FIGURE 17.34

17.19 An A-frame truss is shown in Figure 17.35. From the data given, solve for angles α and β.

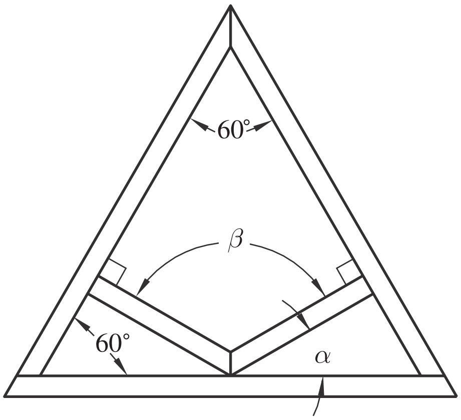


FIGURE 17.35

17.20 In triangle *ABC*, shown in Figure 17.36, the sizes of the acute angles are given as $\angle CAB = 3x$ and $\angle CBA = 2x$. Solve for the two angles in degrees.

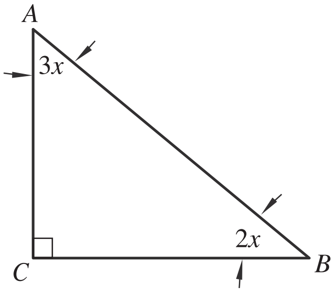


FIGURE 17.36

17.21 In Figure 17.37, *AB* equals *AC*. Angle $BAC = 80°$, *BD* bisects angle *ABC*, and *DC* bisects angle *ACB*. Solve for angle *ABC* and angle *BDC*.

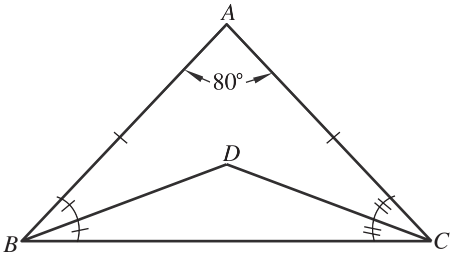


FIGURE 17.37

17.22 Solve for angle β in Figure 17.38.

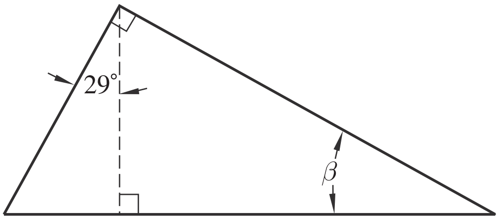


FIGURE 17.38

17.23 Solve for angle α in Figure 17.39.

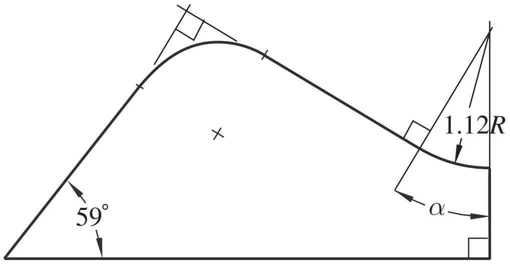


FIGURE 17.39

17.24 Determine $\angle\beta$ on the disc shown in Figure 17.40.

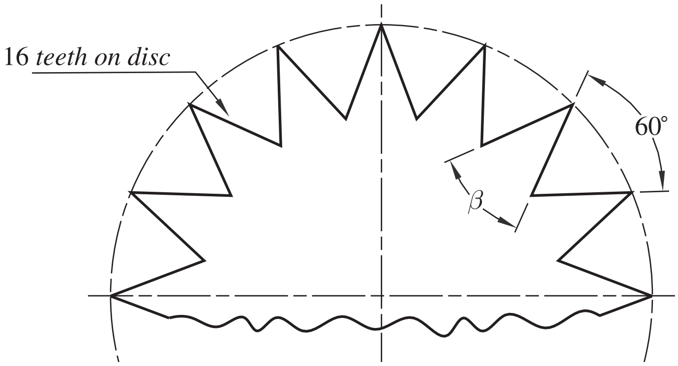


FIGURE 17.40

18

TRIANGLES

Triangles are perhaps the most important of geometric figures in engineering and technology and appropriately receive considerable attention. Triangles are often seen in structures in the form of trusses and guy wires. Mechanical forces acting along the lines of these structures are resolved into component forces using laws of *trigonometry*. This process is called *triangulation*. Surveyors use triangulation to measure elevations and distances. In fact, the Greek mathematician Eratosthenes (circa 276–196 BC) estimated the circumference of the earth with extraordinary accuracy using a large triangle based on the simultaneous measurement of the angles between the sun and the Egyptian cities of Syene and Alexandria. In electrical engineering, triangles are used for calculations involving alternating current and three-phase power.

The list of applications for triangles is very long. In this chapter, the fundamental rules of triangles are presented and the foundation laid for the study of trigonometry. The essential laws of trigonometry are presented in Chapter 20.

18.1 Special Lines in Triangles

To aid in the solution of problems involving triangles, certain auxiliary lines or constructions are sometimes drawn. These lines and constructions include the *angle bisector, circumscribed circle, inscribed circle,* and *altitude.*

All of the special lines and constructions discussed in this section can be determined using only a straight edge and compass. These are called *geometric constructions* and are not discussed in this book. If interested, consult a good drafting book to learn how these constructions are done.

An example of an angle bisector is shown in Figure 18.1 where $\angle B$ in $\triangle ABC$ is bisected by line BD, the angle bisector.

The $\triangle$ symbol stands for triangle.

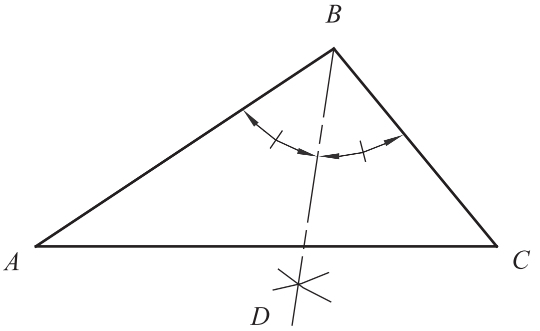


FIGURE 18.1 ***The angle bisector***

In the case of an equilateral triangle the angle bisector also bisects the side opposite the bisected angle, and intersects it at right angles as shown in Figure 18.2.

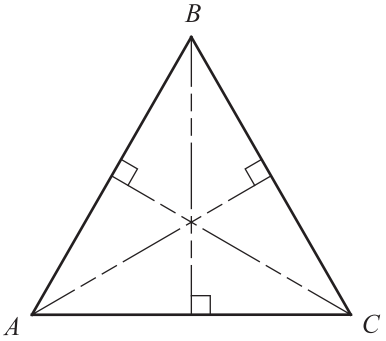


FIGURE 18.2 ***Angle bisector in an equilateral triangle***

The three angle bisectors of an equilateral triangle meet at a common point, which is the center of the *circumscribed circle* shown in Figure 18.3A. As will be discussed in Chapter 19, the smallest circle having three points touching the triangle must be *inside* the triangle. The largest circle having three points touching the triangle must be *outside* the triangle. In both cases the coincident points are *on* the circle.

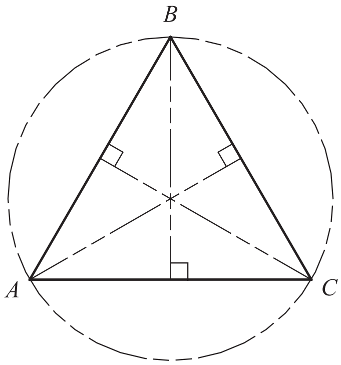


FIGURE 18.3A ***Circle circumscribed about an equilaterial triangle***

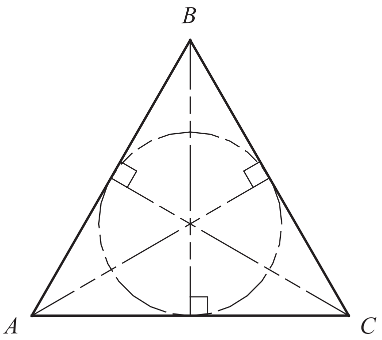


FIGURE 18.3B ***Circle inscribed in an equilaterial triangle***

The center formed by the bisectors of an equilateral triangle is also the center of the inscribed circle shown in Figure 18.3B. The three points of intersection occur at the points where the perpendicular bisectors of the sides of the equilateral triangle meet the circle. The inscribed circle is a very important element in metal cutting tool geometry.

For scalene triangles, the angle bisector of any given angle does *not* bisect the opposite side. Figure 18.4 shows a scalene triangle with line *BD* the bisector of ∠*B*, which does not bisect the opposite side *AC*.

Recall that a scalene triangle has three sides of unequal length.

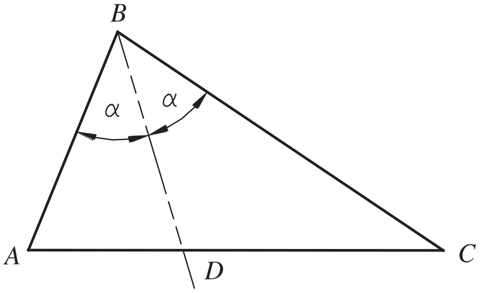


FIGURE 18.4 ***Angle bisector in a scalene triangle***

As shown in Figure 18.5, the bisector of ∠*B* in the isosceles triangle bisects the opposite side, *AC*, the side that is the base.

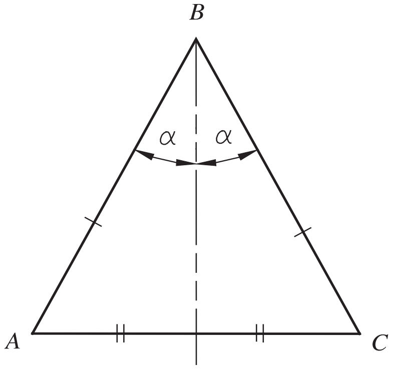


FIGURE 18.5 ***Angle bisector in an isosceles triangle***

An altitude to a side of any triangle is a line drawn from a vertex perpendicular to the opposite side. In the special case of a right triangle, only the altitude from the 90° vertex is visible, as the other two altitudes coincide with the sides of the right triangle. This can be observed from Figure 18.6.

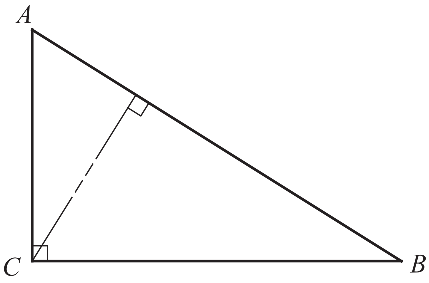


FIGURE 18.6 ***An altitude to the hypotenuse***

The hypotenuse is the longest side of a right triangle. It is always the side opposite the 90° angle.

The altitudes in a typical scalene triangle are shown in Figure 18.7. Altitude *BD* is perpendicular to side *AC* and *within* the boundaries of the triangle. This is always true of the altitude from the oblique triangle's obtuse angle. Altitudes from its acute angles, $\angle A$ and $\angle C$, fall *outside* the boundaries of the triangle and so the sides to which they are drawn must be extended: *BC* is extended to point *E* for altitude *AE*, and *AB* to point *F* for altitude *CF*.

An oblique triangle is any triangle without a 90° angle.

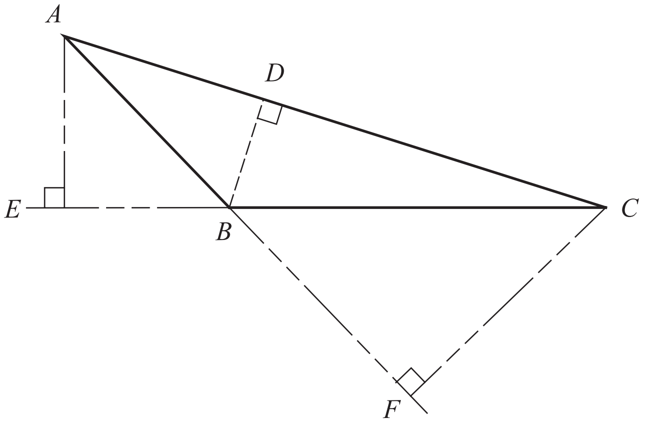


FIGURE 18.7 ***Altitudes in a scalene triangle***

EXERCISES

18.1 What is an angle bisector?

18.2 True or False: An angle bisector in an equilateral triangle bisects the side opposite the bisected angle.

18.3 True or False: An angle bisector in a scalene triangle bisects the side opposite the bisected angle.

18.4 What is the altitude of a triangle?

18.2 Similar Triangles

Two triangles are said to be *similar* if their shapes are the same but their relative sizes are different. In similar triangles, such as the two pictured in Figure 18.8, the angles *A* and *A′* are called *corresponding angles* because they are a congruent pair. The same is true of *B* and *B′*, *C* and *C′*. For triangles to be similar, all the angles of one must have a corresponding congruent angle in the other. So,

$$\angle A \cong \angle A', \qquad \angle B \cong \angle B', \qquad \angle C \cong \angle C'.$$

Equivalently, $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, and $m\angle C = m\angle C'$.

The other fact about similar triangles is that the ratios of their corresponding sides are in proportion to one another. Thus, in Figure 18.8,

$$\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}$$

Other polygons, such as hexagons and octagons may be similar by the same argument.

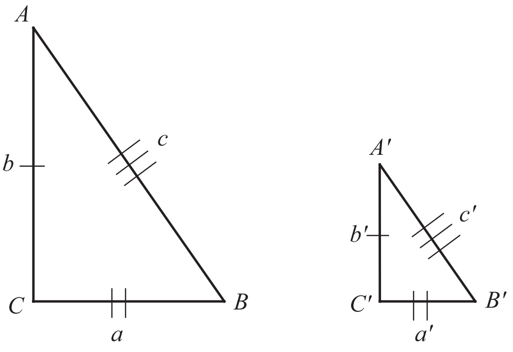


FIGURE 18.8 ***Corresponding sides***

In Figure 18.8, uppercase letters are used conventionally to designate angle vertices and lower-case letters are used for sides. The side opposite ∠A is a, the side opposite ∠B is b, and the side opposite ∠C is c.

The symbol ~ means *is similar to.* Hence, $\triangle ABC \sim \triangle A'B'C'$ is read, "Triangle *ABC* is similar to triangle *A* prime, *B* prime, *C* prime."

The facts about similar triangles are summed up in the definition.

Definition of Similar Triangles

Corresponding angles of similar triangles are equal in measure.

Ratios of corresponding sides of similar triangles are in proportion.

Theorem 18-1, called the "AA (angle-angle) theorem" for similarity, shows that establishing the congruence of two pairs of angles is sufficient to show similarity exists between triangles.

Theorem 18-1 (AA Theorem for Similarity)

Two triangles are similar if two angles of one triangle are equal in measure to two angles of the second triangle.

Thus, if the conditions of Theorem 18-1 are satisfied, then the third pair of angles are also equal in measure since the sum of the measures of the interior angles of any triangle must be 180. Also notice that in the case of right triangles the conditions of Theorem 18-1 are satisfied if an acute angle of one triangle is equal to an acute angle in the other triangle since "right triangle" implies that both triangles already have one angle measure equal, the 90° angle.

Theorem 18-2, called the "SAS (side-angle-side) theorem" for similarity gives another sufficient condition for similarity to exist.

Theorem 18-2 (SAS Theorem for Similarity)

Two triangles are similar if one pair of angles are equal in measure and the sides including these angles have ratios that are proportional.

Theorem 18-3

A line drawn inside a triangle parallel to one of its sides forms a new triangle that is similar to the given triangle.

By Theorem 18-3, $\triangle A'BC'$ is similar to $\triangle ABC$ as shown in Figure 18.9.

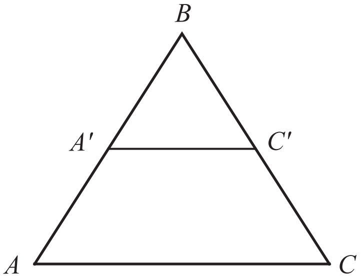


FIGURE 18.9 ***Parallel line drawn inside a triangle***

Theorem 18-4 Two triangles are similar if each side of one triangle is parallel to a side of the other triangle.

EXAMPLE 18.1A: *Similar Triangles*

A carpenter needs to build a triangular jig that is a smaller version of an existing jig. Find the unknown dimension x.

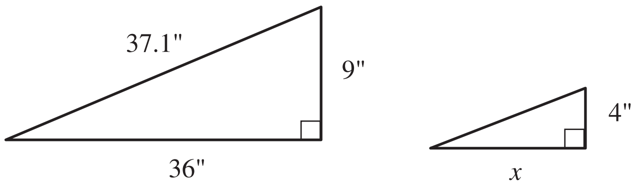


FIGURE 18.10

Solution: Since the two triangles are similar, their corresponding angles have equal measures and their corresponding sides are proportional.

Therefore,

$$\frac{4}{9}=\frac{x}{36}$$

$$9x=(4)(36)=144$$

$$x=\frac{144}{9}=16 \text{ inches}$$

Notice that we could have also used the following proportions to reach the same conclusion:

$$\frac{4}{x}=\frac{9}{36} \quad \text{or} \quad \frac{9}{4}=\frac{36}{x}$$

The form of the proportion that is usually chosen is the one that places the unknown quantity in the numerator and consequently can be solved most easily.

EXAMPLE 18.1B: *Similar Triangles*

A construction inspector needs to estimate the height of a tree. The sun causes the 6-foot-tall inspector to cast a shadow 8 feet long. The tree's shadow is 40 feet long. How tall is the tree?

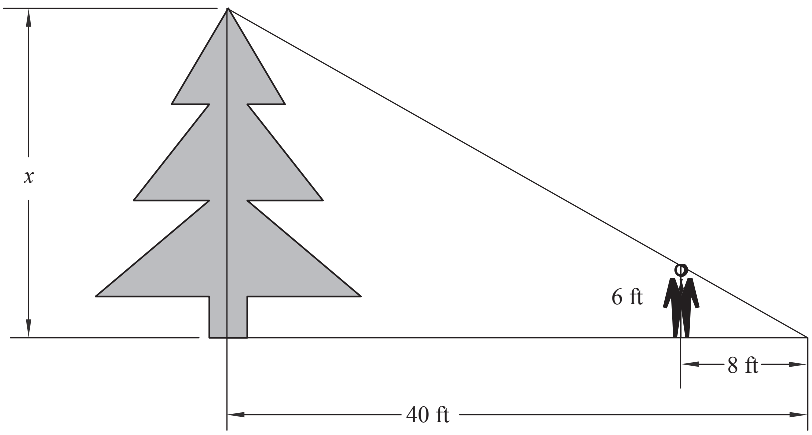


FIGURE 18.11

Solution:

Let *x* represent the height of the tree. The sketch helps us visualize the similarity of the triangles. Both the tree and the inspector make 90° angles with the ground and the tips of their shadows share a common angle. Therefore, by the AA theorem, the third angles of both triangles have equal measures. Hence, the triangles are similar and the following proportion is established:

$$\frac{x}{6} = \frac{40}{8}$$

$$8x = 240$$

$$x = 30 \text{ feet}$$

Theorem 18-5 Mean Proportional

The altitude drawn from the right angle to the hypotenuse of a right triangle forms two triangles. These triangles are similar to the given triangle and to each other.

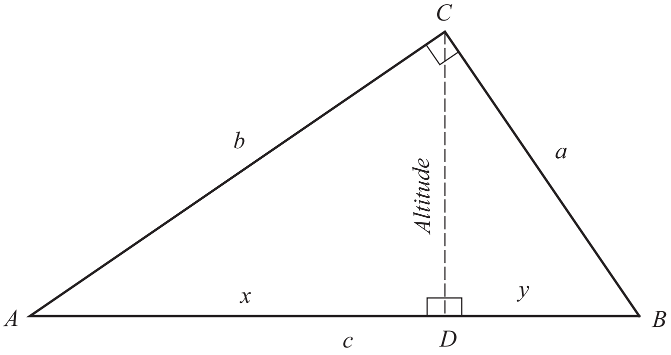


FIGURE 18.12 ***Altitude to the hypotenuse of a right triangle***

When stating the similarity relationship of triangles, name the triangles with their corresponding angles in the same order.

To see the truth of Theorem 18-5, consider Figure 18.12. According to the theorem, $\triangle ABC \sim \triangle ACD$. This is indeed the case because the conditions of the AA theorem (Theorem 18-1) are satisfied. Namely, two angles from both triangles are equal in measure since each triangle has a right angle, and $\angle CAD$ is common to both triangles. An analogous argument shows that $\triangle ABC \sim \triangle CBD$. Furthermore, since $\triangle ACD$ and $\triangle CBD$ are similar to $\triangle ABC$, they are similar to one another.

Theorem 18-6 The length of a leg of a right triangle is the geometric mean between the length of the hypotenuse and the length of the projection of that leg on the hypotenuse.

The term "projection" refers to segments x and y in Figure 18.12. The projection of AC is segment x and the projection of BC is segment y.

By Theorem 18-5, $\triangle ACD \sim \triangle ABC$ and therefore, $\dfrac{x}{AC} = \dfrac{AC}{AB}$.

Also by Theorem 18-2, $\triangle CBD \sim \triangle ABC$ and therefore, $\dfrac{y}{BC} = \dfrac{BC}{AB}$.

EXAMPLE 18.2: ***Projection Length***

If in Figure 18.12, $AC = 4$ feet and $AB = 8$ feet, how long is segment x?

Solution:

Since,

$$\frac{x}{AC} = \frac{AC}{AB}$$

$$x = \frac{(AC)^2}{AB} = \frac{(4\text{ ft})^2}{8\text{ ft}} = 2\text{ feet}$$

EXERCISES

18.5 In Figure 18.13 right triangle ABC is cut by a line DE which is perpendicular to the base BC at E. If $AB = 6$, $BC = 30$, and $EC = 15$, what is the length of DE?

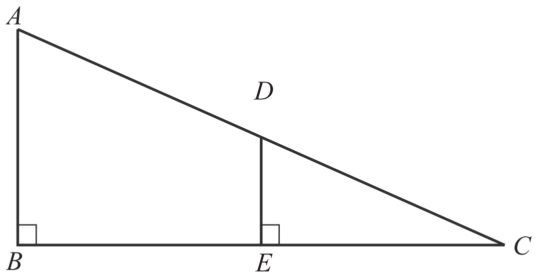


FIGURE 18.13

18.6 In Figure 18.14 similar triangles *ABC* and *DEF* are shown with $AC \perp EF$ and $AB \perp ED$. Solve for $\angle DFE$ when $\angle ABC$ is 70°.

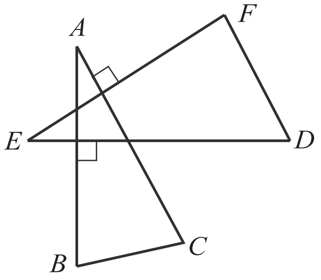


FIGURE 18.14

18.7 In triangle *ABC* shown in Figure 18.15, angle *ABC* is a right angle, and *BD* is drawn perpendicular to *AC*. If angle *DBC* is 30°, what is angle *BAD*?

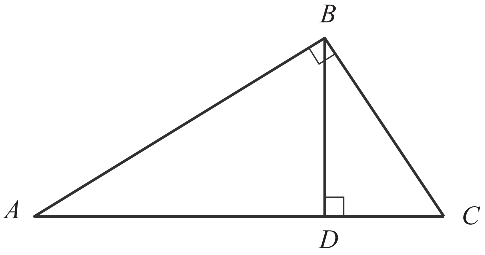


FIGURE 18.15

18.8 In Figure 18.16, $\overline{AC}$ is 25 inches and $\overline{BD}$ is 10 inches. $\overline{EG}$ is parallel to $\overline{AC}$ and is 4 inches above $\overline{AC}$. Find the length of $\overline{EG}$.

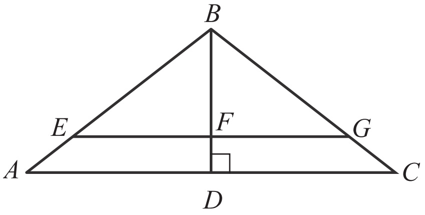


FIGURE 18.16

18.9 In triangle *ABC* shown in Figure 18.17, line *AD* bisects angle *BAC* and is perpendicular to side *BC*. If side *BC* is 6 inches long, what is the length of $\overline{BD}$? of $\overline{DC}$? Are triangles *ABD* and *ACD* congruent? Why?

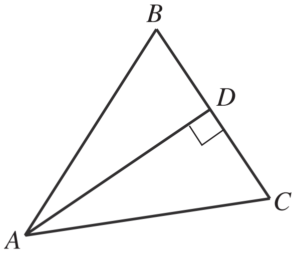


FIGURE 18.17

18.10 In Figure 18.18 the interior angles of triangle *ABC* are equal. Line *BE* bisects angle *ABC* and is perpendicular to line *AC*. Line *CE* bisects angle *ACD*. Solve for angle *BEC*.

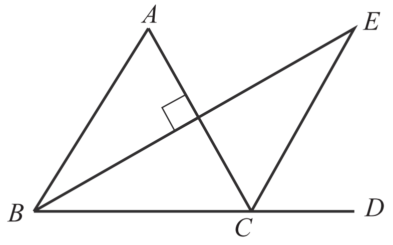


FIGURE 18.18

18.11 In Figure 18.19, *AB* = *AD* and *BC* = C*D*. Solve for angles *BCD* and *BDC*.

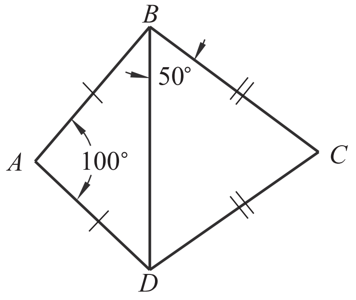


FIGURE 18.19

18.12 In triangle *ABC* shown in Figure 18.20, segments *DE* and *BC* are parallel, *DA* = 4 inches, *AB* = 8 inches, *DE* = 10 inches. Solve for length of side *BC*.

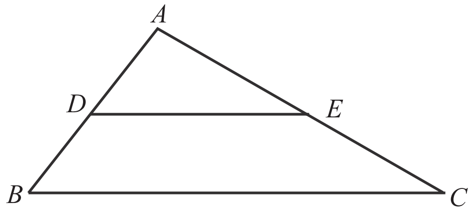


FIGURE 18.20

18.13 In Figure 18.21, *AB* = *BC* = *AC, BD* = *AD*. Angle *BDC* is 45°. Solve for angle *ABD* and angle *BCD*.

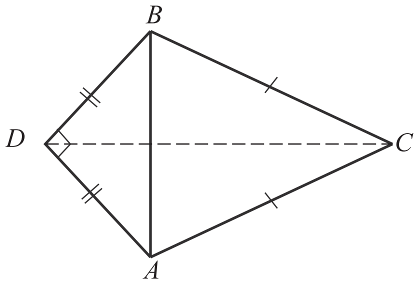


FIGURE 18.21

18.14 Triangle *ABC* in Figure 18.22 is equilateral. Angle *EAD* is 40° and angle *ADE* is 90°. Solve for angle *AEB* and angle *CED*.

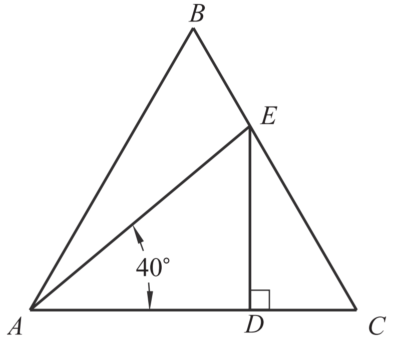


FIGURE 18.22

18.15 In right triangle *BAC* shown in Figure 18.23, lines *DE* and *FG* are parallel to the base *BC* and cut the side *AB* into three congruent segments and the side *AC* into congruent segments. *FG* is 22 inches. Solve for the length of segments *DE* and *BC*.

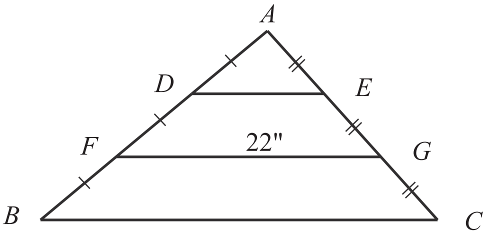


FIGURE 18.23

18.16 In Figure 18.24, $m\angle ABC = m\angle CBD$ and $m\angle BAC = m\angle CAE$. Lines *BD* and *AE* are parallel. Solve for angle *ACB*.

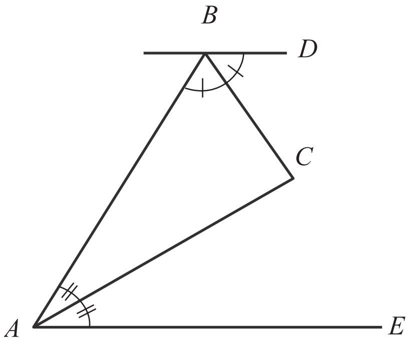


FIGURE 18.24

18.17 A flagpole casts a 15-foot shadow at the same time that a 6-foot post casts a 2-foot shadow. Find the height of the flagpole if the pole and the post are both at right angles to the ground.

18.18 Solve for *CD* in Figure 18.25. Dimensions are in meters.

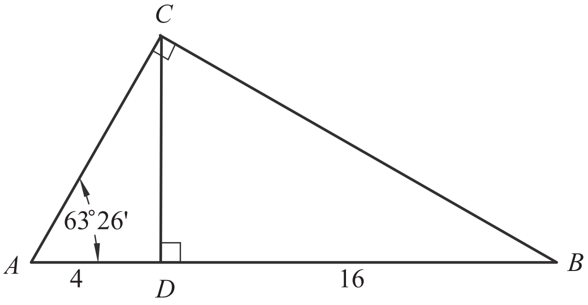


FIGURE 18.25

18.19 In triangle *ABC* shown in Figure 18.25, if angle *CAD* = 63° 26′ solve for angles *ACD, DCB* and *CBD.*

18.20 In Figure 18.26, solve for:

a) a and h when $p = 2$ and $q = 6$
b) c and h when $p = 4$ and $a = 6$
c) q and b when $p = 16$ and $h = 8$
d) p and h when $b = 12$ and $q = 6$

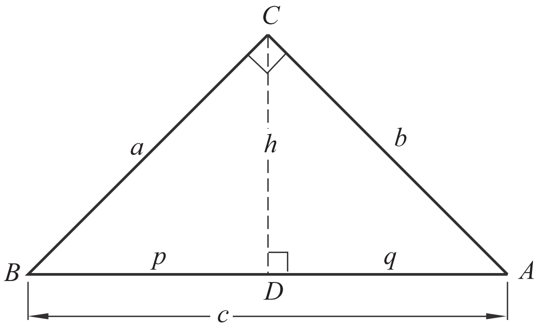


FIGURE 18.26

18.21 Solve for distance *CD* in Figure 18.27. Dimensions are in meters.

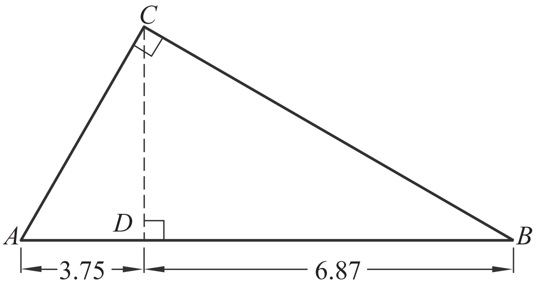


FIGURE 18.27

18.22 Solve for angle β in Figure 18.28.

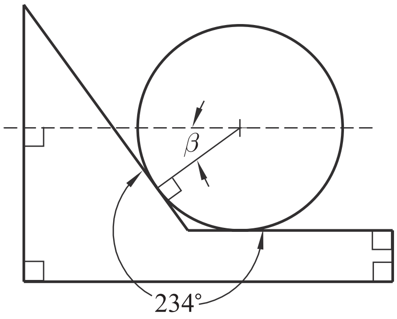


FIGURE 18.28

18.3 Pythagorean Theorem

Pythagoras (circa 569–475 BC) was a Greek philosopher and mathematician. He was credited with discovering one of the most significant theorems in geometry: the Pythagorean theorem. It is the basis for proof of many principles in technology and finds application everywhere from aerospace engineering to solutions of common household problems. Memorization of this theorem is essential.

In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Theorem 18-7 (Pythagorean)

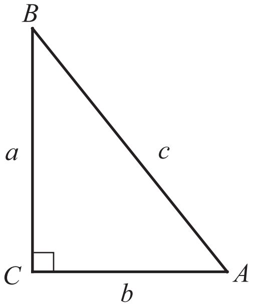


FIGURE 18.29 ***Standard right triangle***

Figure 18.29 shows the classic labeling for right triangles. Angles have uppercase labels and sides have lower case labels. The altitude lies opposite angle *A* and is represented by *a*. Similarly, the base is opposite angle B and is represented by *b*. Angle *C* is reserved for the right angle and the hypotenuse is represented by *c*. Accordingly, the Pythagorean theorem is written symbolically:

$$c^2 = a^2 + b^2$$

For example, if a right triangle has side lengths $a = 5$ cm, and $b = 12$ cm, then hypotenuse length *c* is found by substituting *a* and *b* into the Pythagorean formula and solving for *c*:

$$c^2 = a^2 + b^2 = (5\text{ cm})^2 + (12\text{ cm})^2 = 25\text{ cm}^2 + 144\text{ cm}^2 = 169\text{ cm}^2$$

Taking the square root of each side gives the answer:

$$c = \sqrt{169\text{ cm}^2} = 13\text{ cm}$$

Two of the several proofs for the Pythagorean theorem are discussed below.

Proof 1 of Pythagorean Theorem: In Figure 18.30, altitude *CD* is perpendicular to hypotenuse *AB*. The resulting segments into which *AB* is broken are labeled *x* and *y*.

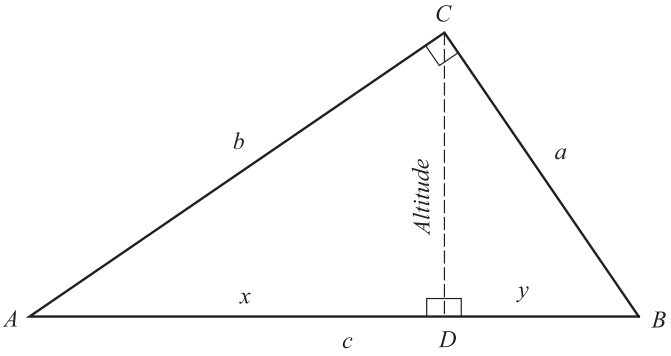


FIGURE 18.30

Observe that $\triangle ABC \sim \triangle CBD$. Likewise, $\triangle ABC \sim \triangle ACD$. Hence, because of the mean proportional theorem for similar triangles (Theorem 18-5),

$$\frac{c}{a} = \frac{a}{y} \text{ and } \frac{c}{b} = \frac{b}{x}$$

Rearranging the equations yields:

$$cy = a^2 \text{ and } cx = b^2$$

Adding these two equations gives

$$cy + cx = a^2 + b^2$$

Factoring out the c term gives

$$c\left(y + x\right) = a^2 + b^2$$

Since $y + x = c$ as can be seen in Figure 18.30, we can make a substitution to get

$$cc = a^2 + b^2$$

or $c^2 = a^2 + b^2$ and the Pythagorean theorem is proved.

Proof 2 of Pythagorean Theorem (Graphical):
A second proof of the Pythagorean theorem is motivated by Figure 18.31.

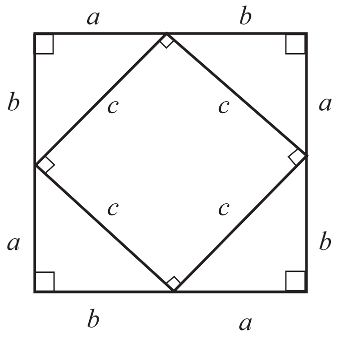


FIGURE 18.31 ***Graphical proof of Pythagorean theorem***

The area of the outer square shown in Figure 18.31 can be written in terms of *a* and *b* as:

$A = (a + b)^2$ since the length of any side of the square is given by $(a + b)$.

The same area can also be thought of as the sum of the areas of the inner square and the four triangles surrounding the inner square. In this case, the area is given as:

$$c^2 + (4)\left(\frac{1}{2}ab\right)$$

Since both of the area equations give the area of the outer square, they can be set equal to one another and solved for c^2, the length of the hypotenuse of any of the right triangles in the figure whose legs are given by *a* and *b*. Accordingly,

$$c^2 + (4)\left(\frac{1}{2}ab\right) = (a+b)^2$$

$$c^2 = (a+b)^2 - (4)\left(\frac{1}{2}ab\right)$$

$$c^2 = a^2 + 2ab + b^2 - 2ab$$

$$c^2 = a^2 + b^2$$

Remember:
$(a+b)^2 = a^2 + 2ab + b^2$

and the Pythagorean theorem is proved.

EXAMPLE 18.3: ***Pythagorean Theorem***

In the right triangle shown in Figure 18.32, let $a = 13$ feet and $b = 9$ feet. Find the length of side *c*.

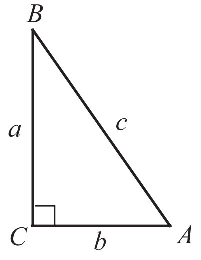


FIGURE 18.32

EXAMPLE 18.3: ***Pythagorean Theorem (Continued)***

Solution:

By the Pythagorean theorem,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 13^2 + 9^2 \\ &= 250 \\ \sqrt{c^2} &= \sqrt{250} \end{aligned}$$

$c = 15.8$ feet (rounded to tenths).

The Pythagorean theorem is slightly more difficult to solve when we are looking for the length of one of the legs. This is because some algebraic transformation is necessary as shown in the following example.

EXAMPLE 18.4: ***Using the Pythagorean Theorem***

In the right triangle shown in Figure 18.33, let $b = 15$ inches and $c = 17$ inches. Find the length of side a.

The Pythagorean theorem requires a subtraction when the unknown is one of the legs.

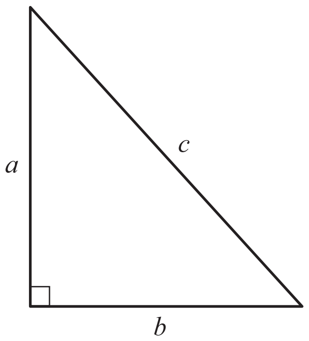


FIGURE 18.33

Solution:

The equation is algebraically transformed so that a^2 is isolated on the left:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 &= c^2 - b^2 \\ a &= \sqrt{c^2 - b^2} \end{aligned}$$

Substitute the given values:

$$a^2 = c^2 - b^2 = 17^2 - 15^2 = 289 - 225 = 64$$

$$a = \sqrt{64} = 8 \text{ inches}$$

EXAMPLE 18.5: ***Pythagorean Theorem Applied to a Sheet Metal Problem***

Solve for length of side *AE* in the sheet metal layout shown in Figure 18.34. All dimensions are in meters.

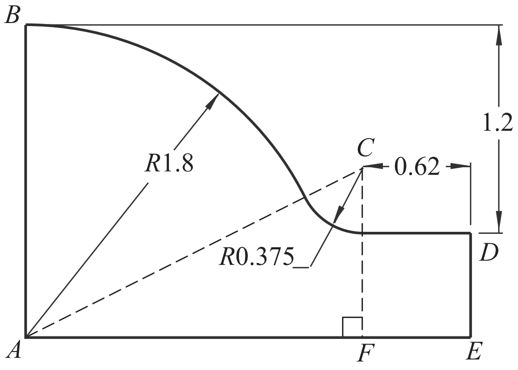


FIGURE 18.34

Solution:

Draw auxiliary line *AC*.

Draw perpendicular auxiliary line *CF* from center *C* to line *AE*.

$$AC = \text{radius } 1.8 + \text{radius } 0.375$$
$$AC = 1.8 + 0.375 = 2.175 \text{ meters}$$

From the Pythagorean theorem (Theorem 18-7),

$$(AC)^2 = (AF)^2 + (CF)^2$$

Rearranging terms, $(AF)^2 = (AC)^2 - (CF)^2$ and $AF = \sqrt{(AC)^2 - (CF)^2}$

AC is already known. We need *CF*, which we see from the figure is

$$CF = AB - 1.2 + 0.375$$
$$CF = 1.8 - 1.2 + 0.375 = 0.975 \text{ meters}$$

Therefore,

$$\begin{aligned} AF &= \sqrt{(AC)^2 - (CF)^2} \\ &= \sqrt{(2.175)^2 - (0.975)^2} \\ &= \sqrt{4.731 - 0.951} \\ &= \sqrt{3.780} \\ &= 1.94 \text{ meters} \end{aligned}$$
$$AE = AF + FE = 1.94 + 0.62 = 2.56 \text{ meters}$$

EXERCISES

18.23 Referring to Figure 18.29:

- **a)** Find c when $a = 12$ and $b = 16$.
- **b)** Solve for a when $b = 6$ and $c = 8$.
- **c)** Find b when $a = 2$ and $c = 4$.
- **d)** Solve for b when $a = 12$ and $c = 20$.
- **e)** If $\angle B$ is 30°, what does $\angle A$ equal?
- **f)** Find a and b if $c = 15$ and a and b are in the ratio of 3 to 4.
- **g)** If a and b are both equal to 7, how much is c?
- **h)** If $a = 5$ and $b = 12$, how long is hypotenuse c?
- **i)** If $c = 17$ and $b = 15$, find a.

18.24 In Figure 18.35 solve for X. Dimensions are in feet.

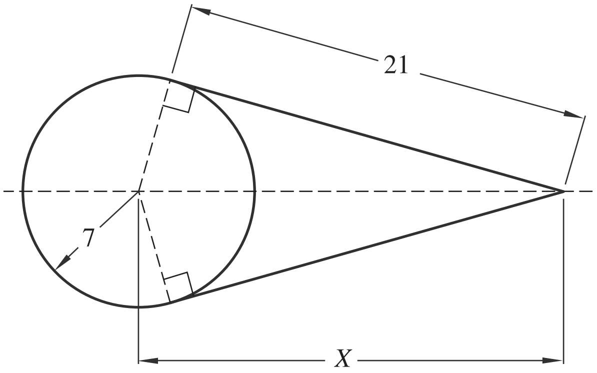


FIGURE 18.35

Hint: Construct a right triangle along vertical centerline.

18.25 In Figure 18.36, find the depth X to which a milling cutter must be sunk to cut a keyway with a width of 0.500 inch and a depth A of 0.375 inch. The shaft is 3.625 inches in diameter.

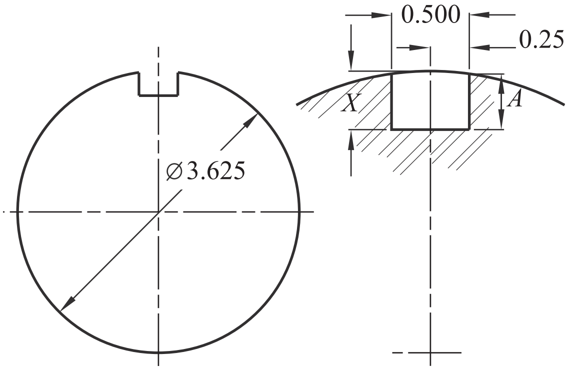


FIGURE 18.36

18.26 Solve for radius *r* in the template shown in Figure 18.37.

Hint: Construct a right triangle along horizontal centerline.

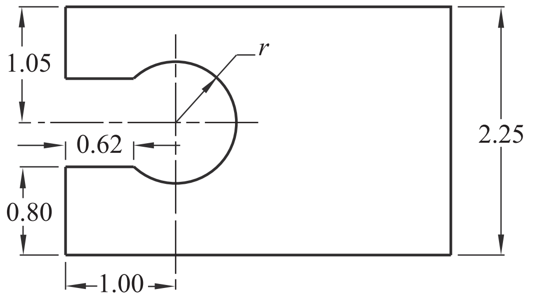


FIGURE 18.37

18.27 Solve for dimension *X* in Figure 18.38.

Hint: Construct two right triangles along horizontal centerline.

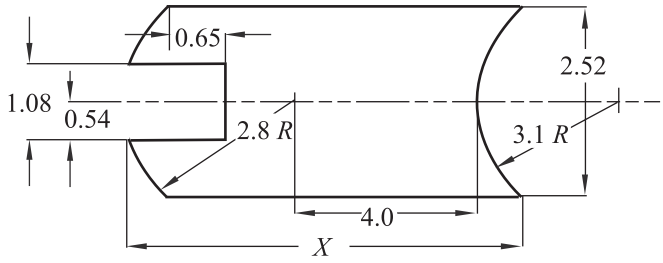


FIGURE 18.38

18.28 In the trapezoid shown in Figure 18.39, *AD* = 5 inches, *CB* = 14 inches, and *DB* = 13 inches. Solve for the length of *DC*.

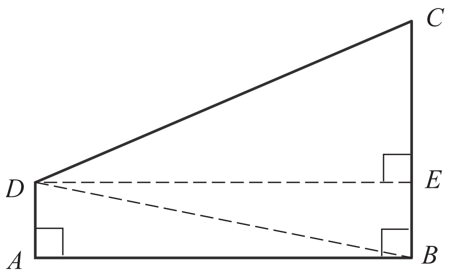


FIGURE 18.39

18.29 In the isosceles trapezoid shown in Figure 18.40, find the length of side *AD* when *AB* = 32 cm, *DC* = 20 cm, and *DE* = 8 cm.

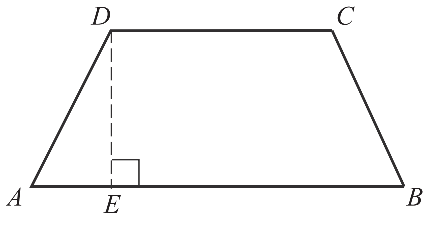


FIGURE 18.40

18.30 What is the perimeter of the blank shown in Figure 18.41? Dimensions are in inches.

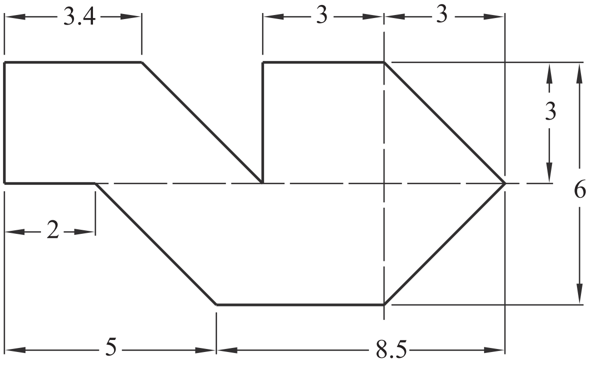


FIGURE 18.41

18.31 A numerical control drilling machine will be used to drill the three holes shown in Figure 18.42. To position the table to drill the upper hole, the distances *X* and *Y* must be known. Calculate these distances rounding to thousandths. Dimensions are in inches.

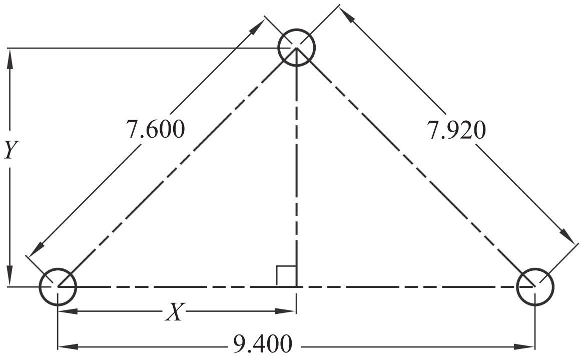


FIGURE 18.42

18.4 Congruent Triangles

The word *congruent* is used to describe plane geometric figures that are identical in size and shape. Just as parts of figures are said to be congruent if their measurements are equal, entire figures that coincide at all points when superimposed are congruent.

The symbol for congruence is ≅.

Congruent figures are copies of one another, much the same as mass-produced metal stampings off the same press are identical to one another. Many problems in geometry can be solved by the use of basic theorems relating to the condition of congruency. These theorems are given here.

Congruent triangles, for example, have congruent corresponding sides and angles.

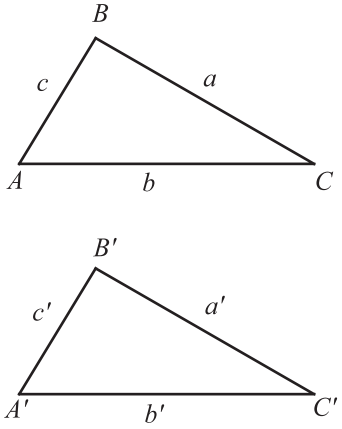


FIGURE 18.43 ***Congruent triangles***

In Figure 18.43, the triangles are defined as being the same size and shape, that is, they are *congruent*. Size relates to length of lines, so the corresponding sides of the triangles are of equal length. Likewise, shape is related to interior angles, therefore the corresponding angles are congruent. So we write

$$\angle A \cong \angle A', \angle B \cong \angle B', \angle C \cong \angle C' \text{ and } a \cong a', b \cong b', c \cong c'.$$

If we study Figure 18.43 carefully, we can observe that only a certain number and combination of sides and angles need be known to determine if two triangles are congruent. This idea leads to the following theorems.

Theorem 18-8 (SAS Theorem for Congruence)

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the figures are congruent.

This is called the SAS, or side-angle-side, theorem for congruence. According to Theorem 18-8, the triangles in Figure 18.44 are congruent when, for example, $b \cong b'$, $c \cong c'$ and $\angle A \cong \angle A'$.

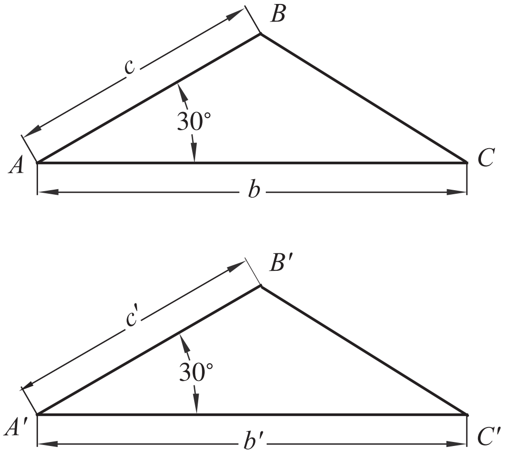


FIGURE 18.44 ***Congruent triangles; side-angle-side***

Theorem 18-9 (ASA Theorem for Congruence)

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, the triangles are congruent.

This is called the ASA, or angle-side-angle, theorem for congruence. According to Theorem 18-9, the triangles in Figure 18.45 are congruent when, for example, $b \cong b'$, $\angle A \cong \angle A'$, and $\angle C \cong \angle C'$.

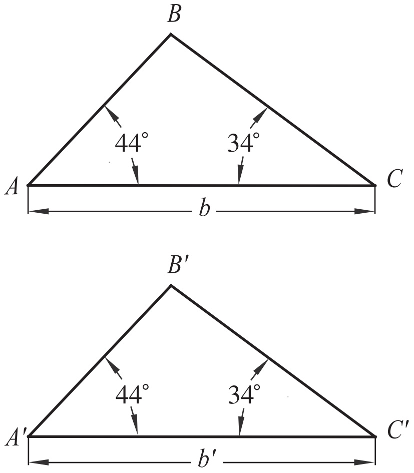


FIGURE 18.45 ***Congruent triangles; angle-side-angle***

Theorem 18-10 (SSS Theorem for Congruence)

If three sides of one triangle are congruent to the three sides of another triangle, the triangles are congruent.

This is called the SSS, or side-side-side, theorem for congruence. According to Theorem 18-10, the triangles in Figure 18.46 are congruent when $a \cong a'$, $b \cong b'$, and $c \cong c'$.

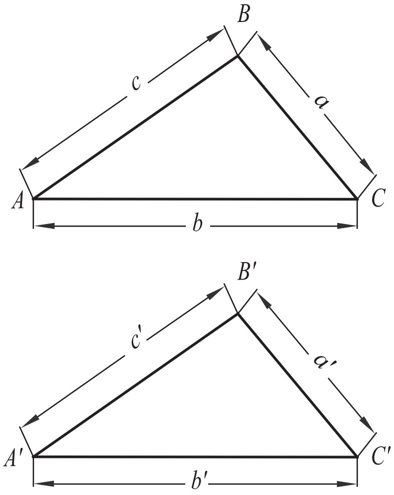


FIGURE 18.46 ***Congruent triangles; side-side-side***

While the foregoing discussion of congruency dealt with triangles, the same propositions can be adapted to other plane figures. For example, two squares whose sides are congruent will be congruent, as will be two hexagons whose angles all have equal measure and whose corresponding sides are equal.

EXERCISES

18.32 In the congruent triangles *ABC* and *A′ B′ C′* shown in Figure 18.47, determine angle *C′* and the length of side *b′*.

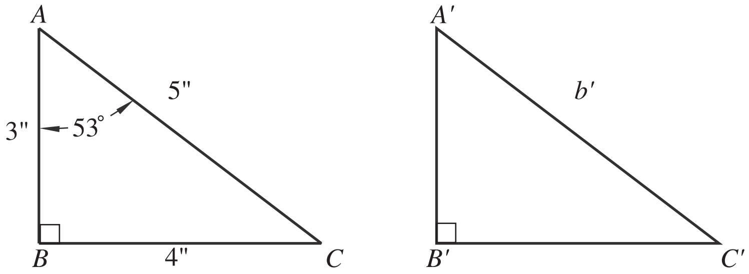


FIGURE 18.47

18.33 In the parallelogram *ABCD* shown in Figure 18.48, *AB* is parallel to *CD,* and *AC* is parallel to *BD*. *AD* bisects *BC*. Are triangle *AEB* and triangle *DEB* congruent?

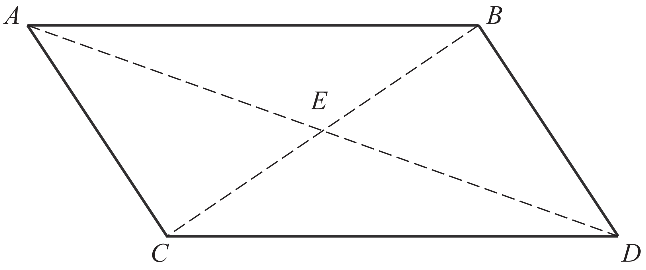


FIGURE 18.48

18.34 In the triangle shown in Figure 18.49, *AB* ≅ *AC*. Solve for angle *ABC* and angle *ACB*.

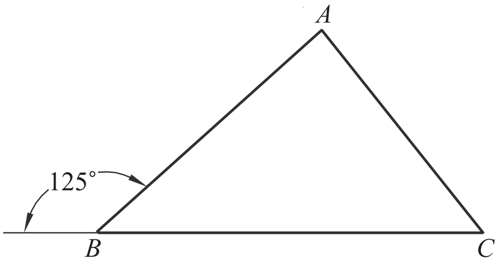


FIGURE 18.49

18.35 In Figure 18.50, $\angle CAE \cong \angle DBE$, $AE = EB$, and angle $\angle AEC \cong \angle DEB$. CE is given as $2x - 5$ and BD as $3y + 2$. Using geometric principles and algebra, solve for x and y.

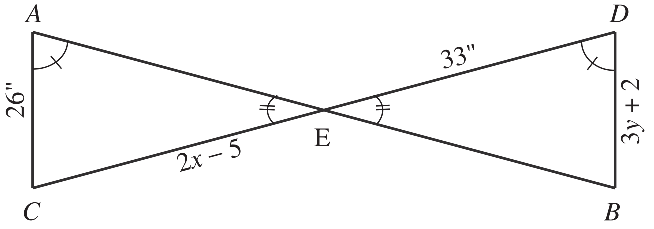


FIGURE 18.50

18.36 In Figure 18.51, $ADC \cong BEC$, $CD = CE$, and $\angle ACD \cong \angle ECB$. AC is given as $3y - 5$, CB as $2y + 7$, AD as $x + 8$, and EB as $3x$. Solve for x and y using geometric principles and algebra.

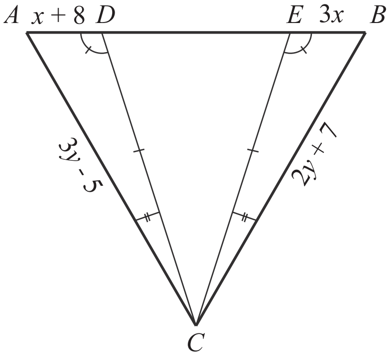


FIGURE 18.51

18.37 In Figure 18.52, $\overline{DE} \cong \overline{BC}$, $\overline{EF} \cong \overline{AC}$, and $\overline{DF} \cong \overline{AB}$. Angle ABC measures twice angle BAC. Solve for angle DFE.

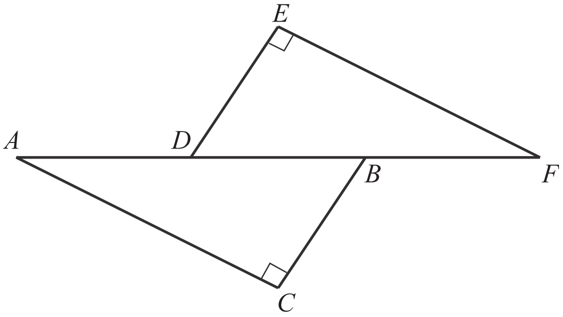


FIGURE 18.52

18.5 The Projection Formula

Acute Oblique Triangle Projection

Oblique triangles, those without a right angle, appear less often in practical shop problems than right triangles. However, for *acute oblique triangles* with the lengths of three sides known, altitude lengths can be calculated using the *projection formula*. Then, with one altitude known, the area of the triangle can be determined. When only two sides and an included angle are known, or when two angles and the included side are known, a trigonometry background is needed to solve for area.

In the acute triangle shown in Figure 18.53, altitude h from $\angle C$ to base c is shown. It cuts the base into two segments, p and p'.

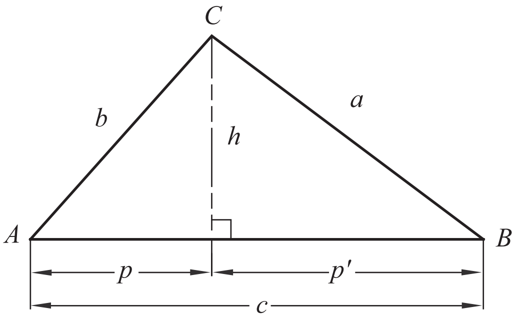


FIGURE 18.53 ***Projection formula for an acute oblique triangle***

The distance p from $\angle A$ to the foot of the altitude h is called the *projection* of AC onto AB. The projection formula for this acute oblique triangle is

$$p = \frac{c^2 + b^2 - a^2}{2c}$$

These projection formulas only apply to acute oblique triangles.

When using the projection formula, great care must be taken to ensure that the sides are labeled properly. If they are not, an incorrect answer will result. The distance p' from angle B to the foot of the altitude h is called the projection of BC onto AB. The formula for this projection is:

$$p' = \frac{a^2 + c^2 - b^2}{2c}$$

EXAMPLE 18.5A: ***Acute Oblique Triangle Projection***

Using Figure 18.53, solve for p, the projection of AC onto AB, when $a = 13$ inches, $b = 9$ inches, and $c = 16$ inches. Also, find the altitude h.

Solution: Using the projection formula for an acute oblique triangle,

$$p = \frac{c^2 + b^2 - a^2}{2c}$$

EXAMPLE 18.5A: ***Acute Oblique Triangle Projection (Continued)***

$$p = \frac{(16^2 + 9^2 - 13^2)}{(2)(16)}$$
$$= \frac{(256 + 81 - 169)}{32}$$
$$= \frac{168}{32} = 5.25 \text{ inches}$$

Using the Pythagorean theorem to find the altitude h,

$$h = \sqrt{b^2 - p^2} = \sqrt{81 - 27.56}$$
$$= \sqrt{53.44} = 7.31 \text{ inches}$$

Obtuse Oblique Triangle Projection

Although less straightforward, a projection formula can be given for *obtuse* oblique triangles. For these, extra lines must be drawn to accommodate the position of the resulting projection. In the obtuse triangle shown in Figure 18.54, an altitude h is shown drawn from $\angle C$. Notice that h is perpendicular to the base AB extended to point D outside the given triangle. The projection of side BC on base AB is p. The projection formula for this obtuse triangle is:

$$p = \frac{c^2 + a^2 - b^2}{2c}$$

This projection formula only applies to obtuse oblique triangles.

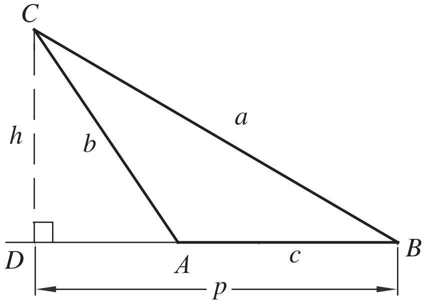


FIGURE 18.54 ***Projection formula for an obtuse oblique triangle***

EXAMPLE 18.6: ***Obtuse Oblique Triangle Projection***

In the obtuse oblique triangle shown in Figure 18.54, solve for p and altitude h when a = 13.75 cm, b = 8.5 cm, and c = 7.5 cm.

Solution: Using the projection formula for an obtuse oblique triangle:

$$p = \frac{c^2 + a^2 - b^2}{2c}$$

Substitute values given in problem:

$$p = \frac{(7.5^2 + 13.75^2 - 8.5^2)}{(2)(7.5)} = \frac{56.25 + 189.06 - 72.25}{15} = \frac{173.06}{15} = 11.54 \text{ cm}$$

EXAMPLE 18.6: ***Obtuse Oblique Triangle Projection (Continued)***

Using the Pythagorean theorem,

$$\begin{aligned} h^2 + p^2 &= a^2 \\ h^2 &= a^2 - p^2 \\ h &= \sqrt{a^2 - p^2} \\ &= \sqrt{(13.75)^2 - (11.54)^2} \\ &= \sqrt{189.06 - 133.17} \\ &= \sqrt{55.89} = 7.48 \text{ cm} \end{aligned}$$

EXERCISES

18.38 For the acute oblique triangle shown in Figure 18.55, solve for p' and altitude h when $a = 9$ inches, $b = 13$ inches, and $c = 16$ inches.

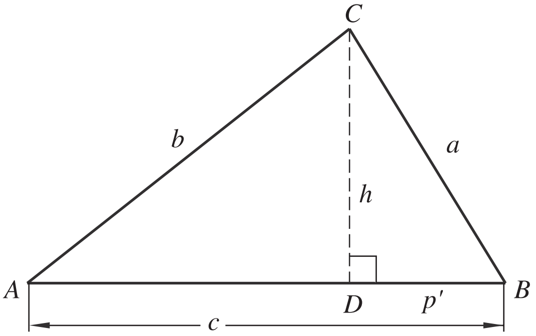


FIGURE 18.55

18.39 For the obtuse oblique triangle shown in Figure 18.56, solve for d and altitude h when $a = 13.75$ cm, $b = 7.5$ cm, and $c = 8.5$ cm. (Hint: First solve for projection p.)

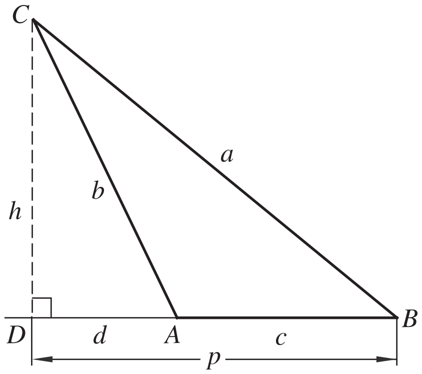


FIGURE 18.56

18.40 In the acute triangle shown in Figure 18.57, solve for the altitude. Dimensions are in millimeters.

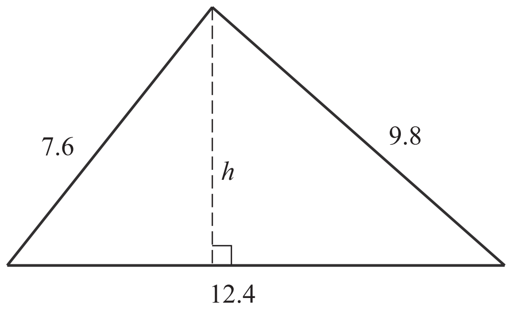


FIGURE 18.57

18.41 For the obtuse triangle shown in Figure 18.58 solve for *X*. Dimensions are in centimeters.

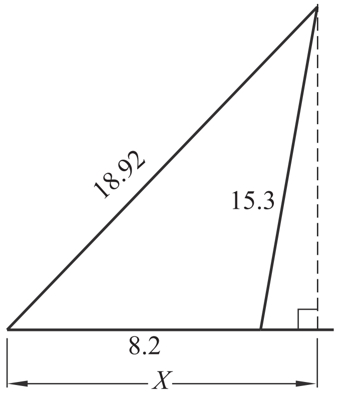


FIGURE 18.58

18.42 For the obtuse triangle shown in Figure 18.59, solve for *h*. Dimensions are in inches.

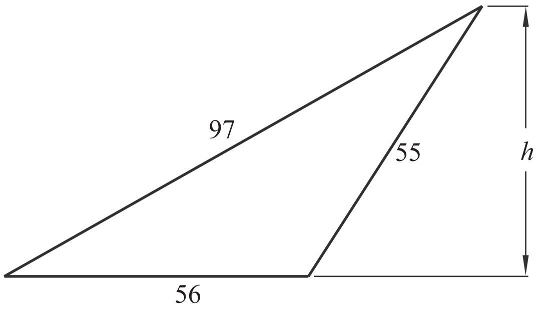


FIGURE 18.59

18.6 Hero's Formula

During the first century A.D., Hero of Alexandria proved a formula for calculating the area of a triangle when only the lengths of the sides are known. His formula, known as *Hero's formula*, is:

Hero's Formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the lengths of the three sides of the triangle and s is the perimeter of the triangle divided by 2, as follows:

$$s = \frac{a+b+c}{2}$$

Hero's formula works for acute, obtuse, and right triangles and is most helpful when triangle height is unknown.

For right triangles, Hero's formula simplifies to $A = \frac{1}{2}bh$.

EXAMPLE 18.7A: ***Hero's Formula***

Given a triangle with sides a = 12 feet, b = 9 feet, and c = 7 feet, find area A.

Solution: Since we lack height h, we will use Hero's formula. First,

$$s = \frac{a+b+c}{2} = \frac{12+9+7}{2} = 14 \text{ feet}$$

Substituting s = 14 feet into Hero's formula gives

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{(14)(14-12)(14-9)(14-7)} \\ &= \sqrt{(14)(2)(5)(7)} = 31.3 \text{ ft}^2 \end{aligned}$$

Notice that the unit under the radical is ft^4*, so the unit of the root is ft*^{2}.

EXAMPLE 18.7B: ***Hero's Formula***

Find the area of a triangle with sides 8 inches, 10 inches, and 12 inches long using Hero's formula.

Solution: Apply Hero's formula where a = 8, b = 10, and c = 12. Then,

$$s = \frac{a+b+c}{2} = \frac{8+10+12}{2} = 15 \text{ inches}$$

Therefore, area is

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{15(15-8)(15-10)(15-12)} \\ &= \sqrt{15 \cdot 7 \cdot 5 \cdot 3} = 39.7 \text{ square inches} \end{aligned}$$

$\sqrt{\text{in}^4} = \text{in}^2$

Although the proof of Hero's formula involves a considerable amount of algebra, it is shown here for the curious student. Referring to Figure 18.60,

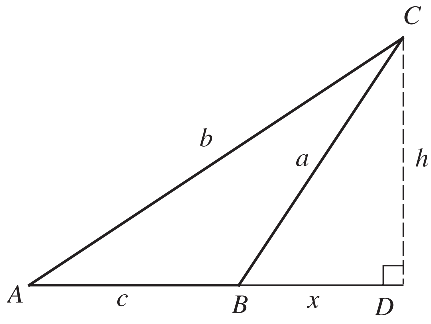


FIGURE 18.60 ***Area using Hero's formula***

Intermediate steps are not shown.

$$\begin{aligned} h^2 &= a^2 - x^2 \\ \text{and } h^2 &= b^2 - (c+x)^2 \\ \therefore \quad a^2 - x^2 &= b^2 - (c+x)^2 \\ a^2 - x^2 &= b^2 - c^2 - 2cx - x^2 \\ 2cx &= b^2 - c^2 - a^2 \\ x &= \frac{b^2 - c^2 - a^2}{2c} \end{aligned}$$

Since $h^2 = a^2 - x^2$, we get from the difference of squares factorization:

$$h^2 = (a - x)(a + x)$$

Substituting our result for x in $h^2 = (a - x)(a + x)$, gives us

$$\begin{aligned} h^2 &= \left[a - \frac{(b^2 - c^2 - a^2)}{2c}\right]\left[a + \frac{(b^2 - c^2 - a^2)}{2c}\right] \\ &= \left[\frac{(a^2 + 2ac + c^2) - b^2}{2c}\right]\left[\frac{b^2 - (a^2 - 2ac + c^2)}{2c}\right] \\ &= \left[\frac{(a+c)^2 - b^2}{2c}\right]\left[\frac{b^2 - (a-c)^2}{2c}\right] \end{aligned}$$

Let $a + b + c = 2s$, then

$$\begin{aligned} a + c - b &= 2s - 2b \quad \text{or} \quad 2(s - b) \\ b + a - c &= 2s - 2c \quad \text{or} \quad 2(s - c) \\ b - a + c &= 2s - 2a \quad \text{or} \quad 2(s - a) \end{aligned}$$

Substituting, after some algebraic steps,

$$\begin{aligned} h^2 &= \left[\frac{2s\cdot 2(s-b)}{2c}\right]\left[\frac{2(s-c)\cdot 2(s-a)}{2c}\right] \\ &= \frac{4s(s-a)(s-b)(s-c)}{c^2} \\ h^2c^2 &= 4s(s-a)(s-b)(s-c) \\ hc &= 2\sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

This substitution results when difference of squares factorizations are done on the last expression for h^2.

Since area of $\triangle ABC = \frac{1}{2}hc$, area of $\Delta ABC = \frac{1}{2}\cdot 2\sqrt{s(s-a)(s-b)(s-c)}$.

Finally, the area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$ and Hero's formula is proven for any triangle.

EXERCISES

18.43 For the acute oblique triangle shown in Figure 18.61, find the area of the triangle when $a = 9$ inches, $b = 16$ inches, and $c = 13$ inches.

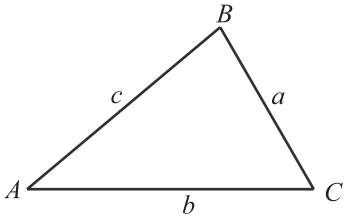


FIGURE 18.61

18.44 For the obtuse oblique triangle shown in Figure 18.62, find the area of the triangle when $a = 13.75$ cm, $b = 8.5$ cm, and $c = 7.5$ cm.

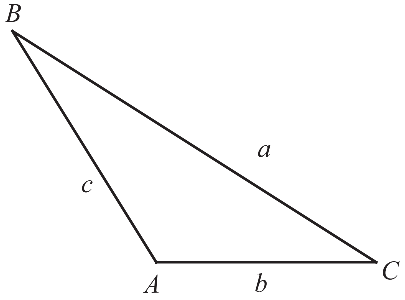


FIGURE 18.62

18.45 For the acute triangle shown in Figure 18.63 solve for altitude *h* and find the area. Dimensions are in feet.

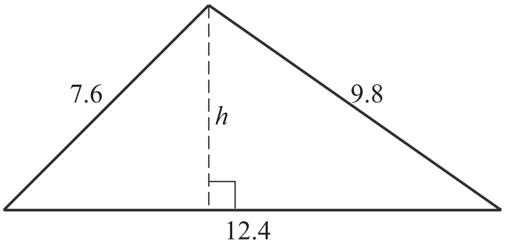


FIGURE 18.63

18.46 Find the area of the obtuse triangle shown in Figure 18.64. Dimensions are in millimeters.

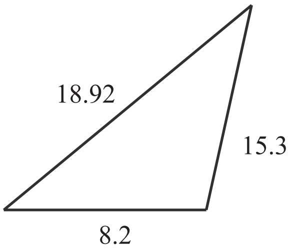


FIGURE 18.64

18.47 Find the area of the obtuse triangle shown in Figure 18.65. Dimensions are in inches.

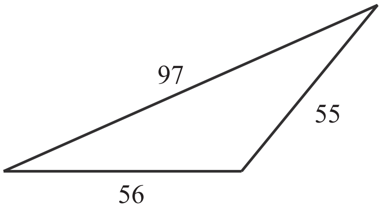


FIGURE 18.65

18.48 Find the total area of the polygon shown in Figure 18.66. Dimensions are in millimeters.

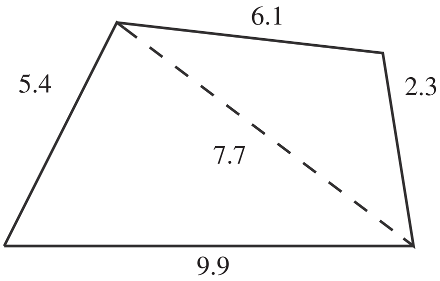


FIGURE 18.66

18.49 Find the total area of the polygon shown in Figure 18.67. Dimensions are in inches.

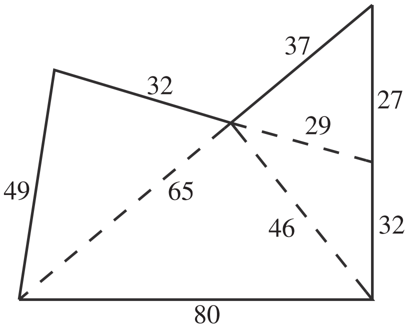


FIGURE 18.67

19

THE CIRCLE

Circles have long intrigued humans. Often called the perfect geometric form, the circle occurs frequently in engineering design, art, architecture, and applied mathematics. Almost 2200 years ago, Archimedes sought to find a square with the same area as a given circle. This problem is called *squaring the circle.*

The circle forms the base for systems of navigation on the sea and in the air. The geometry of circles involves not only the relationships of the principal parts of the circle but also the relation of the circle to lines, angles, and polygons.

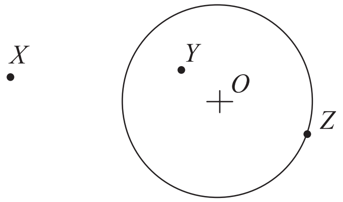
19.1 Definitions

Circle

A *circle* is the set of all points in a plane equidistant from a given point. This distance is the length of the *radius* and the given point is the circle's *center*. A circle is named by its center point. For example, a circle with center at point O is referred to as "circle O."

Circles are symbolic of infinity, eternity, and completeness. They appear in the literature and architecture of many cultures.

Figure 19.1 is a drawing of a circle with nearby points. Note that a point may lie either *on* the circle; or it may lie *inside* the circle, making it an *interior point* of the circle; or it may lie *outside* the circle, making it an *exterior point* of the circle. A common mistake is to view interior points, including the center, of a circle as part of the circle. The circle consists of *only* those points a given distance from the center.



Point X is outside circle O.

Point Y is inside circle O.

Point Z is on circle O.

FIGURE 19.1 ***A circle***

The plural of "radius" is "radii."

The *radius* is any line segment joining the center of a circle with a point on the circle's circumference.

A *chord* is a line segment joining any two points on the circle's circumference.

The *diameter* is any line segment through the center of the circle joining two points on the circumference. The diameter's length is twice the length of the radius. Thus, the diameter is the longest chord that can be drawn in a circle.

The words "radius," "diameter," and "circumference" can mean either these physical aspects of the circle or the specific measurement assigned to each.

An angle formed by two radii is a *central angle*. We say that a chord or an angle *subtends* an arc, or an arc is *intercepted* by an angle or a chord.

An *arc* is a portion of the circle lying between two points on the circle. A chord's endpoints split the circle into two arcs, the *minor arc* and the *major arc*, as shown in Figure 19.2. Since the diameter is the longest chord in a circle, it subtends two arcs equal in measure. These arcs are *semicircles*.

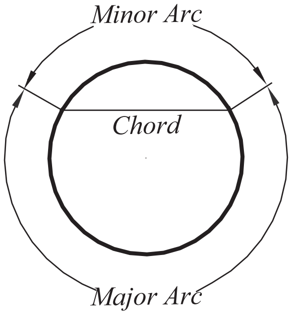


FIGURE 19.2 ***Arcs of a chord***

A *semicircle* is an arc measuring one-half the circumference of a circle, a half circle.

A wedge-of-pie shaped area bounded by an arc and its two related radii is a *sector*.

The area bounded by an arc and its chord is a *segment*.

A *secant* is a line that intersects the circle at two points. The part of a secant that is not outside the circle is a chord.

A line that intersects a circle at one and only one point is called a *tangent*. Note that a tangent does not cut through the circle. Referring to Figure 19.3, a tangent at point *A* is perpendicular to the radius *OA*. In fact, it is the only line perpendicular to radius *OA* through point *A*.

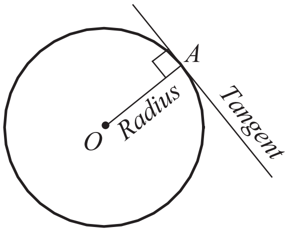


FIGURE 19.3 ***Tangent line and radius***

An *inscribed angle* in a circle has its vertex on the circle and its sides in the interior of the circle. It subtends an arc, as shown in Figure 19.4 where angle *ACB* is an inscribed angle and angle *AOB* is the corresponding *central angle*. In fact, angle *AOB* is said to be the central angle *subtended* by inscribed angle *ACB* and arc *AB* is the *intercepted arc*. Figure 19.4 illustrates an angle inscribed in a circle.

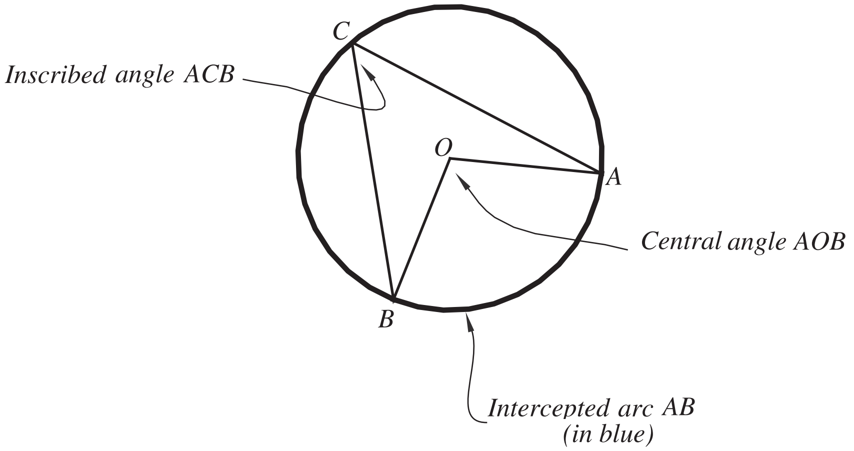


FIGURE 19.4 ***An inscribed angle***

The important geometric features of a circle are summarized in Figure 19.5.

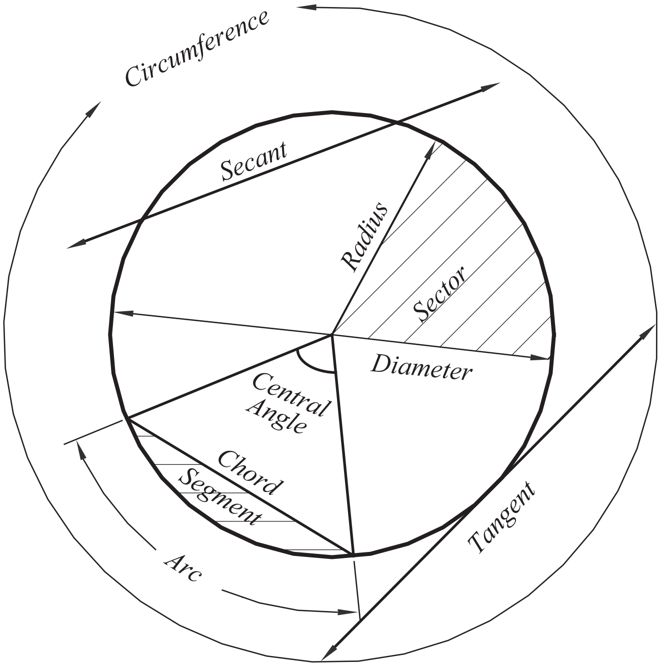


FIGURE 19.5 ***Important geometric features of a circle***

Circle/Circle Relationships

Concentric circles are circles with a common center but different radii, as shown in Figure 19.6.

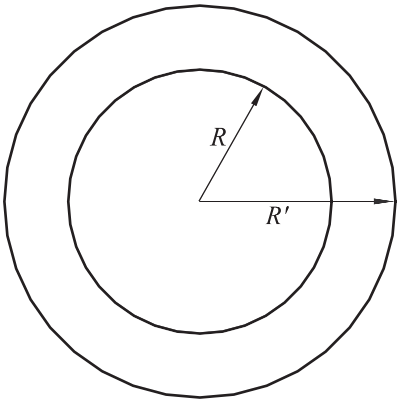


FIGURE 19.6 ***Concentric circles***

Externally tangent and *internally tangent* circles are shown in Figures 19.7A and 19.7B.

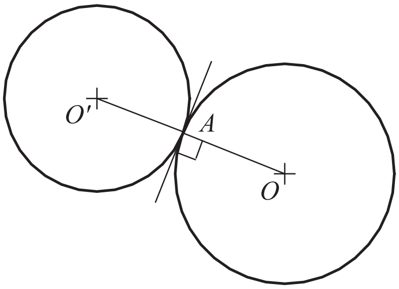


FIGURE 19.7A ***Externally tangent circles***

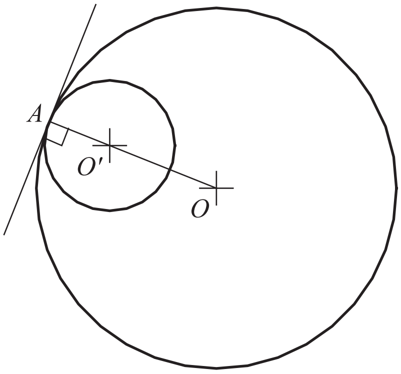


FIGURE 19.7B ***Internally tangent circles***

Polygon/Circle Relationships

A polygon is said to be *inscribed* in a circle if all its sides are chords of a circle. This is the same as requiring that each vertex of the polygon be a point on the circle, as shown in Figure 19.8A. We could also say that the circle *circumscribes* the polygon.

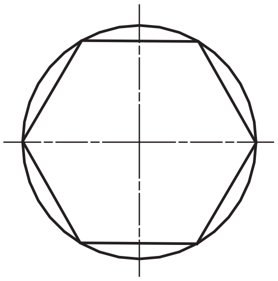


FIGURE 19.8A ***A regular hexagon inscribed in a circle***

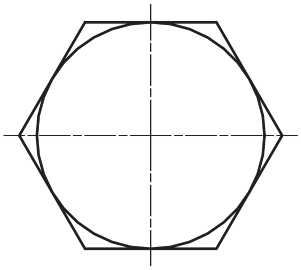


FIGURE 19.8B ***A regular hexagon circumscribed on a circle***

A polygon is *circumscribed* on a circle if all of its sides are tangent to the circle. A circumscribed polygon is shown in Figure 19.8B. The circle is *inscribed* in the polygon. Although regular hexagons are shown in Figures 19.8A and 19.8B, an inscribed or circumscribed polygon does not have to be regular. Figure 19.9 shows a nonregular quadrilateral inscribed in a circle.

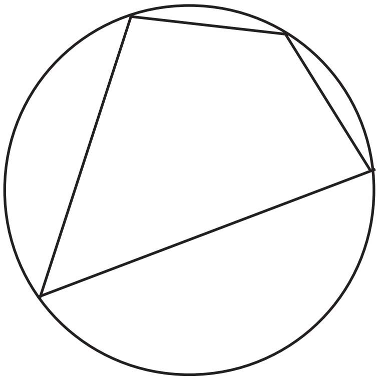
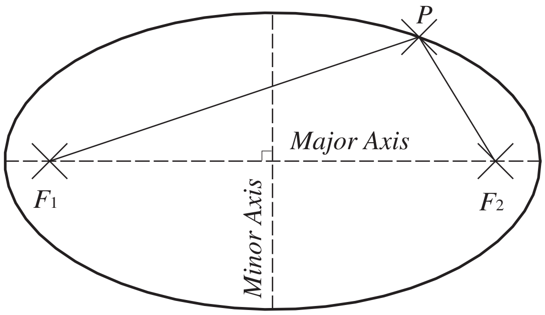


FIGURE 19.9 ***A nonregular quadrilateral inscribed in a circle***

Ellipse

An ellipse is shown in Figure 19.10. Like a circle, an ellipse is a closed curve. Unlike a circle, which has a single *focus*, its center, an ellipse has two focal points, called *foci* (pronounced $fo' \cdot s\bar{i}$), making it appear flattened, or egg-shaped. An ellipse can be drawn by positioning pins at two points, F_1 and F_2, and loosely tying a length of string between these points, the foci. Using a pencil point to hold the string taut, follow the string to form the shape shown in Figure 19.10. Notice that the distance $F_1P + F_2P$ remains constant.

The ellipse also has a center. It is the point of intersection of the major and minor axes. These are special line segments that intersect at right angles in the ellipse. The major axis runs through the foci and center; the minor axis is the perpendicular bisector of the major axis.



$$\overline{F_1P} + \overline{F_2P} = \text{constant}$$

FIGURE 19.10 ***Important features of an ellipse***

A circle is a special kind of ellipse in which the two focal points are coincident.

Some examples of ellipses in technical fields are:

- The shape of the end section on a pipe that has been cut on a slant.
- The projection of a circle in an isometric view of a part to be machined.
- The orbits of the planets around the sun.

Circles and ellipses are examples of *conic sections*. The other conic sections are the parabola and the hyperbola. Conic sections are the shapes obtained by passing a plane through a right circular cone or cones at various angles.

EXERCISES

19.1 Define the terms:

a) Circle
b) Chord
c) Semicircle
d) Central angle
e) Sector
f) Radius
g) Secant
h) Inscribed angle
i) Diameter
j) Arc
k) Tangent
l) Intercepted arc

19.2 What is an inscribed polygon?

19.3 What is a circumscribed polygon?

19.4 How many foci does an ellipse have?

19.5 Draw an ellipse whose major and minor axes are equal. What figure has been drawn?

19.2 Theorems Involving Circles

In this section several practical theorems about circles are stated and motivations regarding the proofs of some of these theorems are offered.

A diameter perpendicular to a chord bisects the chord and the two arcs that the chord subtends.

Theorem 19-1

Figure 19.11 illustrates Theorem 19-1. Observe the congruence of the minor arcs that the diameter subtends.

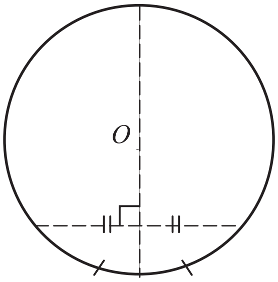


FIGURE 19.11 ***Theorem 19-1***

EXAMPLE 19.1: ***Theorem 19-1***

Chord *AB* is 9 inches long and drawn perpendicular to diameter *CD* What are the lengths of segment *AE* and segment *EB*?

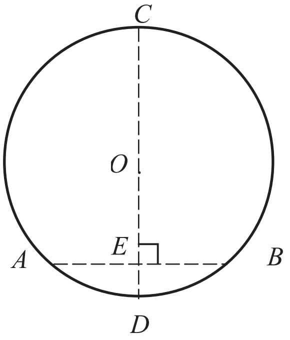


FIGURE 19.12

Solution: Since diameter *CD* is perpendicular to chord *AB*, Theorem 19-1 maintains that segment *AE* and segment *EB* are equal in measure. Therefore,

$$AE = EB = \frac{9}{2} = 4.5 \text{ inches}$$

In the same or congruent circles, congruent chords are equidistant from the circle's center.

Theorem 19-2

Theorem 19-2 is illustrated in Figure 19.13 where segments *AB* and *CD* are of equal length. Segments *OE* and *OF* are radii drawn perpendicular to the chords and are therefore of equal length.

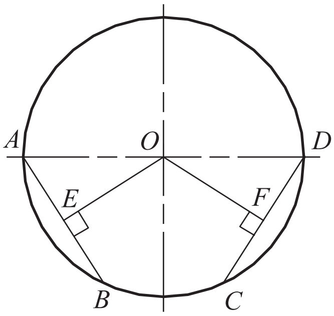


FIGURE 19.13 ***Congruent chords are equally distant from center of circle***

EXAMPLE 19.2: ***Theorem 19-2***

The shortest distance from the center of a hex bolt to the hex bolt's side is $\frac{1}{4}$ inch. What is the distance across the flats?

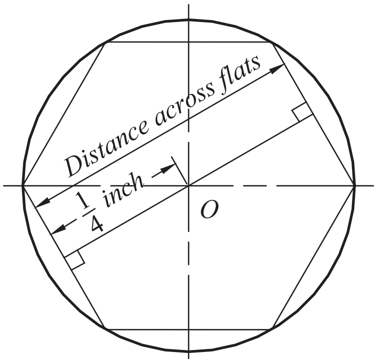


FIGURE 19.14

Solution:

Since the six sides of a hex bolt are equal in length, a circle could be drawn circumscribing the hex bolt, making each of the sides congruent chords. By Theorem 19-2, all sides of the hex bolt are equidistant from the hex bolt's center. Since the distance from the midpoint of one of the sides is given as $\frac{1}{4}$ inch from the center, the distance across the flats must be twice the distance given, or $\frac{1}{2}$ inch.

The *converse* of a theorem is formed by reversing the order of the statements in the original theorem. For example, the converse of Theorem 19-2 would state that in the same or congruent circles, chords that are equally distant from the circle's center are congruent.

A tangent to a circle is perpendicular to the radius drawn to the point of tangency. Conversely, a line in the plane of a circle that is perpendicular to a radius and that intersects the radius on the circle is tangent to the circle.

Theorem 19-3

As a motivation for Theorem 19-3, recall that a tangent to a circle is a line located outside the circle but that touches the circumference at a single point. In Figure 19.15, line *AC* is perpendicular to radius *OB* at point *B*, which is on the circle. Therefore, line *AC* is tangent to the circle.

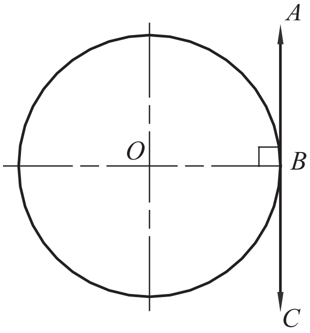


FIGURE 19.15 ***Theorem 19-3***

A *corollary* to a theorem is a statement that is related closely to and follows immediately from the theorem. Related to Theorem 19-3 is the following corollary.

If two or more tangents are drawn to a circle, the lines drawn perpendicular to each tangent at the point of tangency will intersect each other at the center of the circle, unless the lines are collinear.

Corollary to Theorem 19-3

In Figure 19.16, diameters *EC* and *E′C′* are perpendicular to tangent lines *AB* and *A′B′*, respectively, and they pass through the respective points of tangency, *E* and *E′*. The center of the circle is located at *O*, the point of intersection of diameters *EC* and *E′C′*.

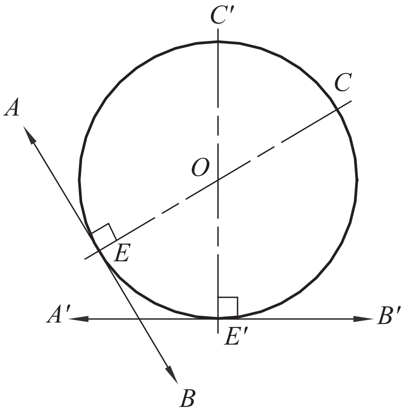


FIGURE 19.16 ***Tangents to a circle***

Theorem 19-4

If two distinct tangents are drawn to a circle from the same external point, the line segments from the external point to the points of tangency are equal in length; they form congruent right triangles when a line is drawn from the center of the circle to the external point.

The fact that the triangles described by Theorem 19-4 are congruent can be-shown by the side-angle-side theorem from Chapter 17.

Figure 19.17 illustrates Theorem 19-4. Segments *AB* and *AC* are drawn tangent to the circle from an external point *A*. Segment *AB* is congruent to segment *AC* and angle *BAO* is congruent to angle *CAO*.

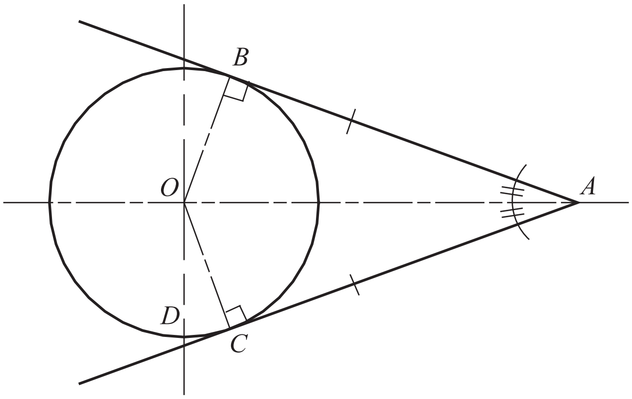


FIGURE 19.17 ***Equal length of tangents***

Theorem 19-4 is used frequently in machine shop practice.

EXAMPLE 19.3: ***Theorem 19-4***

In Figure 19.17, segments *AB* and *AC* are tangent to the circle. Given that the sum of the measures of all the angles in a triangle is 180 and that $m\angle BAC = 30$, find $m\angle COD$.

Solution:

From Theorem 19-4, line *AO* bisects $\angle BAC$ and $\angle OAC = \frac{1}{2}\angle BAC$.

Therefore, $m\angle OAC = \frac{30}{2} = 15$.

$\angle ACO$ is a right angle by Theorem 19-3 and therefore measures 90. Since we are given that the sum of the measures of interior angles in triangle *AOC* is 180, it follows that $m\angle AOC = 180 - (15 + 90) = 180 - 105 = 75$.

Since $\angle AOD$ is a right angle, $m\angle AOC + m\angle COD = 90$.

Therefore, $m\angle COD = 90 - 75 = 15$.

If two or more distinct pairs of tangents are drawn from points external to a circle, the bisectors of the angles formed by each pair of tangents will intersect at the center of the circle.

Corollary to Theorem 19-4

In Figure 19.18, the bisectors *AB* and *A′B′* of the angles formed by the tangents drawn from external points *A* and *A′* intersect at the center of the circle, *O*.

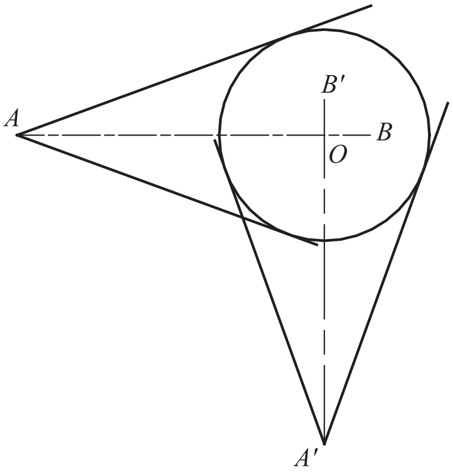


FIGURE 19.18 ***Intersection of bisectors at the center of the circle***

Theorem 19-4 is the basis of the operation of a tool called a center head, which is used to find the center of the end faces of cylindrical bars or shafts.

In the same or congruent circles, congruent central angles subtend congruent chords.

Theorem 19-5

In the same or congruent circles, congruent central angles subtend congruent arcs.

Theorem 19-6

Theorems 19-5 and 19-6 are illustrated in Figure 19.19. Given that central angles *AOD* and *BOD* are congruent, it then follows that chord *AD* is congruent to chord *BD*, and arc *AD* is congruent to arc *BD*.

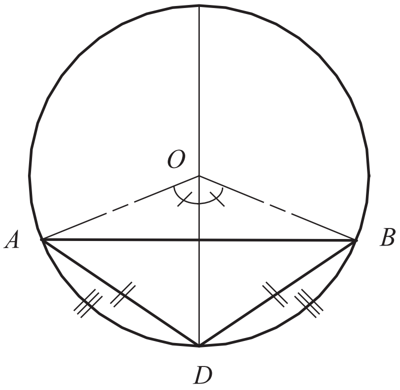


FIGURE 19.19 ***Congruent central angles subtending congruent chords and arcs***

Theorem 19-7

The measure of the diameter of a circle inscribed in a right triangle is equal to the difference of the sum of the measure of the legs of the triangle (sides adjacent to the right angle) and the measure of the hypotenuse.

Figure 19.20 illustrates Theorem 19-7, which states that diameter EG is equal to the difference of the length of hypotenuse AC and the sum of the lengths of legs AB and BC.

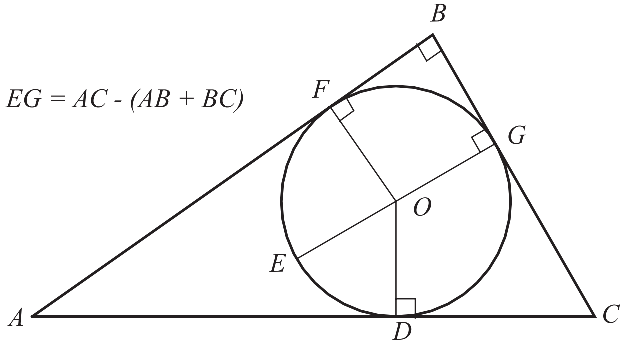


FIGURE 19.20 ***Diameter of a circle inscribed in a right triangle***

Here we will offer a *proof* of Theorem 19-7. As its name indicates, a mathematical proof must prove a theorem is true. It cannot merely illustrate the theorem. This proof uses facts known or implied from previous definitions and theorems in combination with basic algebraic principles. After several steps, the theorem is shown to be true for the given figure. This figure is a sufficiently general case to demonstrate that any right triangle in which a circle is inscribed would work.

Proof of Theorem 19-7

Consider Figure 19.20. Notice that $EG = EO + OG$ since the diameter of a circle has measure equal to the sum of the measures of any two radii. Notice also that $EO \cong OF \cong OG$ since all are radii of the same circle.

$BF \cong BG$ by Theorem 19-4. This is due to the fact that circumscribing a circle with a polygon entails drawing the polygon's sides tangent to the circle. Thus the theorem on congruent tangents applies to this situation.

OFBG is a four-sided figure with three of its angles given or deduced to be right angles. Hence the fourth angle is a right angle since the sum of the measures of the interior angles of a quadrilateral is 360 by the $n - 2$ *rule*. Therefore, the figure is a rectangle. Noting that the radii and the tangent line segments are adjacent sides and congruent and knowing that opposite sides of a rectangle are congruent, we can conclude that all sides of this rectangle are congruent, making it a square. Thus, $OF \cong FB \cong BG \cong OG$.

Therefore, $FB + BG = FO + OG = EO + OG = EG =$ diameter.

The sum of the legs of the right triangle $ABC = AF + FB + BG + GC$.

The length of the hypotenuse $AC = AD + DC = AF + GC$, again by Theorem 19-4, which allows that $AD \cong AF$ and $DC \cong GC$.

Putting the pieces together, we see that the sum of the legs of the right triangle minus the hypotenuse equals

$$\begin{aligned} AF + FB + BG + GC - (AF + GC) &= \cancel{AF} + FB + BG + \cancel{GC} - \cancel{AF} - \cancel{GC} \\ &= FB + BG \end{aligned}$$

which was earlier shown to equal EG, the length of the diameter.

So we can write $EG = AB + BC - AC$ proving Theorem 19-7.

EXAMPLE 19.4: ***Diameter of a Circle Inscribed in a Right Triangle***

Consider a right triangle with hypotenuse measuring 5 feet and legs measuring 3 feet and 4 feet. What is the diameter of the inscribed circle?

Solution:

The answer follows directly from Theorem 19-7. Diameter equals the sum of the lengths of the legs subtracted from the length of the hypotenuse. Therefore, diameter = 3 feet + 4 feet – 5 feet = 2 feet.

A central angle has the same measurement in degrees as its intercepted arc. ***Observation***

Angle measurement is known as *angle degrees* while arc measurement is *arc degrees* or *degrees of arc*. In Figure 19.21, the angle measure of $\angle AOB$ equals the arc measure of $\overset{\frown}{AB}$.

Therefore, we say central angle degrees = degrees of arc.

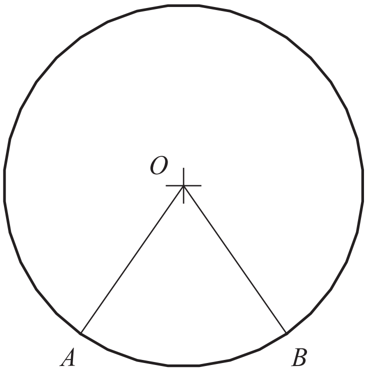


FIGURE 19.21 ***Measurement of a central angle***

The measure of an inscribed angle is half the measure of its intercepted arc. ***Theorem 19-8***

To illustrate Theorem 19-8, consider Figure 19.22.

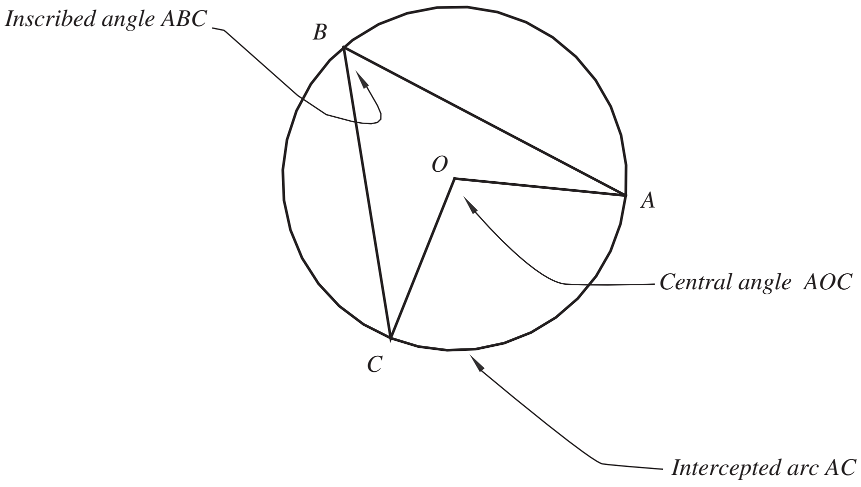


FIGURE 19.22

Notice that both angles *ABC* and *AOC* subtend arc *AC*. That is, arc *AC* is the intercepted arc of both the inscribed and central angles shown. From our earlier observation, the measure of angle *AOC* equals the measure of arc *AC*. Theorem 19-8 says that the measure of inscribed angle *ABC* equals one-half the measure of arc *AC*, which also equals one-half the measure of angle *AOC*. The significance of Theorem 19-8 is that the measure of the inscribed angle is half the measure of the central angle that shares its *endpoints*.

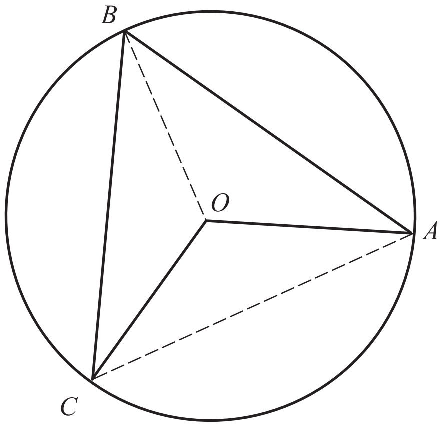


FIGURE 19.23

In reference to Figure 19.23, a proof of Theorem 19-8 can be setup as follows.

Statement	Reason
$m\angle ABC = m\angle CBO + m\angle ABO$	By inspection
$\overline{OB} \cong \overline{OA} \cong \overline{OC}$	Radii in the same circle
$\angle CBO \cong \angle BCO$	Isosceles triangles
$\angle ABO \cong \angle BAO$	Isosceles triangles

Statement	Reason
$m\angle AOC + (m\angle AOB + m\angle BOC) = 360$	Circle has 360°
$(m\angle CBO + m\angle BCO) + (m\angle ABO + m\angle BAO) + (m\angle AOB + m\angle BOC) = 360$	Quadrilateral *ABCO*
$2\,(m\angle CBO) + 2\,(m\angle ABO) + (m\angle AOB + m\angle BOC) = 360$	Substitution
$2\,(m\angle CBO) + 2\,(m\angle ABO) + (m\angle AOB + m\angle BOC) = m\angle AOC + (m\angle AOB + m\angle BOC)$	Transitive, both 360
$2\,(m\angle CBO) + 2\,(m\angle ABO) = m\angle AOC$	Subtraction
$2\,(m\angle CBO + m\angle ABO) = m\angle AOC$	Distributive property
$2\,m\angle ABC = m\angle AOC$	Substitution
$m\angle ABC = \frac{1}{2} m\angle AOC$	Division
	Theorem proved *(Q.E.D.)*

Sometimes the abbreviation Q.E.D. is used at the end of formal proofs. It is a Latin abbreviation for "quod erat demonstrandum" meaning "which was to have been proven."

EXAMPLE 19.5: ***Theorem 19-9***

A 52-degree angle is inscribed in a circle. How many degrees does the central angle measure?

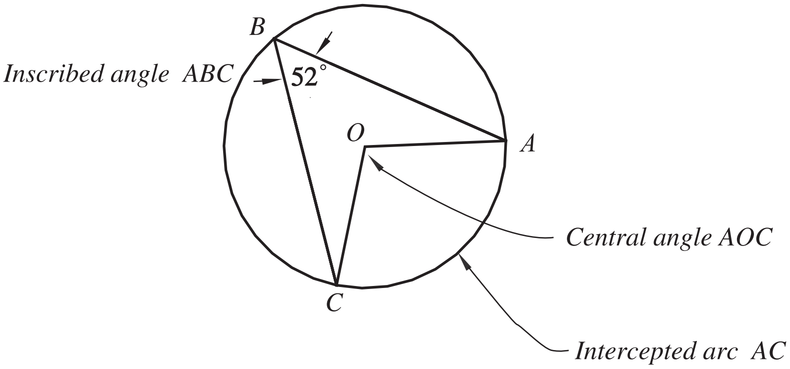


FIGURE 19.24

Solution:

From Theorem 19-8, the central angle must be twice the inscribed angle since the theorem states that the inscribed angle is half of the intercepted arc. Therefore, the measure of the central angle is (52)(2) = 104 or 104°.

Theorem 19-8 can be applied to show that an angle inscribed in a semicircle must be a right angle.

Figure 19.25 demonstrates that a central angle and an inscribed angle can relate in another way. As long as an inscribed angle shares endpoints with a central angle, the relation of the measures of these two angles holds as stated.

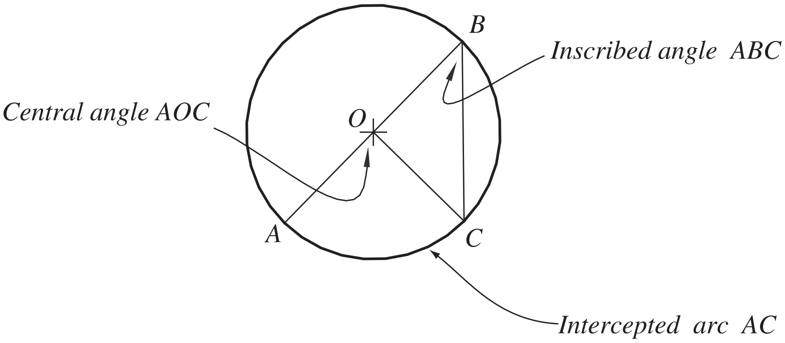


FIGURE 19.25 ***Relation between an inscribed angle and its intercepted arc***

This observation leads directly to the next theorem.

Theorem 19-9 All inscribed angles subtending the same arc are equal.

Figure 19.26 demonstrates Theorem 19-9: $\angle ABE \cong \angle ACE \cong \angle ADE$.

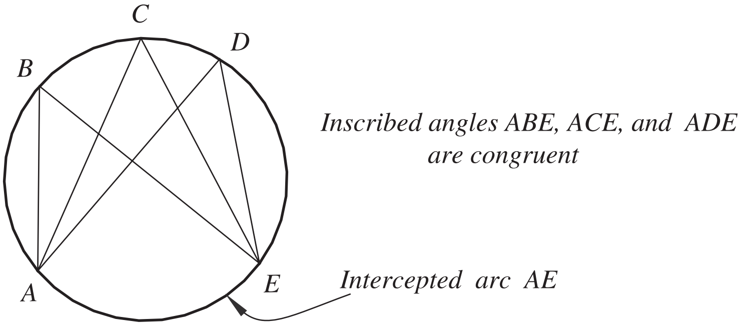


FIGURE 19.26 ***Inscribed angles subtending the same arc are congruent***

Theorem 19-10 The line passing through the centers of two (or more) circles that are internally or externally tangent also passes through the point of tangency.

Figure 19.27 illustrates Theorem 19-10. One pair of circles is internally tangent and the other pair is externally tangent at point *T*. The line *OO′* passes through the centers of both circles and through point *T*. Line *OO′* is perpendicular to tangent *AB*.

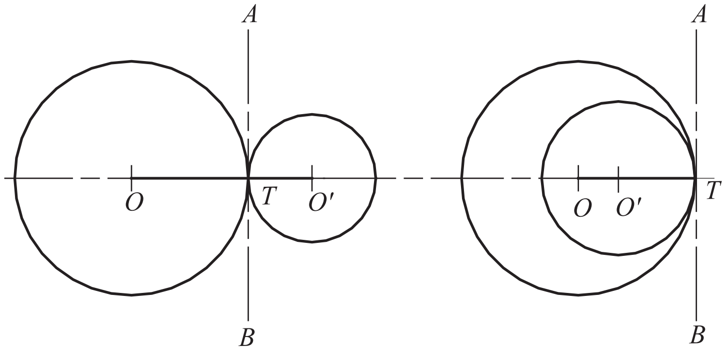


FIGURE 19.27 ***Line of centers of two tangent circles***

Theorem 19-10 is particularly useful when working on problems involving spur gears.

If two chords intersect within a circle, the product of the lengths of two segments of one chord equals the product of the lengths of two segments of the other chord.

Theorem 19-11

Figure 19.28 shows how Theorem 19-11 is applied. Chords *AB* and *CD* intersect at point *E*. The product of the two segments of chord *AB* is $AE \cdot EB$ and is equal to $CE \cdot ED$, the product of the two segments of chord *CD*. Thus, for Figure 19.28, Theorem 19-11 can be stated as $AE \cdot EB = CE \cdot ED$.

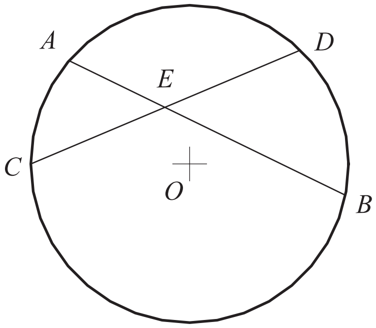


FIGURE 19.28 ***Intersection of two chords***

EXAMPLE 19.6: *Using geometric relationships to understand shop drawings*

Shop drawings are difficult to understand without knowledge of the relationships of lines, angles, and arcs because many details of drawings are not explicitly shown unless they are essential. Items that can be inferred from geometric relationships are often omitted to prevent obscuring more important information.

Consider the front view of the gauge shown in Figure 19.29. Note the following items, most of which are assumed:

- Line a is parallel to line *c*, and line *b* is parallel to line *d*.
- Line *a* is perpendicular to line *b*, and line *c* is perpendicular to line *b* and also to line *d*.
- Lines *a* and *c* are parallel; therefore, they are everywhere equidistant.
- Lines *a* and *c* are perpendicular to *b*, and the angles formed at the points of intersection are 90-degree angles.
- Line *a* is tangent to the arc having a 1.00-inch radius.
- The arc having a 1.0-inch radius and the arc having a 0.50-inch radius are tangent to each other.
- Line *d* is tangent to the arc having a 0.50-inch radius.

Notice that the design for this gauge could not be drawn, nor the data from it used, if the fundamental facts about the relationships between lines, angles, and arcs were not understood.

EXAMPLE 19.6: *Using geometric relationships to understand shop drawings (Continued)*

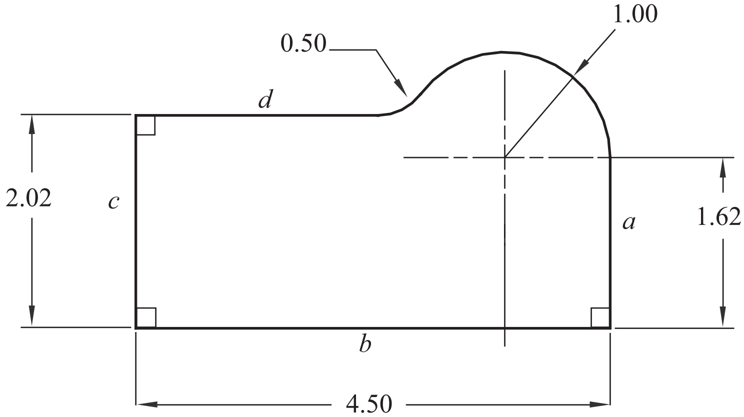


FIGURE 19.29 ***Relationships of lines, angles, and arcs***

EXERCISES

19.6 In Figure 19.30, solve for $\angle\alpha$.

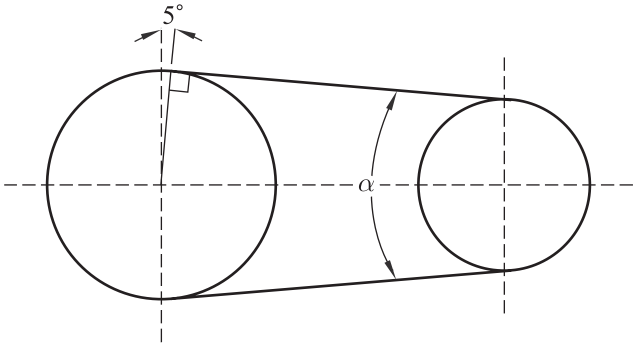


FIGURE 19.30

19.7 The circle in Figure 19.31 is inscribed in a scalene triangle. Solve for $\angle\beta$.

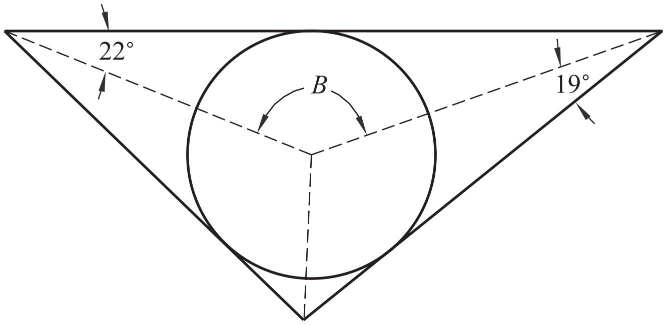


FIGURE 19.31

19.8 In Figure 19.32, lines *l* and *m* are tangent to both circles. Determine the size of angles *A*, *B*, *C*, *D,* and *E*.

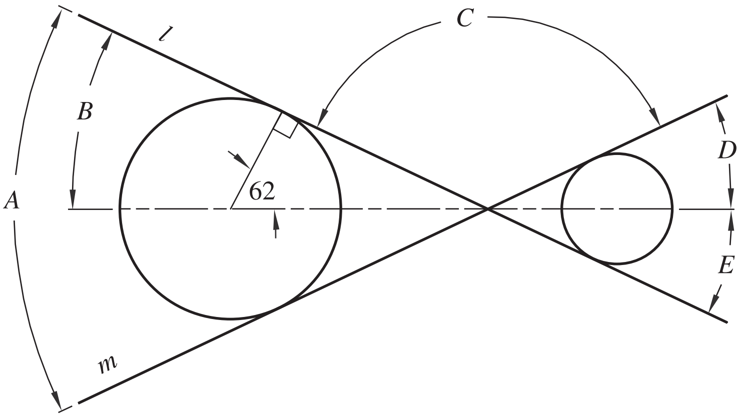


FIGURE 19.32

19.9 In Figure 19.33, segments *AP*, *BQ*, and *AB* are tangent to the circle, and $\overline{AP} \cong \overline{BQ}$. Solve for *Y* when segment *AP* measures 6 inches.

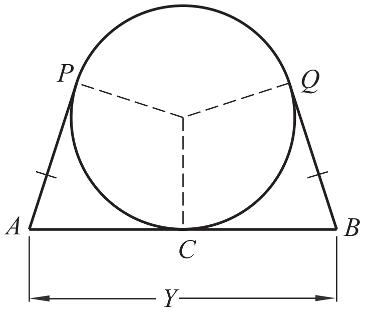


FIGURE 19.33

19.10 In Figure 19.34, segments *AP* and *AQ* are tangent to the circle. Solve for ∠*PAO* when *m*∠*POQ* = 140.

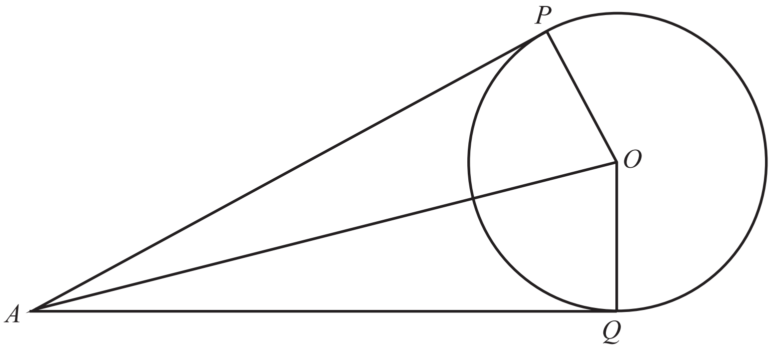


FIGURE 19.34

19.11 In Figure 19.35, the circle is inscribed in quadrilateral *ABCD*. Solve for the length of segment *AB* when *BC* = 11 cm, *CD* = 5.5 cm, *CN* = 2 cm, and *AD* = 9.5 cm.

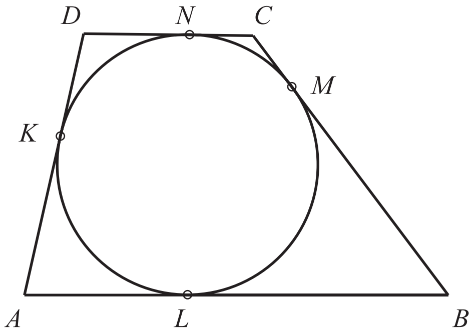


FIGURE 19.35

19.12 In Figure 19.36, the quadrilateral has two points of tangency to the circle. Solve for ∠α.

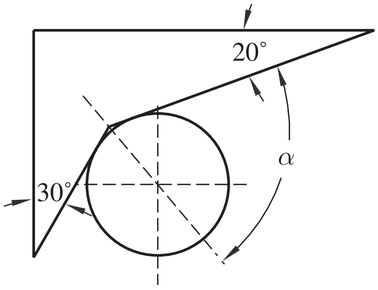


FIGURE 19.36

19.13 In Figure 19.37, lines *l* and *m* are tangent to both circles. Determine the distance *X*. The given dimension is in inches.

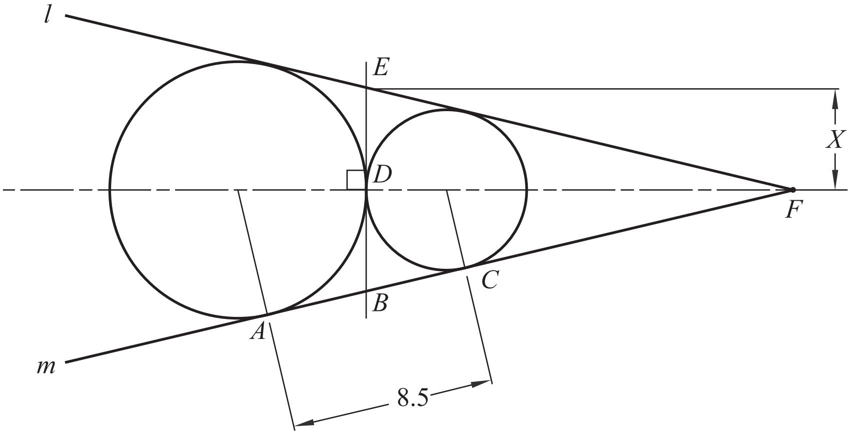


FIGURE 19.37

19.14 In Figure 19.38, the polygon is tangent to the circle at the three points shown. Determine the $m\angle\alpha$.

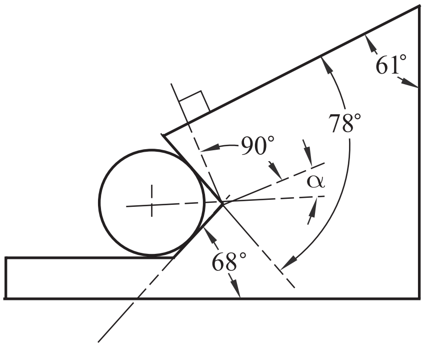


FIGURE 19.38

19.15 In Figure 19.39, solve for $\angle ACO$.

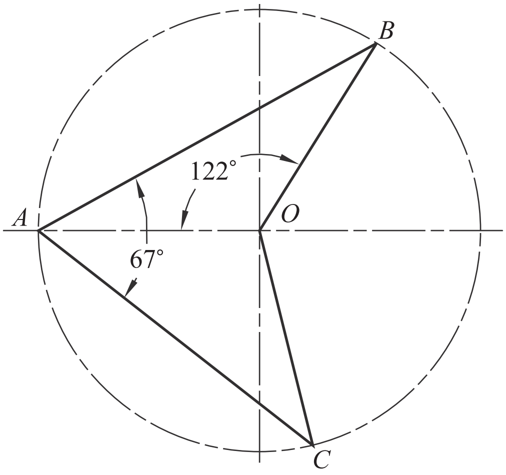


FIGURE 19.39

19.16 In Figure 19.40, find the diameter of an inscribed circle in a right triangle with sides 8.6 inches and 10.2 inches long.

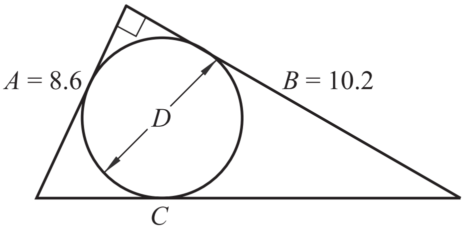


FIGURE 19.40

19.17 In Figure 19.41, solve for ∠*AB*C of the triangle inscribed in the circle.

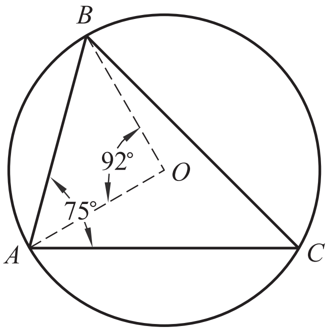


FIGURE 19.41

19.18 In Figure 19.42, solve for diameter *D* in the circle inscribed in the right triangle. Dimensions are in meters.

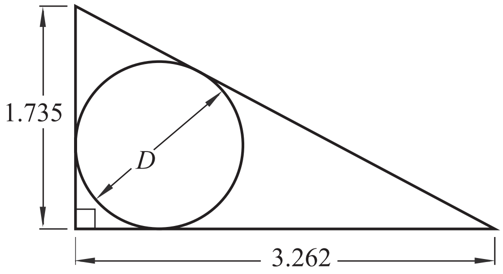


FIGURE 19.42

19.19 In Figure 19.43, solve for dimension *X*. Dimensions are in inches.

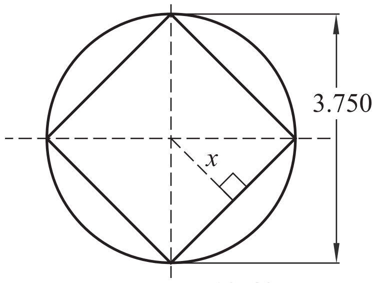


FIGURE 19.43

19.20 In Figure 19.44, solve for dimension *A* in the regular hexagon. Dimensions are in millimeters.

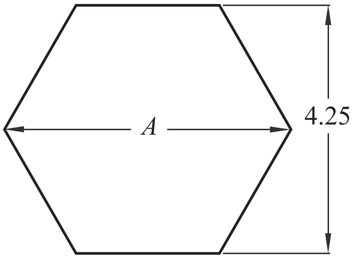


FIGURE 19.44

19.21 In Figure 19.45, solve for dimension *X*, a chord of the circle, and *Y*, the circle's diameter. Dimensions are in centimeters.

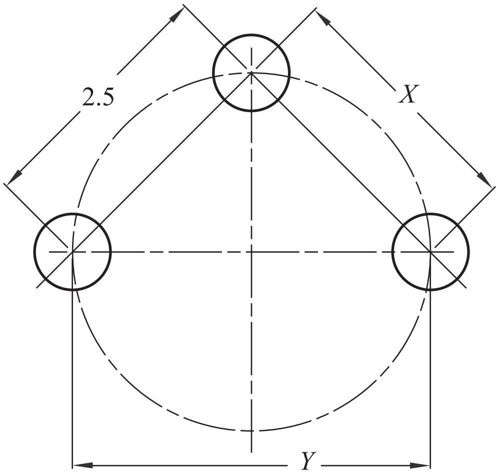


FIGURE 19.45

19.22 In Figure 19.46, solve for the length of the other two sides of the right isosceles triangle circumscribed around the circle. Dimensions are in inches.

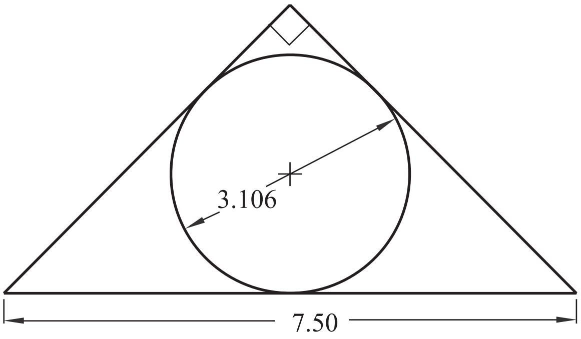


FIGURE 19.46

19.23 In Figure 19.47, solve for $\angle ABC$ and $\angle BCA$ in the inscribed quadrilateral.

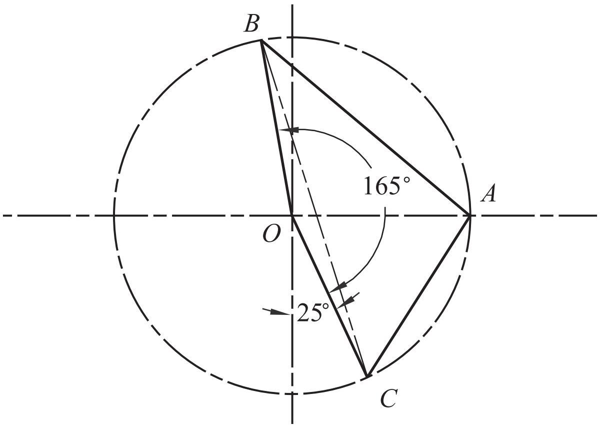


FIGURE 19.47

19.24 In Figure 19.48, solve for $\angle\alpha$.

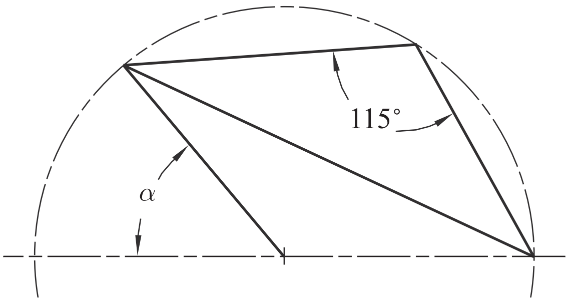


FIGURE 19.48

19.25 In Figure 19.49, solve for dimensions *X* and *Y*. Dimensions are in inches.

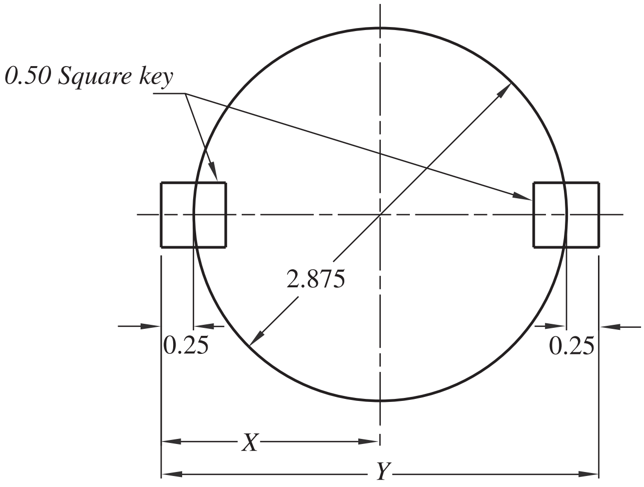


FIGURE 19.49

19.26 In Figure 19.50, solve for $\angle\alpha$ and $\angle\beta$.

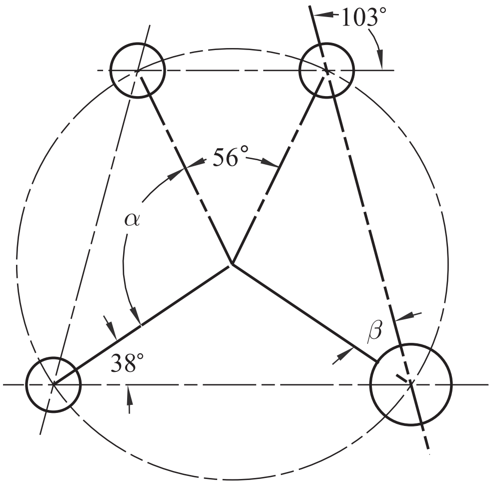


FIGURE 19.50

19.27 In Figure 19.51, solve for angles α, β and ω. The small circles are not equally spaced on hole circle nor are they symmetrically located (top half to bottom half).

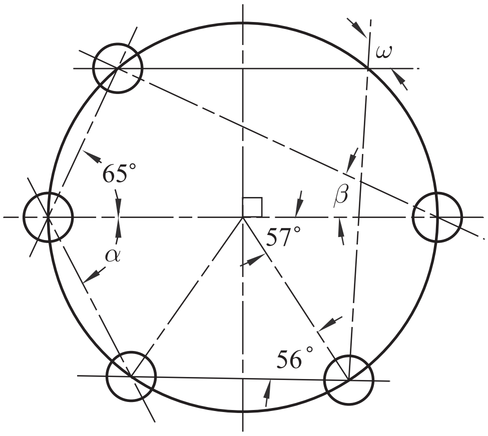


FIGURE 19.51

19.28 In Figure 19.52, a sector of a circle is shown in which $\overline{CD}$ bisects $\overline{AB}$. Determine the radius of the circle. Dimensions are in inches.

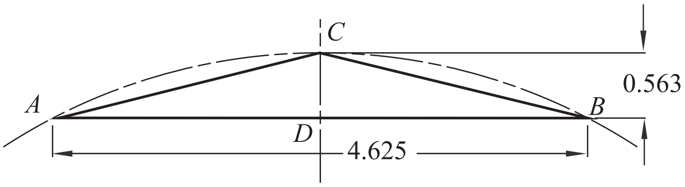


FIGURE 19.52

19.29 Solve for X in Figure 19.53. Dimensions are in millimeters.

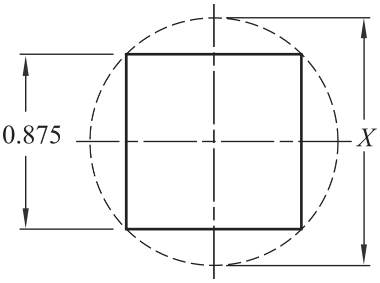


FIGURE 19.53

19.30 In Figure 19.54, solve for the distance across the flats. The figure represents a regular hexagon inscribed in a circle. Dimensions are in inches.

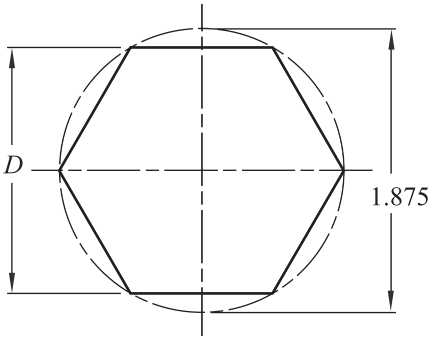


FIGURE 19.54

19.31 Determine the diameter of the circle circumscribed around the hexagon in Figure 19.55. Dimensions are in centimeters.

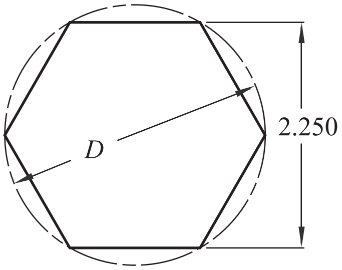


FIGURE 19.55

19.32 In Figure 19.56, determine the diameter. Dimensions are in inches.

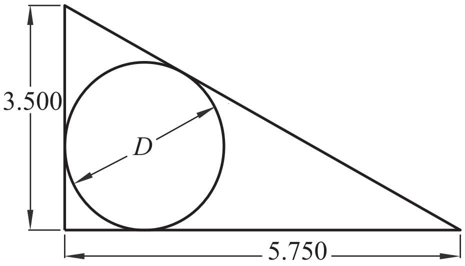


FIGURE 19.56

19.33 Determine $\angle\alpha$ in Figure 19.57. The triangle is inscribed in the circle as shown.

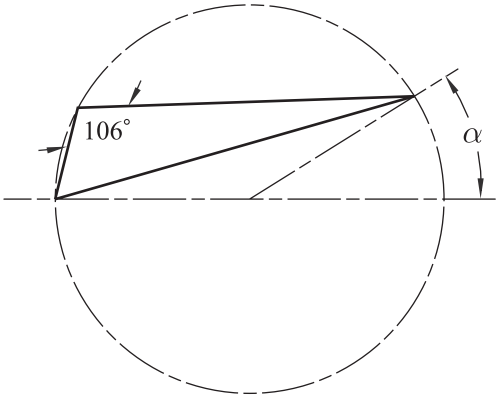


FIGURE 19.57

20

TRIGONOMETRY FUNDAMENTALS

Up to this point we have been learning the language of mathematics and laying the foundation for trigonometry. Along the way we have discussed *arithmetic,* relationships and operations of numbers; *algebra,* a type of generalized arithmetic involving both constant numbers and variables; and *geometry,* the study of forms and relationships of plane figures.

Trigonometry is a branch of mathematics that includes arithmetic, algebra, and geometry, but is primarily concerned with the relationships of lines and angles in triangles. It is the basis of measurements used in surveying, engineering, shop mechanics, geodesy, and astronomy. The word *trigonometry* comes from the Greek words *trigonon,* a triangle, and *metron,* to measure.

trigon ≡ △
metron ≡ measure

20.1 Some Key Definitions Used in Trigonometry

We begin our exploration of trigonometry by focusing strictly on right triangles, which form the basis for all trigonometric calculations. Recall that a right triangle is any triangle with a 90° angle. The standard right triangle is illustrated in Figure 20.1. Note that the three angles are labeled by the capital letters A, B, and C. Sometimes the three-letter convention is used to name the angles, in which case the vertex is the center letter, as in $\angle ABC$ also referred to as $\angle B$. When the three vertices are used to name a right triangle, the vertex containing the right angle is sometimes underlined as in triangle $A\underline{C}B$. By convention, $\angle C$ is always the 90° angle.

The word trigonometry is often informally shortened to "trig."

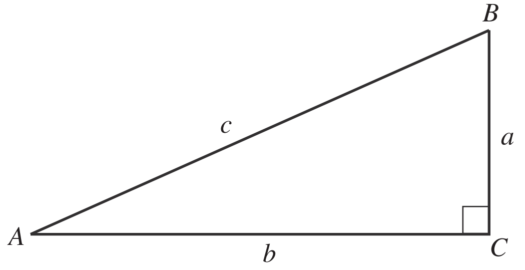
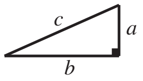


FIGURE 20.1 ***Conventional right triangle designations***

Pythagorean Theorem

$a^2 + b^2 = c^2$



The sides of triangles are identified with lowercase letters. Side *a*, for example, is the side opposite acute $\angle A$ and side *b* is the side opposite acute $\angle B$. The side labeled *c* is always opposite to the 90° angle and is therefore always the hypotenuse. Of course, the term hypotenuse is already familiar from our study of the Pythagorean theorem and it is easily recognized as the longest side in a right triangle. Sides *a* and *b* are the *legs* of the right triangle.

"Opposite" and "adjacent" are always relative to the specified angle.

In reference to sides *a* and *b*, note the importance of understanding what is meant by "opposite side" and "adjacent side." These are relative terms, which change depending on which acute angle of a right triangle is under consideration. In Figure 20.1, side *a* is adjacent to $\angle B$ while side *b* is opposite to $\angle B$. Likewise, side *b* is adjacent to $\angle A$ while side *a* is opposite to $\angle A$. The reference chosen depends on the problem to be solved. The hypotenuse does not change and is always opposite the right angle.

Greek letters such as α, β, and θ (alpha, beta, and theta) are commonly used to denote the acute angles in right-triangles.

Table 20.1 provides a summary of terms used in trigonometry.

TABLE 20.1: **Summary of Terms Used in Trigonometry**

Term	Definition
Acute angle	Any angle less than 90°
Right angle	An angle of 90°
Obtuse angle	Any angle greater than 90° (never found in a right triangle)
Hypotenuse	The side opposite the 90° angle
Opposite side	The side opposite a specified angle
Adjacent side	A side of a specified angle

Trigonometric Functions

The underlying premise for trigonometry is that for *any* right triangle, the ratios of the lengths of any two sides relative to a given acute angle is the same. In other words, regardless of the size of a triangle, the proportions of the sides remain constant if the angles remain the same. Taking two sides at a time, we can readily see that six such ratios are possible for a right triangle. Each ratio is a specific trigonometric function.

The six trigonometric functions are: *sine* (abbreviated *sin*); *cosine* (abbreviated *cos*); *tangent* (abbreviated *tan*); *cotangent* (abbreviated *cot*); *secant* (abbreviated *sec*) and *cosecant* (abbreviated *csc.*)

The functions, named in reference to a standard right triangle, are defined and shown in Figures 20.2 through 20.13. As before, we follow convention and letter the right angle *C* and its opposite side, the hypotenuse, as *c*. The legs, or the shorter sides of the triangle, are labeled *a* and *b* and are opposite the acute angles *A* and *B,* respectively.

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

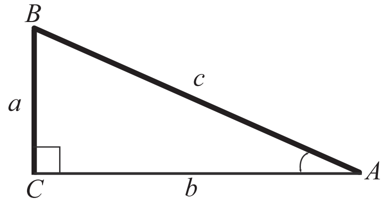


FIGURE 20.2

SOHCAHTOA *(soh cah toa) is a made-up word to help remember the primary trigonometric relationships: "**S**ine is **O**pposite over **H**ypotenuse, **C**osine is **A**djacent over **H**ypotenuse, **T**angent is **O**pposite over **A**djacent."*

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

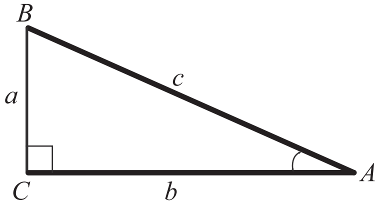


FIGURE 20.3

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{b}$$

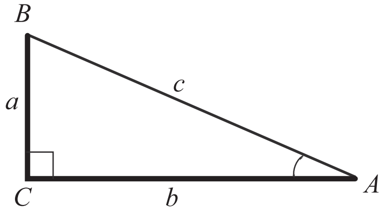


FIGURE 20.4

$$\cot A = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{b}{a}$$

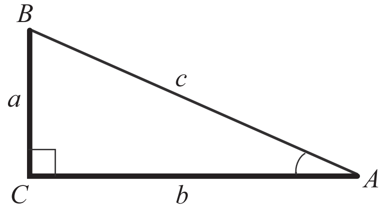


FIGURE 20.5

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{c}{b}$$

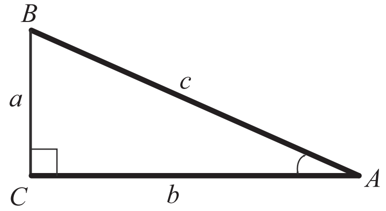


FIGURE 20.6

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{c}{a}$$

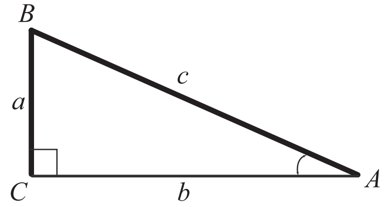


FIGURE 20.7

$$\sin B = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{b}{c}$$

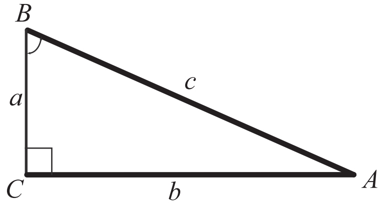


FIGURE 20.8

$$\cos\ B = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$

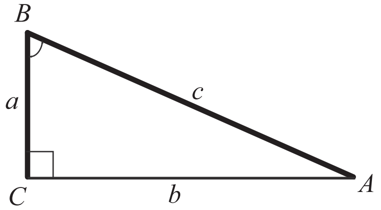


FIGURE 20.9

$$\tan B = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{b}{a}$$

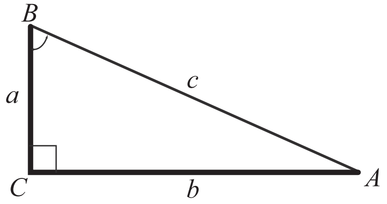


FIGURE 20.10

$$\cot B = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{a}{b}$$

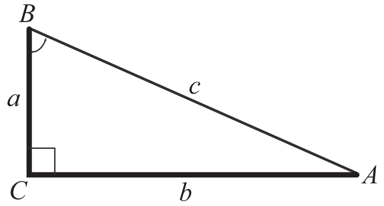


FIGURE 20.11

$$\sec B = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{c}{a}$$

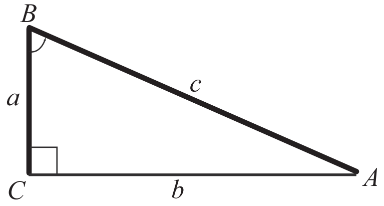


FIGURE 20.12

$$\csc\ B = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{c}{b}$$

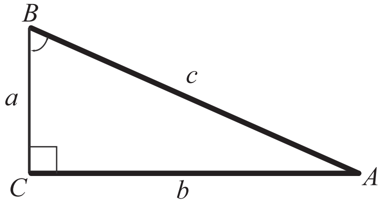


FIGURE 20.13

These six functions are the basis for all work in trigonometry and should be learned thoroughly to facilitate the solving of shop problems.

EXAMPLE 20.1A: *Finding Values for Trigonometric Functions*

Consider the 3-4-5 triangle shown in Figure 20.14. Find the sine, cosine, tangent, cotangent, secant and cosecant of the acute angles.

A "3-4-5" triangle is a triangle with its sides in the ratio of 3 to 4 to 5.

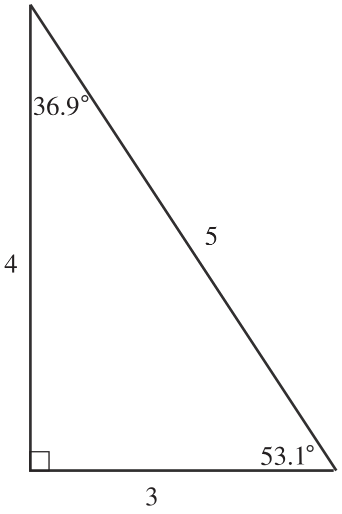


FIGURE 20.14

Notice that $3^2 + 4^2 = 5^2$, as the Pythagorean theorem claims.

Solution: The two acute angles are 36.9° and 53.1°.

$$\sin 36.9° = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{3}{5} \qquad \sin 53.1° = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos\ 36.9° = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos\ 53.1° = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{3}{5}$$

$$\tan 36.9° = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{3}{4} \qquad \tan 53.1° = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{4}{3}$$

$$\cot 36.9° = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{4}{3} \qquad \cot 53.1° = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{3}{4}$$

$$\sec 36.9° = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{5}{4} \qquad \sec 53.1° = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{5}{3}$$

$$\csc\ 36.9° = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{5}{3} \qquad \csc\ 53.1° = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{5}{4}$$

EXAMPLE 20.1B: *Finding Values for Trigonometric Functions*

Consider the 3-4-5 triangle shown in Figure 20.15. While the angles are the same as in the previous example, the lengths of the sides are different. Find the sine, cosine, tangent, cotangent, secant, and cosecant of the acute angles.

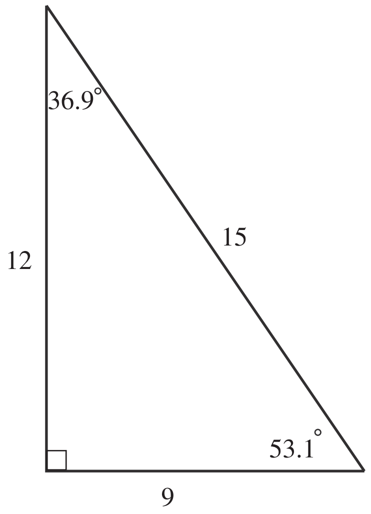


FIGURE 20.15

Solution: Again, the two acute angles are 36.9° and 53.1°.

$$\sin 36.9° = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{9}{15} = \frac{3}{5} \qquad \sin 53.1° = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{12}{15} = \frac{4}{5}$$

$$\cos 36.9° = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{12}{15} = \frac{4}{5} \qquad \cos 53.1° = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{9}{15} = \frac{3}{5}$$

$$\tan 36.9° = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{9}{12} = \frac{3}{4} \qquad \tan 53.1° = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{12}{9} = \frac{4}{3}$$

$$\cot 36.9° = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{12}{9} = \frac{4}{3} \qquad \cot 53.1° = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{9}{12} = \frac{3}{4}$$

$$\sec 36.9° = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{15}{12} = \frac{5}{4} \qquad \sec 53.1° = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{15}{9} = \frac{5}{3}$$

$$\csc 36.9° = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{15}{9} = \frac{5}{3} \qquad \csc 53.1° = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{15}{12} = \frac{5}{4}$$

Because the trigonometric functions are ratios of side lengths whose units are the same, the values of the functions are unitless. For example,

$$\sin 36.9° = \frac{9\,\cancel{\text{cm}}}{15\,\cancel{\text{cm}}} = \frac{9}{15} \text{ or } \frac{3}{5}$$

Notice that although the sizes of the triangles in Examples 20.1A and 20.1B changed, the *proportions* of corresponding sides were the same. Hence, the angles were the same, and so were the values of the trigonometric functions for each acute angle. This fact is the key to how trigonometry works.

Since the angles in Examples 20.1A and 20.1B are given, we could also find the values of the trigonometric functions directly from a scientific calculator without actually using the ratios of the sides. Verify the answers using a scientific calculator.

Several important relationships between certain pairs of trigonometric functions are useful for solving problems. These relationships, known as the *complementary* and *reciprocal relationships*, are discussed next.

Complementary Trigonometric Relationships

From our knowledge of geometry, we know that the sum of the measures of the interior angles of a triangle must add to 180. The measures of the acute angles of right triangle, therefore, must add to 90. Referring to the standard triangle shown in Figure 20.1, A and B must always be complements. Accordingly, $m\angle A = 90 - m\angle B$ and we can formulate the complementary trigonometric relationships given in Table 20.2.

TABLE 20.2: Complementary Trigonometric Identities

$\sin A = \cos(90 - A)$	$\sin B = \cos(90 - B)$
$\cos A = \sin(90 - A)$	$\cos B = \sin(90 - B)$
$\tan A = \cot(90 - A)$	$\tan B = \cot(90 - B)$
$\cot A = \tan(90 - A)$	$\sec B = \csc(90 - B)$
$\sec A = \csc(90 - A)$	$\cot B = \tan(90 - B)$
$\csc A = \sec(90 - A)$	$\csc B = \sec(90 - B)$

Another way to look at the complementary trigonometric relationships is to observe that $\sin A$ is identical to $\cos B$. This is due to the fact that for both functions the ratios involve the same sides. Likewise, $\tan A = \cot B$. These relationships are known as the trigonometric *cofunction identities* and are summarized in Table 20.3 in reference to Figure 20.1.

TABLE 20.3: Trigonometric Cofunction Identities

$\sin A = \cos B = \frac{a}{c}$	$\cos A = \sin B = \frac{b}{c}$
$\tan A = \cot B = \frac{a}{b}$	$\cot A = \tan B = \frac{a}{b}$
$\sec A = \csc B = \frac{c}{a}$	$\csc A = \sec B = \frac{c}{b}$

EXAMPLE 20.2: ***Trigonometric Cofunction Identities***

Fill in the blanks in the following statements and verify results with a scientific calculator:

sin 23° = cos____ cos 39° = sin____ tan 10° = cot____

EXAMPLE 20.2: ***Trigonometric Cofunction Identities (Continued)***

Solution:

$$\sin 23^\circ = \cos(90^\circ - 23^\circ) = \cos 67^\circ = 0.3907$$

$$\cos 39^\circ = \sin(90^\circ - 39^\circ) = \sin 51^\circ = 0.7771$$

$$\cot 10^\circ = \tan(90^\circ - 10^\circ) = \tan 80^\circ = 5.6713$$

Reciprocal Trigonometric Relationships

Recall that the reciprocal of a fraction is obtained by interchanging the numerator and the denominator. For instance, the reciprocal of $\frac{5}{6}$ is $\frac{6}{5}$. The reciprocal of a whole number such as 2 is $\frac{1}{2}$ since the denominator of a whole number is simply "1." Zero has no reciprocal, since $\frac{0}{1}$ inverts to $\frac{1}{0}$, which is undefined. The definitions of the six trigonometric functions reveal that each function is the reciprocal of one of the other functions. These relationships are called the trigonometric *reciprocal identities* and are summarized in Table 20.4.

TABLE 20.4: **Reciprocal Trigonometric Identities**

$\sin A = \dfrac{1}{\csc A}$	$\csc A = \dfrac{1}{\sin A}$
$\cos A = \dfrac{1}{\sec A}$	$\sec A = \dfrac{1}{\cos A}$
$\tan A = \dfrac{1}{\cot A}$	$\cot A = \dfrac{1}{\tan A}$

On many calculators the reciprocal key is identified as 1/x or x^{-1}.

Because of the reciprocal relationships, many scientific calculators only have keys for sine, cosine, and tangent. The cosecant, secant, and cotangent functions are readily found by using the reciprocal function found on most scientific calculators.

EXAMPLE 20.3: ***Reciprocal Trigonometric Identities***

Fill in the blanks in the following statements and verify results with a scientific calculator:

$$\csc 14^\circ = \frac{1}{?} \qquad \sec 60^\circ = \frac{1}{?} \qquad \cot 90^\circ = \frac{1}{?}$$

Solution:

$$\csc 14^\circ = \frac{1}{\sin 14^\circ} = \frac{1}{0.2419} = 4.1336$$

$$\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{0.5} = 2$$

$$\cot 90^\circ = \frac{1}{\tan 90^\circ} = \frac{1}{0} = \text{undefined}$$

EXERCISES

20.1 Explain how each of the following words are used in trigonometry:

a) Adjacent side
b) Opposite side
c) Hypotenuse

20.2 Name the six trigonometric functions and the ratios they represent.

20.3 Use a scientific calculator to find values rounded to the fourth decimal place for each of the following trigonometric functions:

a) $\sin 30^\circ$
b) $\sin 36.45^\circ$
c) $\cot 70.30^\circ$
d) $\sin 5.22^\circ$
e) $\tan 76^\circ$
f) $\cos 3.52^\circ$
g) $\tan 45^\circ$
h) $\tan 22.40^\circ$
i) $\tan 66.35^\circ$
j) $\tan 66.66^\circ$
k) $\csc 22.22^\circ$
l) $\tan 19.22^\circ$
m) $\cos 45^\circ$
n) $\cos 55.10^\circ$
o) $\sec 15.5^\circ$
p) $\cot 45^\circ$
q) $\sin 60.75^\circ$

20.4 Using the triangle shown in Figure 20.16, express each of the six trigonometric functions of angles α and β as fractions.

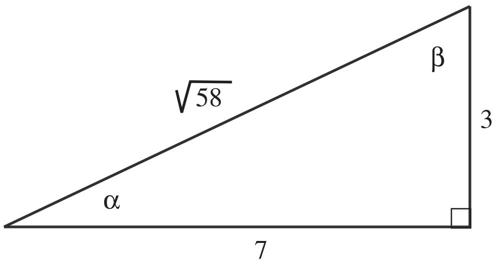


FIGURE 20.16

a) $\sin \alpha =$ __
b) $\csc \alpha =$ __
c) $\sin \beta =$ __
d) $\csc \beta =$ __
e) $\cos \alpha =$ __
f) $\sec \alpha =$ __
g) $\cos \beta =$ __
h) $\sec \beta =$ __
i) $\tan \alpha =$ __
j) $\cot \alpha =$ __
k) $\tan \beta =$ __
l) $\cot \beta =$ __

20.5 Find trigonometric values for the indicated condition.

a) $\sin A$ when $\cos B = 0.89101$
b) $\sin A$ when $\csc A = 1.1223$
c) $\tan A$ when $\cot A = 0.78128$
d) $\tan A$ when $\cot B = 0.78128$
e) $\sec A$ when $\cos A = 0.7431$

20.6 Complete the following statements using a trigonometric function:

a) $\csc 82^\circ = \frac{1}{?}$
b) $\sec 19^\circ = \frac{1}{?}$
c) $\cot 7^\circ = \frac{1}{?}$
d) $\frac{1}{\cos 14^\circ} = ?$
e) $\frac{1}{\cot 75^\circ} = ?$
f) $\frac{1}{\sin 30^\circ} = ?$

20.2 Solving Sides of Triangles Using Trigonometric Functions

Most problems in trigonometry involve finding the length of an unknown side of a triangle when given one angle and one side or finding an unknown angle in a triangle when the lengths of two sides are given. The solution procedure is always the same:

Procedure to solve triangle using trigonometry.

- Sketch the triangle.
- Identify the known and unknown quantities and show on the sketch.
- Decide which trigonometric function to use.
- Set up an equation using a trigonometric function.
- Solve the equation for the unknown quantity.

The next examples illustrate how to solve for sides of a triangle using trigonometric functions.

EXAMPLE 20.4A: ***Solving a Right Triangle***

For the triangle shown in Figure 20.17, solve for $\angle B$ and the lengths of sides a and c.

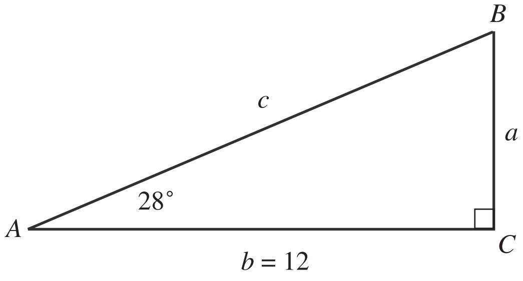


FIGURE 20.17

Solution:

STEP 1: Find $\angle B$

Since the sum of the angle measures of a triangle must add to 180, the measure of $\angle B$ is:

$$m\angle B = 180 - 90 - 28 = 62°$$

So, $\angle B = 62°$.

STEP 2: Find side a

The length of side a can be solved using either $\angle A$ or $\angle B$ as a reference. If we choose $\angle A$, then the adjacent side is the known side and the opposite side is the unknown side. This means that we can use the tangent function by writing:

EXAMPLE 20.4A: ***Solving a Right Triangle (Continued)***

$$\tan 28^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{12}$$

Rearranging $\Rightarrow a = 12\tan 28^\circ$.

Using a scientific calculator, we find that tan 28° = 0.5317.

Finally, $a = (12)(0.5317) = 6.38$.

If we choose to start from $\angle B$ instead, then the opposite side becomes the known side and the adjacent side is the unknown side. Again, this means that we can use the tangent function by writing:

$$\tan 62^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{12}{a}$$

Rearranging $\Rightarrow a = \dfrac{12}{\tan 62^\circ}$.

Using a scientific calculator, we find that tan 62° = 1.8807.

Finally, $a = \dfrac{12}{\tan 62^\circ} = \dfrac{12}{1.8807} = 6.38$, which proves what we found using $\angle A$.

STEP 3: Find side c

Likewise, the length of side c can be solved using either $\angle A$ or $\angle B$ as a reference. If we choose $\angle A$, then the adjacent side is the known side and the hypotenuse is the unknown side. This means that we can use the cosine function by writing:

$$\cos 28^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{c}$$

Rearranging $\Rightarrow c = \dfrac{12}{\cos 28^\circ}$

Using a scientific calculator, we find that cos 28° = 0.8829.

Finally, $c = \dfrac{12}{\cos 28^\circ} = \dfrac{12}{0.8829} = 13.59$

Of course we could have found side c using the Pythagorean theorem since we already knew the lengths of sides a and b:

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2} = \sqrt{6.38^2 + 12^2} = 13.59$$

EXAMPLE 20.4B: ***Solving a Right Triangle***

For the triangle shown in Figure 20.18, solve for $\angle B$ and the lengths of sides a and b.

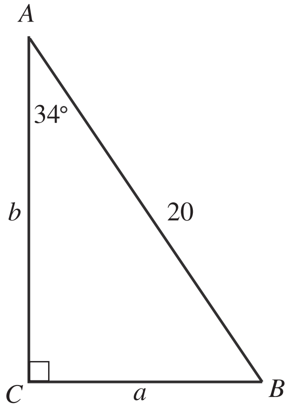


FIGURE 20.18

Solution:

STEP 1: Find $\angle B$

Since the sum of the angle measures of a triangle must add to 180, the measure of $\angle B$ is:

$$m\angle B = 180 - 90 - 34 = 56$$

So, $\angle B = 56°$

STEP 2: Find side a

The length of side a can be solved using either $\angle A$ or $\angle B$ as a reference. If we choose $\angle A$, then the hypotenuse is the known side and the opposite side is the unknown side. This means that we can use the sine function by writing:

$$\sin 34° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{20}$$

Rearranging $\Rightarrow a = 20 \sin 34°$

Using a scientific calculator, we find that $\sin 34° = 0.5592$

Finally, $a = (20)(0.5592) = 11.18$

As an exercise, confirm that the same result is achieved using $\angle B$ as the reference.

STEP 3: Find side b

Likewise, the length of side b can be solved using either $\angle A$ or $\angle B$ as a reference. If we choose $\angle A$, then the hypotenuse is the known side and the adjacent side is the unknown side. This means that we can use the cosine function by writing:

EXAMPLE 20.4B: *Solving a Right Triangle (Continued)*

$$\cos 34° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{20}$$

Rearranging ⇒ $b = 20 \cos 34°$

Using a scientific calculator, we find that $\cos 34° = 0.8290$

Finally, $b = (20)(0.8290) = 16.58$

For practice, confirm that the same result is achieved using the Pythagorean theorem.

EXAMPLE 20.5: *Solving a Complicated Trigonometry Problem*

Calculate dimension *X* on the gauge shown in Figure 20.19. All length dimensions are in inches.

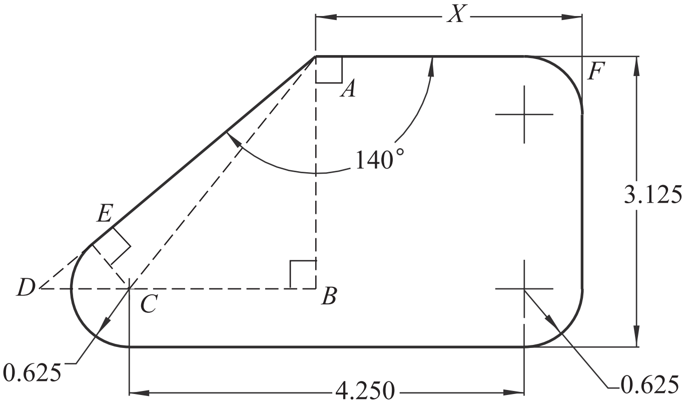


FIGURE 20.19

Solution:

STEP 1: Construct $\overline{AB}$ from point *A*, parallel to the right edge of the gauge.

STEP 2: Draw $\overline{CB}$ from *C*, parallel to the base ($m\angle B = 90$).

STEP 3: Extend lines *BC* and *AE* to *D*. Triangle *ABD* is a right triangle.

STEP 4: Draw line *CE* from point *C* perpendicular to $\overline{AD}$. Triangle *DEC* is a right triangle.

STEP 5: Calculate the length of *AB*: $AB = 3.125 - 0.625 = 2.500$ inches

STEP 6: Calculate $\angle BAD$:

$$\begin{aligned} m\angle BAD &= 140 - m\angle BAF \\ &= 140 - 90 = 50 \end{aligned}$$

So, $\angle BAD = 50°$.

STEP 7: Calculate the length of $\overline{BD}$ in triangle *ABD*:

$$\tan \angle BAD = \frac{BD}{AB}$$

EXAMPLE 20.5: ***Solving a Complicated Trigonometry Problem (Continued)***

$$BD = AB \tan \angle BAD$$

$$(2.50)(1.1917) = 2.979 \text{ inches}$$

STEP 8: In triangle *DEC*, calculate the length of *DC*:

Notice that $m\angle ECD = m\angle BAD$ because both $\angle BCD$ and $\angle BAD$ are complimentary to $\angle ADB$.

$$EC = 0.625 \text{ inch}$$

$$m\angle ECD = m\angle BAD = 50$$

$$\cos \angle ECD = \frac{EC}{DC}$$

$$DC = \frac{EC}{\cos 50^\circ} = \frac{0.625}{0.64279} = 0.972 \text{ inch}$$

STEP 9: Calculate the length of *BC*:

$$BC = BD - DC = 2.979 - 0.972 = 2.007 \text{ inches}$$

STEP 10: Calculate length *X*:

$$X = 4.250 + 0.625 - BC = 4.250 + 0.625 - 2.007 = 2.868 \text{ inches}$$

Inverse Trigonometric Functions

Now that we know how to use the trigonometric functions to find the length of a side of a right triangle, we can discuss how to find an unknown angle.

The six trigonometric functions involve the ratios of the lengths of the sides of right triangles. We have seen that for a given angle, the value of a trigonometric function is the same regardless of the size of the triangle. Consequently, if we know the ratio of any two sides of a right triangle, we can operate backwards and determine the measure of an angle by using the correct *inverse trigonometric function.*

For instance, we can easily show that the sin 30° = 0.5 using a scientific calculator. Suppose we are given a right triangle, whose opposite side is 5 inches and hypotenuse is 10 inches, and asked to find the measure of one of the acute angles, $\angle A$. This amounts to asking for the angle whose sine is $\frac{5}{10} = 0.5$, which we know to be 30°. In general, we solve this type of problem using the inverse trigonometric functions as follows:

$\sin^{-1}$ is the symbol for the inverse sine function. It does not mean sine raised to the negative 1 power.

$$\sin A = 0.5$$
$$A = \sin^{-1}(0.5)$$
$$A = 30^\circ$$

Each of the six trigonometric functions has an inverse function. Sometimes this function is called the *arcfunction* as in arcsine, arccosine, and arctangent. Some scientific calculators use $\sin^{-1}$ to denote the inverse sine while others use the INV key, or some other designation.

EXAMPLE 20.6: *Finding an Unknown Angle Using Trigonometry*

A telephone pole is supported by a 50-foot guy wire attached to the telephone pole 30 feet above the ground. What is the angle α between the guy wire and the ground? What is the angle β between the guy wire and the telephone pole?

Solution:

In general,
$\sin\alpha = x \Rightarrow \sin^{-1}x = \alpha.$

Begin by making a sketch of the word problem as follows:

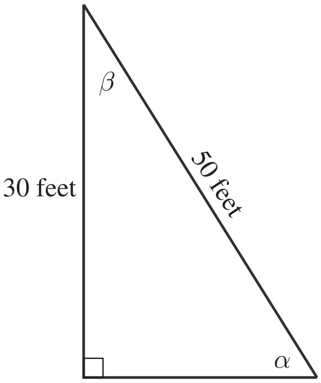


FIGURE 20.20

STEP 1: Solve for α.

Using α as a reference, the opposite side and the hypotenuse are the known sides. This suggests using the inverse sine function to find α as follows:

$$\sin\alpha = \frac{30}{50}$$

$$\alpha = \sin^{-1}\left(\frac{30}{50}\right)$$

Using a scientific calculator, we find that $\alpha = 36.9°$.

STEP 2: Solve for β.

Notice that $\alpha + \beta = 90$.

Using β as a reference, the adjacent side and the hypotenuse are the known sides. This suggests using the inverse cosine function to find β as follows:

$$\cos\beta = \frac{30}{50}$$

$$\beta = \cos^{-1}\left(\frac{30}{50}\right)$$

Using a scientific calculator, we find that $\beta = 53.1°$.

EXERCISES

20.7 With a scientific calculator, use the correct inverse trigonometric function to find the indicated angle measured in degrees.

a) x when $\sin x = 0.52250$
b) x when $\tan x = 2.63954$
c) x when $\sec x = 1.2147$
d) x when $\sin x = 0.70613$
e) x when $\csc x = 1.4140$
f) A when $\sin A = 0.03490$
g) A when $\cos A = 0.92388$
h) A when $\cos A = 0.36650$
i) B when $\csc B = 1.6034$
j) x when $\cos x = 0.78534$
k) x when $\cot x = 3.79833$
l) x when $\csc x = 1.2206$
m) x when $\cos x = 0.48359$
n) x when $\sec x = 1.2209$
o) A when $\tan A = 0.14945$
p) A when $\sec A = 1.1879$
q) B when $\cot B = 2.412$
r) B when $\sin B = 0.78801$

20.8 Given $\sin x = 0.6391$, find x and then find $\cos x$.

20.9 Given $\tan x = 2.8649$, find x and then find $\cot x$.

20.10 Using the right triangle shown in Figure 20.21, solve for the unknown sides, hypotenuse, and angles.

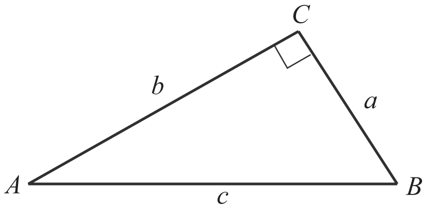


FIGURE 20.21

a) $a = 6$, $c = 12$
b) $b = 20$, $\angle B = 3°$
c) $b = 4$, $\angle A = 60°$
d) $a = 992$, $\angle B = 76°$
e) $c = 200$, $\angle B = 21°$
f) $c = 30.69$, $b = 18.25$
g) $c = 72.15$, $\angle A = 39°$
h) $c = 2194$, $b = 1312.7$
i) $b = 50.94$, $\angle B = 43°$
j) $b = 91.92$, $\angle a = 2.15$
k) $b = 12$, $\angle A = 29°$
l) $c = 8.462$, $\angle B = 86°$
m) $\mathrm{a} = 7$, $\angle A = 18°$
n) $b = 9$, $\angle B = 34°$

20.11 Using the sine function, solve for x when the side opposite x measures 25 feet and the hypotenuse measures 50 feet. See Figure 20.22.

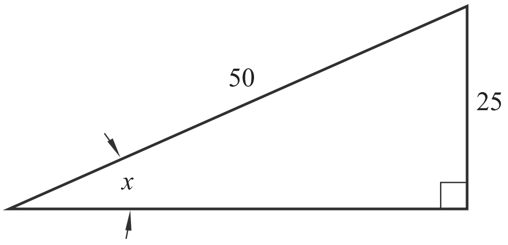


FIGURE 20.22

20.12 A flagstaff breaks off 22 feet from the top and, with the top and bottom still holding together, the top of the staff touches the ground 11 feet from the foot of the flagstaff. What angle does the upper portion (hypotenuse) make with the ground?

Hint: Draw a diagram of the situation.

20.13 An A-shaped roof has a span AA' of 24 feet. The ridgepole is 12 feet above AA'. What angle does side AR make with AA'? See Figure 20.23.

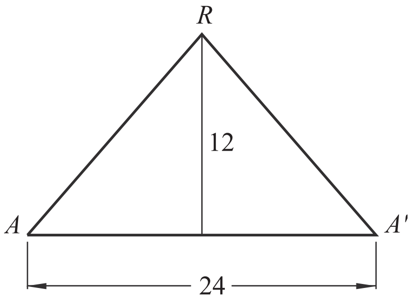


FIGURE 20.23

20.14 A rafter on one side of a roof covers a horizontal run of 12 feet and a vertical rise of 10 feet. What angle does it make with the horizontal (x) and vertical (y)? See Figure 20.24.

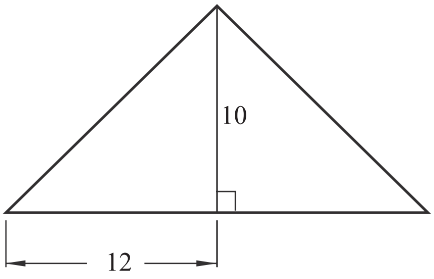


FIGURE 20.24

20.15 Using the cosine function, solve for side b in the right triangle whose hypotenuse c is 28 inches and $\angle A$ is 46°. See Figure 20.25.

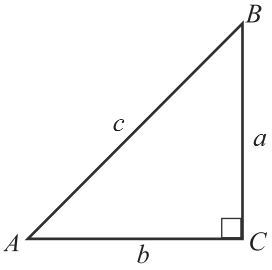


FIGURE 20.25

20.16 A builder wishes to measure a pond. Right triangle *ACB* is constructed as shown in Figure 20.26. If $AB = 928$ feet and $\angle A$ is 29°, find the distance between point *B* and *C*.

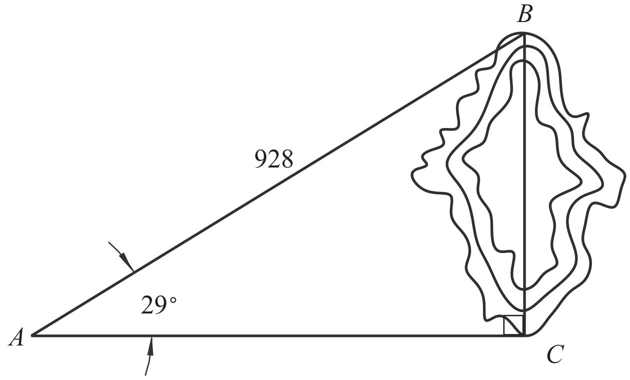


FIGURE 20.26

20.17 A regular hexagon is inscribed in a circle with radius 9 inches. What is the distance from the center of the hexagon to the midpoint of a side? See Figure 20.27.

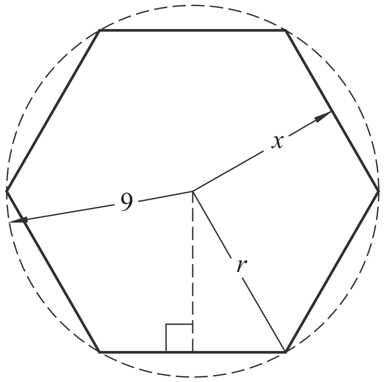


FIGURE 20.27

20.18 An equilateral triangle is inscribed in a circle having a radius of 12 inches. How far is it from the center to a side (i.e., the shortest distance)?

20.19 Using the tangent function, solve for side *a* when $b = 12$ inches and $\angle A$ is 35°. See Figure 20.28.

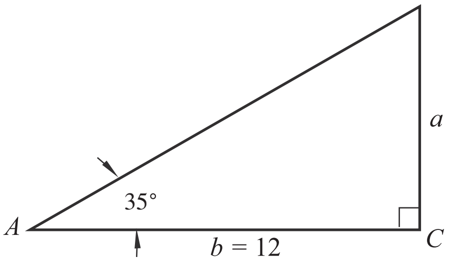


FIGURE 20.28

20.20 Calculate θ, which is the angle between the geometrical tangent *AB* to the 3-inch circle and the line *AO* passing through the center of the circle. The length of *AB* is 10.625 inches. See Figure 20.29.

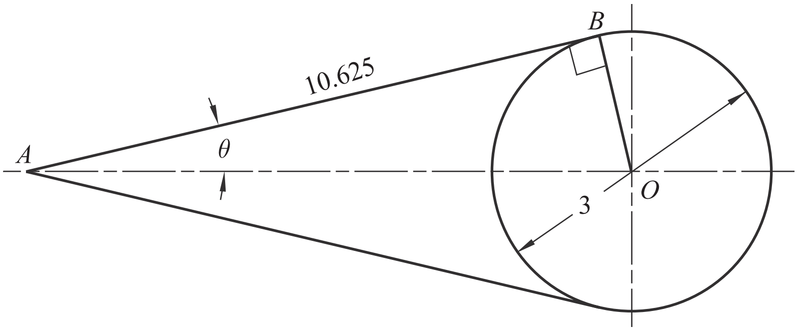


FIGURE 20.29

20.21 A person standing 120 feet from the base of a smokestack finds that the angle of elevation (the angle a line from the eye to the object makes with the horizontal) is 50°. If the person's eye is 5 feet 8 inches from the ground, what is the height of the stack?

20.22 The principal dimensions of a truss are shown in Figure 20.30. Calculate $\angle CHG$ and the length of *HC*. All dimensions are in feet.

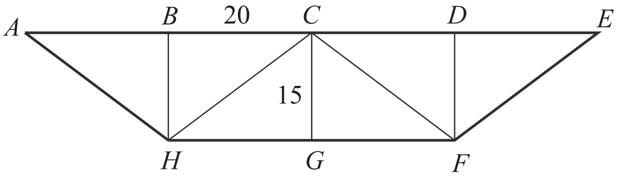


FIGURE 20.30

20.23 What is the angle of inclination of a stairway from the floor if the steps have a tread (run) of 8 inches and a rise of $6\frac{1}{2}$ inches?

20.24 A grade of 1% in a roadbed indicates a rise of 1 foot in a horizontal distance of 100 feet. What is the angle of slope of a roadbed that has a grade of 6%? Of another roadbed with a grade of 0.64%?

20.25 From the top of a smokestack 300 feet high, a point *A* on the ground has angle of depression of 35°, as shown in Figure 20.31. How far is the point from the base of the stack?

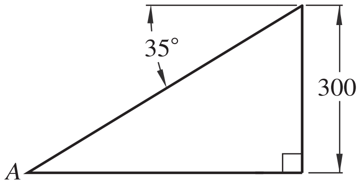


FIGURE 20.31

20.26 A tree known to be 50 feet high standing on the far bank of a stream is observed from the opposite bank to have an angle of elevation of 20°. The angle is measured on a line 5 feet above the ground. How wide is the stream?

20.27 A building standing on level ground has a fire escape platform 20 feet from the ground. From a point *O* on the ground, the angle of elevation to the platform is 38°. The angle of elevation from the same point *O* to the top of the building is 75°. What is the building height?

20.28 A ladder rests against the side of a building, and makes an angle of 28° with the ground. The foot of the ladder is 20 feet from the building wall. How long is the ladder?

20.29 In measuring the distance *AB* across a pond, a surveyor ran a line *AC*, making an angle of 50° with *AB*, and a line *BC* perpendicular to *AC*. (See Figure 20.32.) The surveyor measured *AC* and found it was 880 feet. What was the length *AB* of the pond?

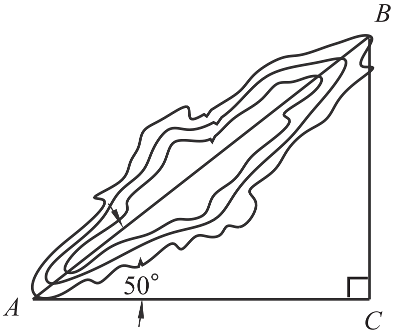


FIGURE 20.32

20.30 The span of the roof in Figure 20.33 is 40 feet and the roof timbers *AB* make an angle of 40° with the horizontal. Find the length of *AB*.

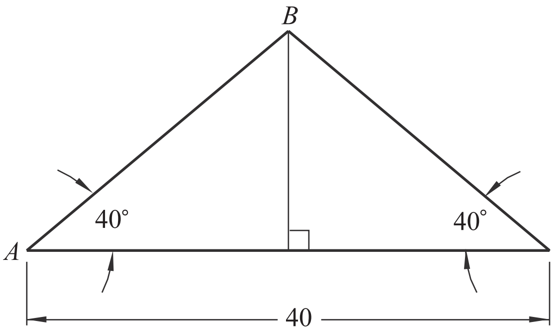


FIGURE 20.33

20.31 A typical rigging to load and unload ships at a pier is shown in Figure 20.34. Solve for $\angle\alpha$ when DW is parallel to BC and AF. All dimensions are in feet.

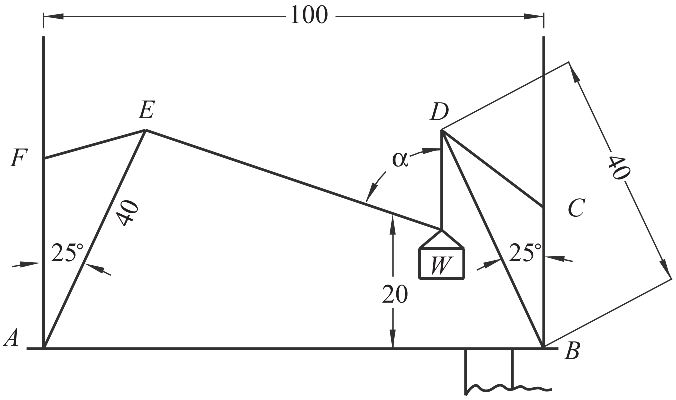


FIGURE 20.34

20.32 From the top of a rock, a cord is stretched to a point on the ground, making an angle of 40° with the horizontal plane. The cord measures 84 feet. Assuming the cord to be straight, how high is the rock?

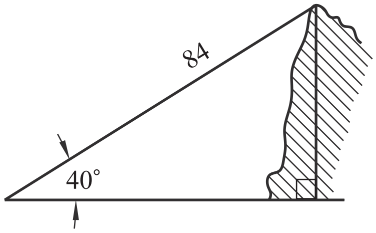


FIGURE 20.35

20.33 The edge of the Great Pyramid is 609 feet and makes an angle of 52° with the base. What is the height of the pyramid?

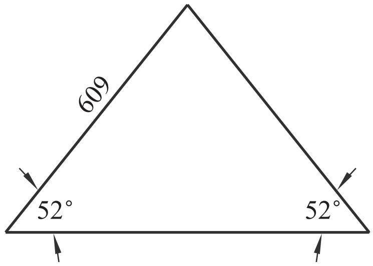


FIGURE 20.36

20.34 Solve for diameter *D*. The dimension shown is in inches.

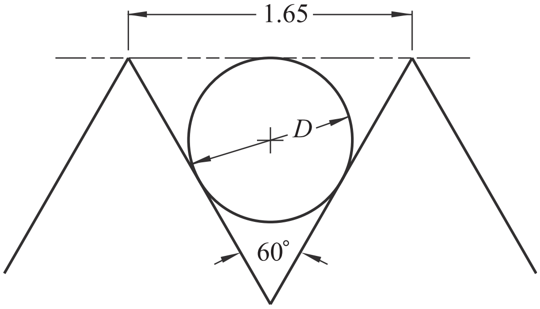


FIGURE 20.37

20.35 Determine angle *A*. The dimensions shown are in inches.

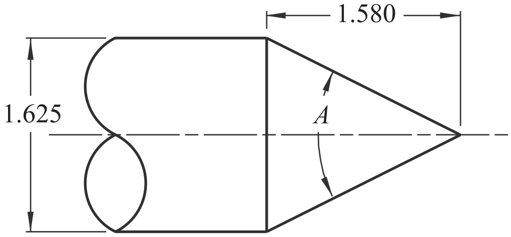


FIGURE 20.38

20.36 Solve for length *L* in Figure 20.39. Dimensions are in meters.

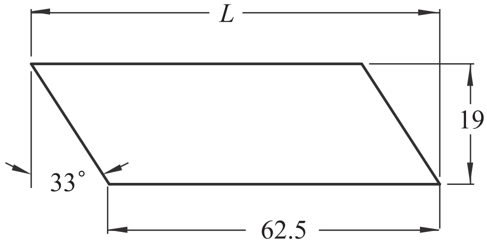


FIGURE 20.39

20.37 Determine angle *A* in Figure 20.40. Dimensions are in meters.

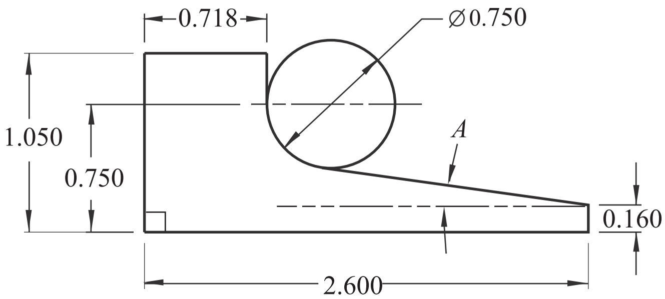


FIGURE 20.40

20.38 Solve for distance *Z* in Figure 20.41. Dimensions are in centimeters.

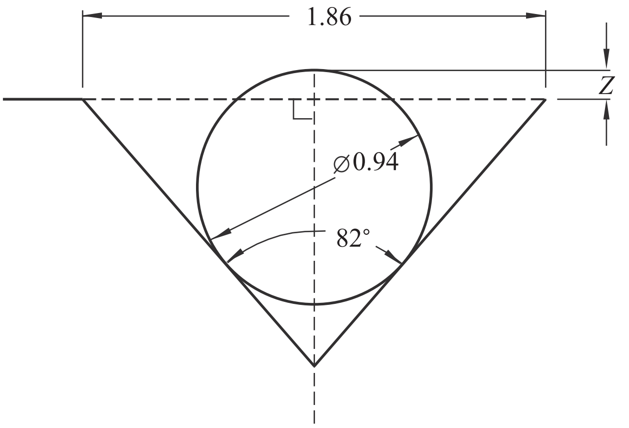


FIGURE 20.41

20.39 Determine radius *r* in Figure 20.42. Dimensions are in millimeters.

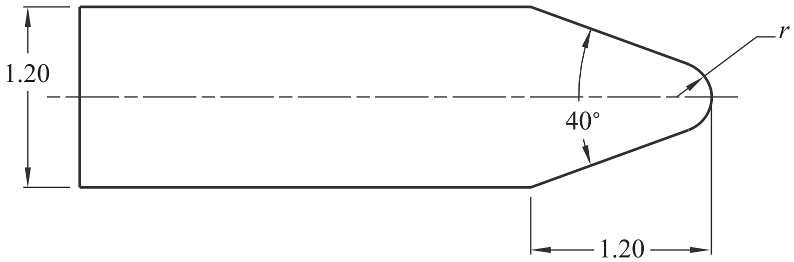


FIGURE 20.42

20.40 Solve for length *L* and angle *A* in Figure 20.43. Dimensions are in millimeters.

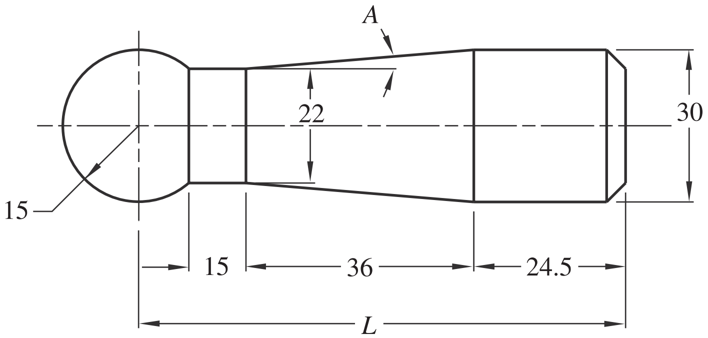


FIGURE 20.43

20.41 Determine angle *A* in Figure 20.44. Dimensions are in inches.

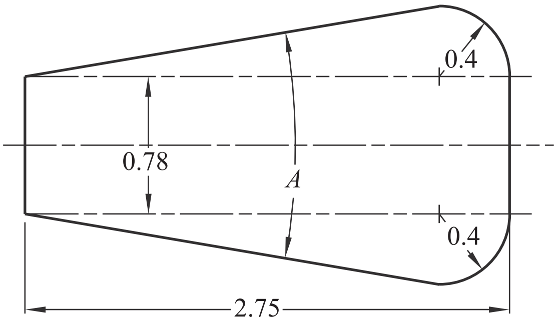


FIGURE 20.44

20.42 Determine the height *H* in Figure 20.45. Dimensions are in millimeters.

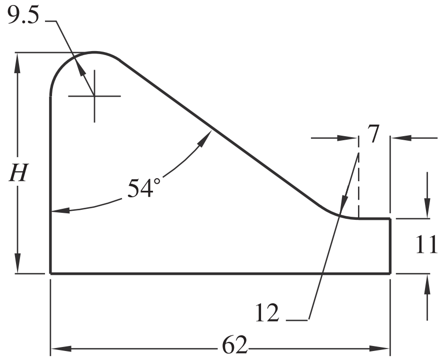


FIGURE 20.45

20.43 Solve for dimension *A* in Figure 20.46. Dimensions are in inches.

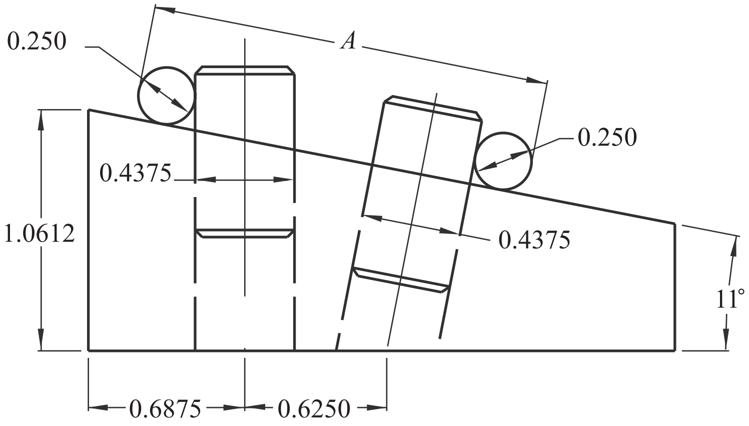


FIGURE 20.46

20.44 Solve for dimension *A* in Figure 20.47. Dimensions are in inches.

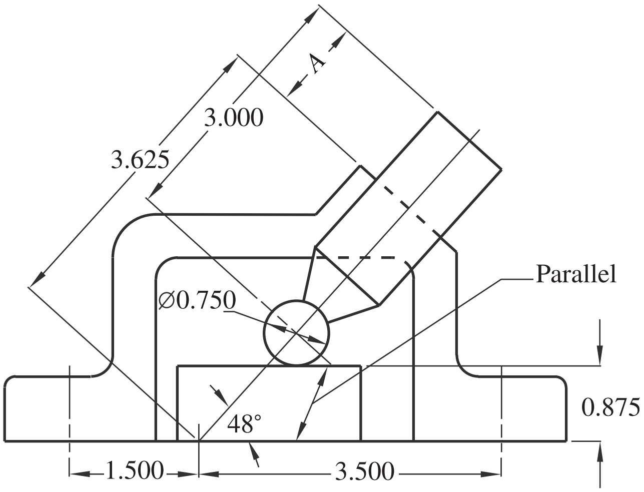


FIGURE 20.47

20.45 Solve for dimension *X* in Figure 20.48. Dimensions are in inches.

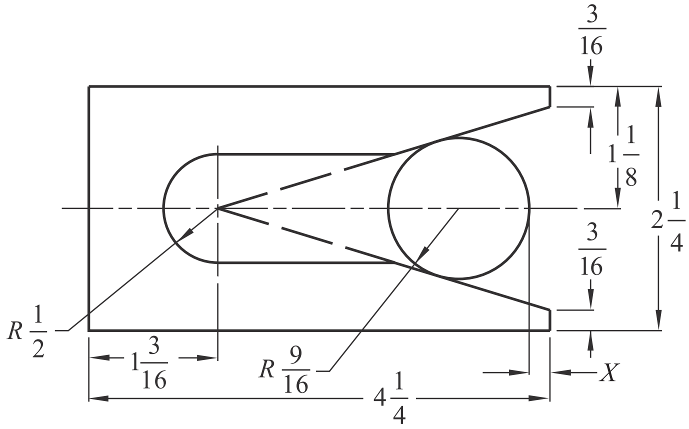


FIGURE 20.48

20.46 Find length *Z* in Figure 20.49. Dimensions are in inches.

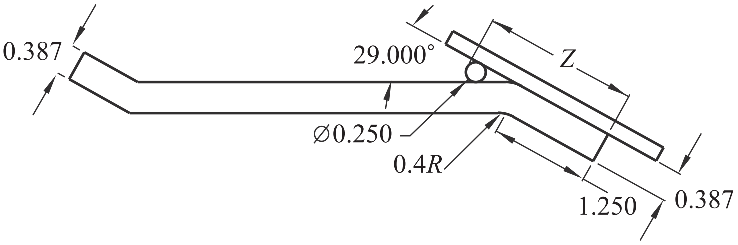


FIGURE 20.49

20.47 Solve for distance *A* in Figure 20.50. Dimensions are in centimeters.

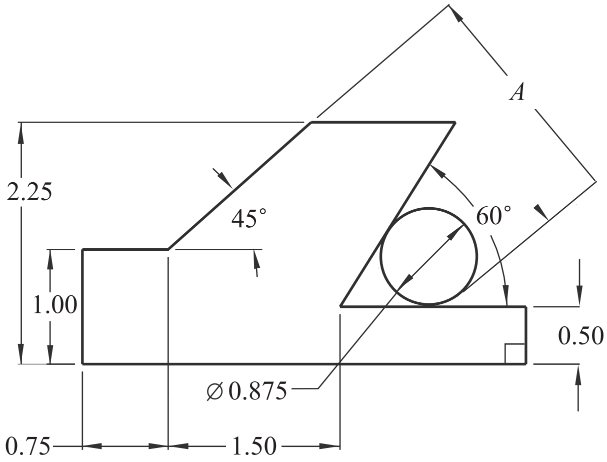


FIGURE 20.50

20.48 Solve for angle *A* and length *L* in Figure 20.51. Dimensions are in inches.

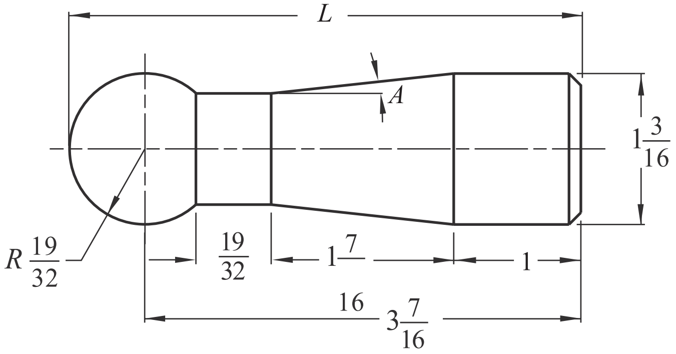


FIGURE 20.51

20.49 Solve for dimension *H* in Figure 20.52. Dimensions are in inches.

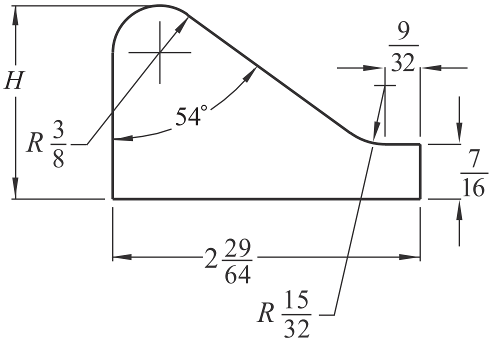


FIGURE 20.52

20.50 Solve for dimension *X* in Figure 20.53. Dimensions are in millimeters.

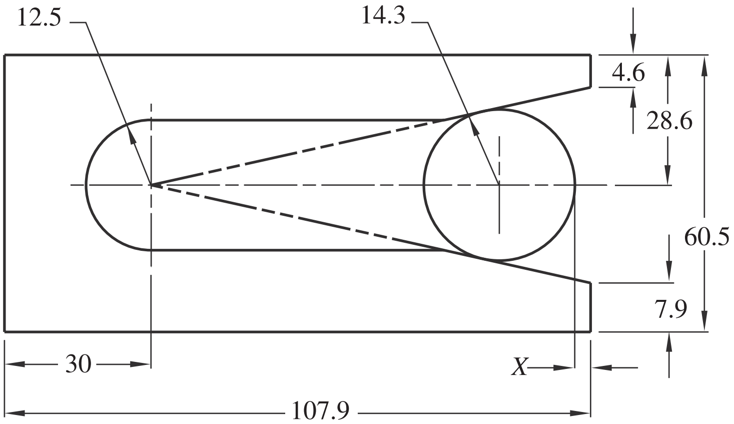


FIGURE 20.53

20.3 Special Triangles and the Unit Circle

30-60-90 and 45-45-90 Degree Triangles

The trigonometric function values for some common angles should be committed to memory. Consider the 30-60-90 and 45-45-90 degree triangles shown in Figures 20.54A and 20.54B.

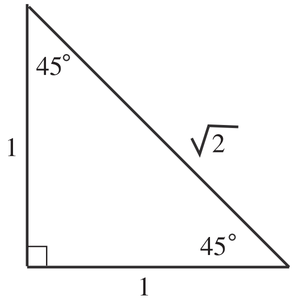


FIGURE 20.54A
45-45-90 degree triangle

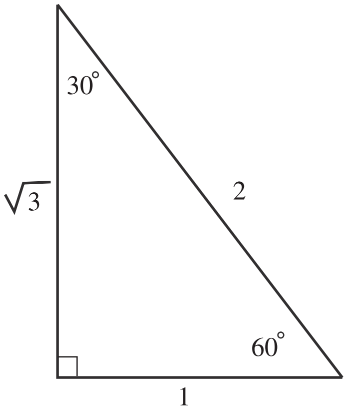


FIGURE 20.54B
30-60-90 ***degree triangle***

By looking at Figure 20.54A and Figure 20.54B we can readily find the values of the six trigonometric functions for 30, 60, and 45 degrees and generate the entries for Table 20.5.

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ *by multiplying the ratio like this:*

$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Likewise,

$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

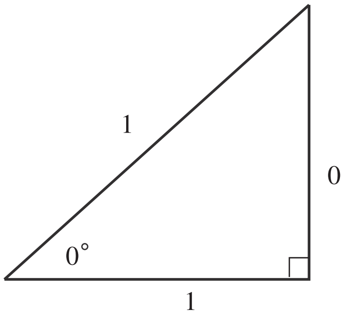
TABLE 20.5: Trigonometric Function Values for 30°, 45°, and 60°

Relationship		30°	45°	60°
sin	$\frac{\text{opp}}{\text{hyp}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\text{adj}}{\text{hyp}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{\text{opp}}{\text{adj}}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\frac{1}{1} = 1$	$\frac{\sqrt{3}}{1} = \sqrt{3}$
cot	$\frac{\text{adj}}{\text{opp}}$	$\frac{\sqrt{3}}{1} = \sqrt{3}$	$\frac{1}{1} = 1$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
sec	$\frac{\text{hyp}}{\text{adj}}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{1} = 2$
csc	$\frac{\text{hyp}}{\text{opp}}$	$\frac{2}{1} = 2$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

Trigonometric Function Values for 0 and 90 Degrees

The trigonometric function values for 0 and 90 degrees are more abstract. To visualize the trigonometric functions for these angles we draw imaginary right triangles in which one side has a length of 0. The triangle for 0 degrees is shown in Figure 20.55.

From Figure 20.55 we can calculate the trigonometric function values of 0 degrees as shown.



$$\sin 0° = \frac{0}{1} = 0 \qquad \cos 0° = \frac{1}{1} = 1$$

$$\tan 0° = \frac{0}{1} = 0 \qquad \cot 0° = \frac{1}{0} = \infty$$

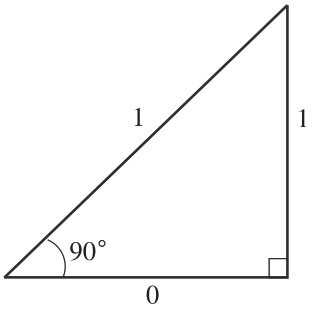
$$\sec 0° = \frac{1}{1} = 1 \qquad \csc 0° = \frac{1}{0} = \infty$$

FIGURE 20.55 ***Imaginary triangle for 0-degree trigonometric functions***

∞ *is the symbol for "undefined."*

The triangle shown in Figure 20.56 shows the imaginary triangle used to find the values for the trigonometric functions of 90 degrees.

From Figure 20.56 we can calculate the trigonometric function values of 90 degrees as shown.



$$\sin 90^\circ = \frac{1}{1} = 1 \qquad \cos 90^\circ = \frac{0}{1} = 0$$

$$\tan 90^\circ = \frac{1}{0} = \infty \qquad \cot 90^\circ = \frac{0}{1} = 0$$

$$\sec 0^\circ = \frac{1}{0} = \infty \qquad \csc 0^\circ = \frac{1}{1} = 1$$

FIGURE 20.56 ***Imaginary triangle for 90-degree trigonometric functions***

The Unit Circle

A circle with radius 1 drawn on Cartesian coordinates can be a useful way to visualize values for trigonometric functions of angles greater than 90 degrees. This circle is called a *unit circle* and is shown in Figure 20.57 with a right triangle in the first quadrant.

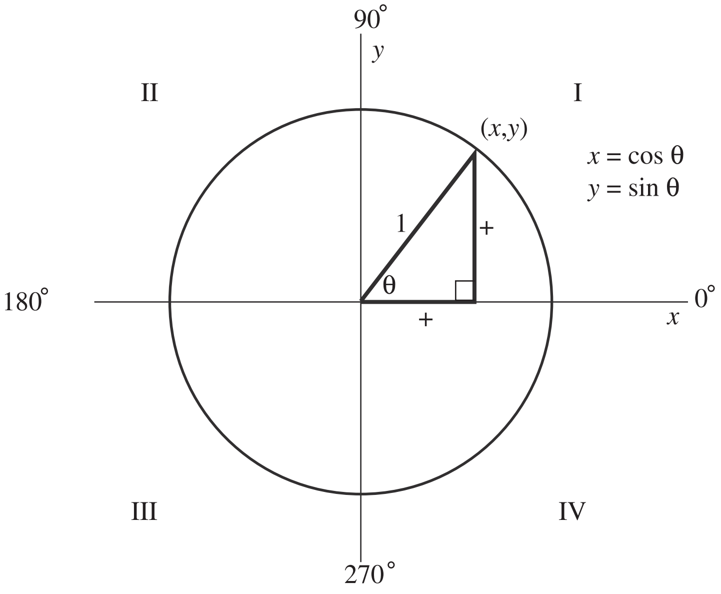


FIGURE 20.57 ***Unit circle with right triangle in Quadrant I***

Quadrant I

Using any point on the unit circle, a right triangle can be constructed by dropping a perpendicular from the point on the circle to the *x*-axis. The hypotenuse of any right triangle drawn this way will always be a radius of the circle and therefore equal to 1. Accordingly, the trigonometric functions of any resulting angle θ will simply be:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y \qquad \text{(always positive in Quadrant I)}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x \qquad \text{(always positive in Quadrant I)}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \qquad \text{(always positive in Quadrant I)}$$

Notice that in Quadrant I all of the *x* and *y* values are positive so all of the trigonometric functions yield positive results. We can also readily confirm the values for the trigonometric functions for 0 and 90 degrees:

$$\sin 0^\circ = \frac{0}{1} = 0 \qquad \cos 0^\circ = \frac{1}{1} = 1 \qquad \tan 0^\circ = \frac{0}{1} = 0$$

$$\sin 90^\circ = \frac{1}{1} = 1 \qquad \cos 90^\circ = \frac{0}{1} = 0 \qquad \tan 90^\circ = \frac{1}{0} = \infty$$

Quadrant II

Angles between 90 and 180 degrees fall in Quadrant II, where all of the *x* values are negative and all the *y* values are positive, as shown in Figure 20.58.

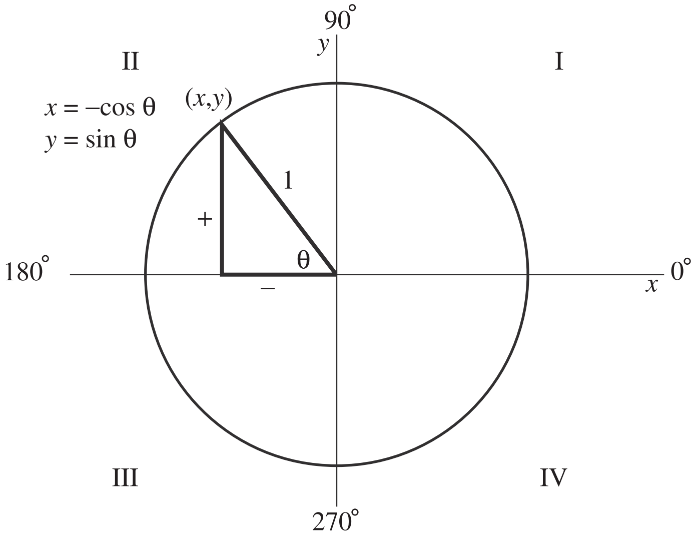


FIGURE 20.58 ***Unit circle with right triangle in Quadrant II***

Inspection of Figure 20.58 reveals that for angles between 90 and 180 degrees:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y \qquad \text{(always positive in Quadrant II)}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{-x}{1} = -x \qquad \text{(always negative in Quadrant II)}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{-x} \qquad \text{(always negative in Quadrant II)}$$

For the special case of 180 degrees, Figure 20.58 shows that

$$\sin 180° = \frac{0}{1} = 0 \qquad \cos 180° = \frac{-1}{1} = -1 \qquad \tan 180° = \frac{0}{1} = 0$$

where opposite side is 0, while adjacent and hypotenuse are 1.

Quadrant III

Angles between 180 and 270 degrees fall in Quadrant III where all of the *x* and *y* values are negative as shown in Figure 20.59.

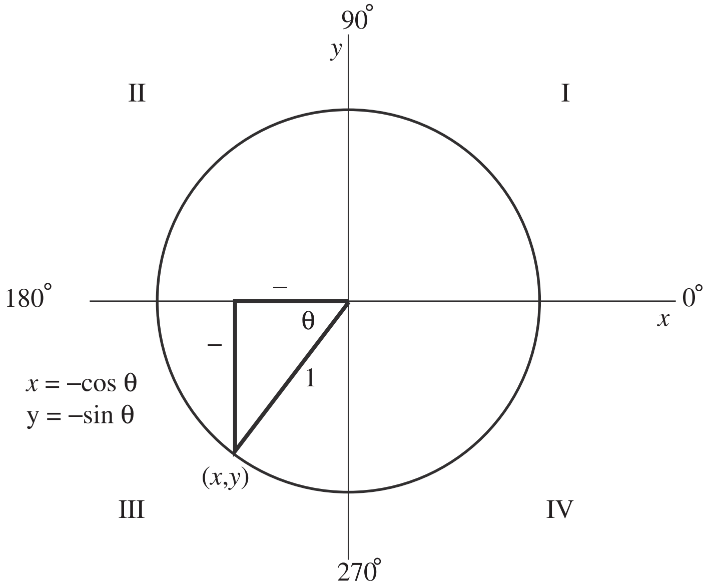


FIGURE 20.59 ***Unit circle with right triangle in Quadrant III***

Inspection of Figure 20.59 reveals that for angles between 180 and 270 degrees:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{-y}{1} = -y \qquad \text{(always negative in Quadrant III)}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{-x}{1} = -x \qquad \text{(always negative in Quadrant III)}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{-y}{-x} \qquad \text{(always positive in Quadrant III)}$$

For the special case of 270 degrees, Figure 20.59 shows that:

$$\sin 270° = \frac{-1}{1} = -1 \qquad \cos 270° = \frac{0}{1} = 0 \qquad \tan 270° = \frac{-1}{0} = \infty$$

Quadrant IV

Angles between 270 and 360 degrees fall in Quadrant IV where all of the x values are positive and all the y values are negative as shown in Figure 20.60.

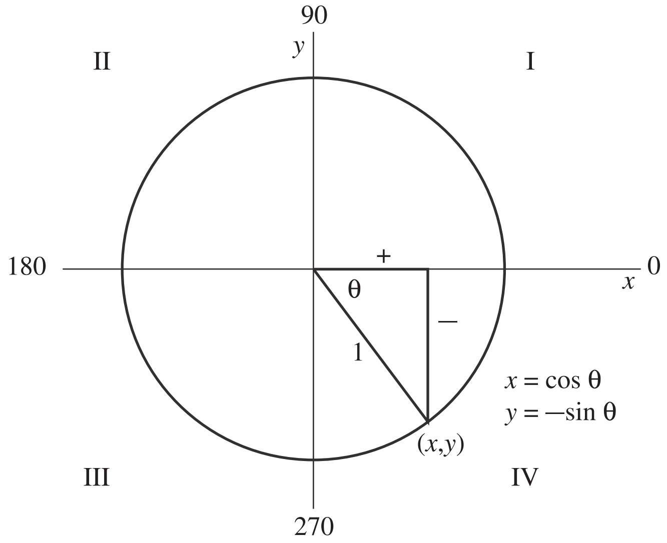


Figure 20.60 ***Unit circle with right triangle in Quadrant IV***

Inspection of Figure 20.60 reveals that for angles between 270 and 360 degrees:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{-y}{1} = -y \qquad \text{(always negative in Quadrant IV)}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x \qquad \text{(always positive in Quadrant IV)}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{-y}{x} \qquad \text{(always negative in Quadrant IV)}$$

The special case of 360 degrees is the same as for 0 degrees.

EXERCISES

20.51 Complete the table by stating for each quadrant whether the indicated trigonometric function is positive or negative.

Quadrant

	I	**II**	**III**	**IV**
sin θ				
cos θ				
tan θ				
csc θ				
sec θ				
cot θ				

20.52 Use a scientific calculator to find the value of each trigonometric function.

a) sin 175°
b) tan 98°
c) tan 301°
d) cot 132°
e) cos 206°
f) csc 323°
g) sin 35°
h) cot 342°
i) sec 351°
j) sin 259°
k) csc 320°
l) sin 355°

20.4 Graphing the Trigonometric Functions

Table 20.6 summarizes the values for the sine, cosine, and tangent functions of common angles. The triangles that yield these values are seen in Figures 20.54A, 20.54B, 20.55, and 20.56. Notice that the angles are given in degree and radian form. Radian measure is discussed in Chapter 7.

The values are expressed as fractions with rationalized denominators. This table is useful to commit to memory.

When using a scientific calculator to find values for trigonometric functions, make sure the calculator is set to DEGREE mode for angles in degrees or RADIAN mode for angles in radians.

TABLE 20.6: **Values for Trigonometric Functions of Common Angles**

Quadrant	**I**					**II**			
θ (degrees)	**0°**	**30°**	**45°**	**60°**	**90°**	**120°**	**135°**	**150°**	**180°**
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

Quadrant	**III**				**IV**			
θ (degrees)	**210°**	**225°**	**240°**	**270°**	**300°**	**315°**	**330°**	**360°**
θ (radians)	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin θ	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
cos θ	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan θ	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

Now that we have established how to find values for trigonometric functions of any angle, we can generate graphs for trigonometric functions.

Sine Function Graph

The graph for the sine function between 0 and 360 degrees is given in Figure 20.61. The data for the graph are from Table 20.6

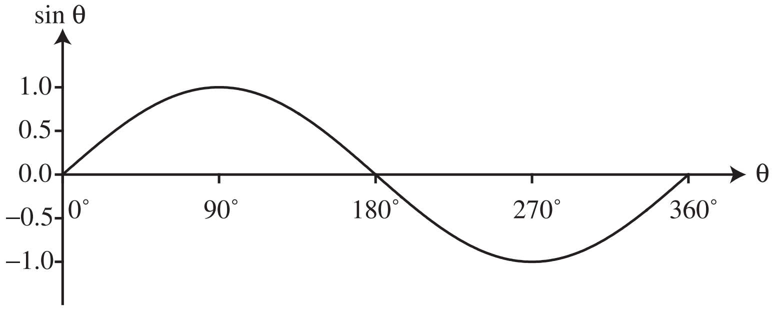


FIGURE 20.61 ***Graph of sine function between 0 and 360 degrees***

The *amplitude* of the sine graph varies between –1 and +1. The sine function is *periodic*, repeating every 360 degrees.

Cosine Function Graph

The graph for the cosine function between 0 and 360 degrees is given in Figure 20.62. The data for the graph are from Table 20.6.

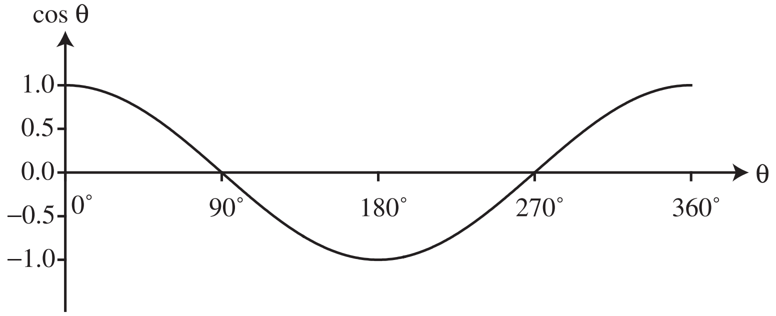


FIGURE 20.62 ***Graph of cosine function between 0 and 360 degrees***

As with the sine graph, the amplitude of the cosine graph varies between –1 and +1. The cosine function is also periodic and repeats every 360 degrees with a 90-degree offset to the left, or phase shift, from the sine graph.

Tangent Function Graph

The graph for the tangent function between 0 and 360 degrees is given in Figure 20.63. The data for the graph are from Table 20.6.

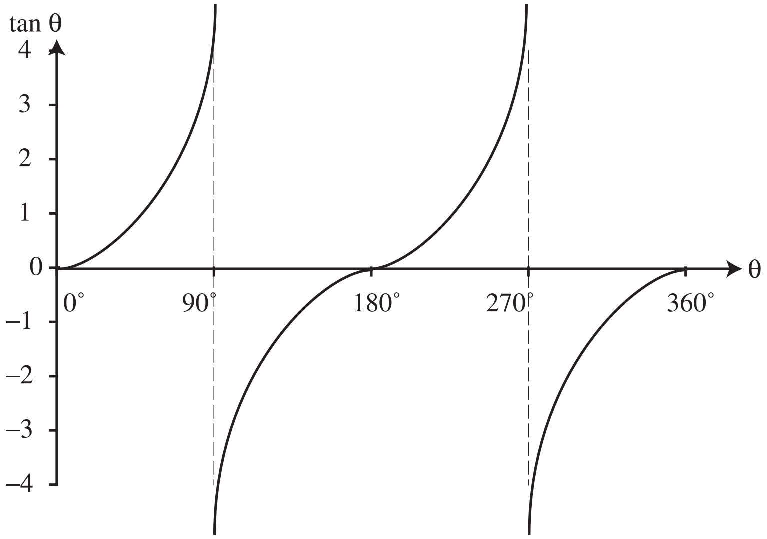


FIGURE 20.63 ***Graph of tangent function between 0 and 360 degrees***

The tangent function approaches positive infinity for angles less than, but very close to, 90 degrees and 270 degrees; it approaches negative infinity for angles greater than, but very close to, 90 and 270 degrees. At 90 and 270 degrees, the function is *undefined*. The dashed lines through 90 and 270 are called *asymptotes*.

Effect of Multiplying a Trigonometric Function by a Constant

Multiplying a trigonometric function by a constant changes its amplitude, as illustrated in Example 20.7.

EXAMPLE 20.7: ***Graph of a Trigonometric Function Multiplied by a Constant***

Construct the graph for $2\sin\theta$ between 0 and 360 degrees.

Solution:

STEP 1: First, build a table of values selecting points every 90 degrees.

θ **(degrees)**	$2\sin\theta$
0	0
90	2
180	0
270	–2
360	0

STEP 2: Use the data in the table to draw the graph. The graph of sin θ is shown in blue for reference.

EXAMPLE 20.7: *Graph of a Trigonometric Function Multiplied by a Constant (Continued)*

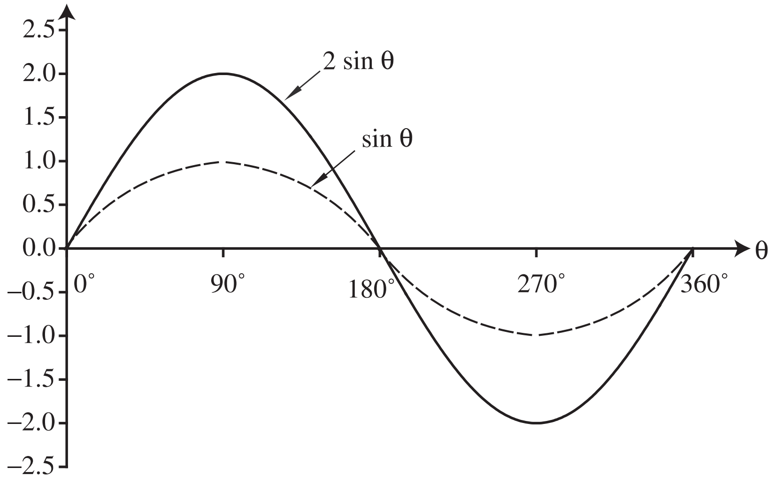


FIGURE 20.64

The period of a trigonometric function can be changed by multiplying the angle by a constant, as illustrated in Example 20.8.

EXAMPLE 20.8: *Changing the Period of a Trigonometric Function*

Construct the graph for $\sin 2\theta$ between 0 and 360 degrees.

Solution:

STEP 1: First, build a table of values selecting points every 45 degrees.

θ (degrees)	$\sin 2\theta$
0	0
45	1
90	0
135	–1
180	0
225	1
270	0
315	–1
360	0

STEP 2: Use the data in the table to draw the graph. The graph of $\sin \theta$ is shown in blue for reference.

Example 20.8: *Changing the Period of a Trigonometric Function (Continued)*

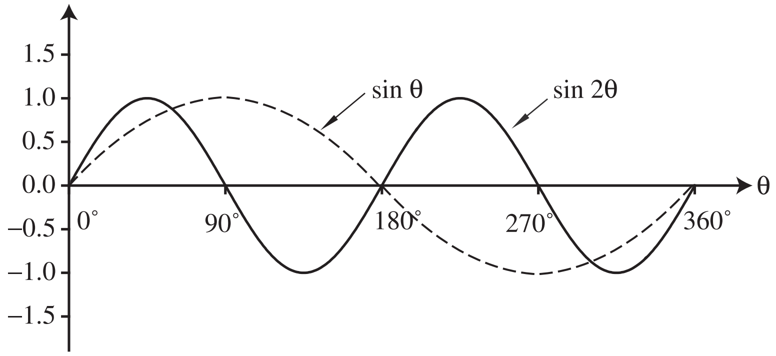


Figure 20.65

Here we have shown how the amplitude and period of the sine function can be changed by using a constant. The amplitudes and periods of the other trigonometric functions can be changed in the same manner.

This chapter has discussed right angle trigonometry. The next chapter generalizes the concepts of trigonometry to oblique (non-right) triangles.

EXERCISES

20.53 Graph the indicated sine functions between 0 and 360 degrees.

a) $3\sin\theta$ **b)** $1.5\sin\theta$ **c)** $2\sin 2\theta$ **d)** $2.5\sin 3\theta$ **e)** $5\sin 2\theta$

20.54 Graph the indicated cosine functions between 0 and 360 degrees.

a) $3\cos\theta$ **b)** $1.5\cos\theta$ **c)** $2\cos 2\theta$ **d)** $2.5\cos 3\theta$ **e)** $5\cos 2\theta$

20.55 Graph the indicated tangent functions between 0 and 360 degrees.

a) $3\tan\theta$ **b)** $1.5\tan\theta$ **c)** $2\tan 2\theta$ **d)** $2.5\tan 3\theta$ **e)** $5\tan 2\theta$

20.56 Graph the indicated trigonometric functions between 0 and 360 degrees.

a) $\sin\theta + \cos\theta$ **b)** $2\sin\theta + \cos\theta$ **c)** $\sin 2\theta + \cos 2\theta$

21

OBLIQUE ANGLE TRIGONOMETRY

Thus far our study of trigonometry has been limited to right triangles. Application of trigonometry to *oblique triangles* is studied in this chapter.

Oblique triangles are often encountered in shop problems. Distances between holes, lengths of sides, and angles between centerlines are usually indicated in the layout of machine parts. The oblique triangles formed by sides and centerlines may be solved by construction of right triangles on the obliques or by the use of special formulas given in this chapter.

Oblique triangles are triangles in which none of the angles are right angles. *Acute oblique triangles* have all angles less than 90 degrees. *Obtuse oblique triangles* have one angle greater than 90 degrees.

21.1 Solving Oblique Triangles Using Right Triangles

One way to solve a problem involving an oblique triangle is by reconfiguring the problem into one involving right triangles. Auxiliary lines are added to construct the necessary right triangles by dropping perpendiculars from angle vertices to their opposite sides. This construction is shown in Figure 21.1 for an acute oblique triangle and in Figure 21.2 for an obtuse oblique triangle. The resulting right triangles may then be solved by applying the rules for right angle trigonometry.

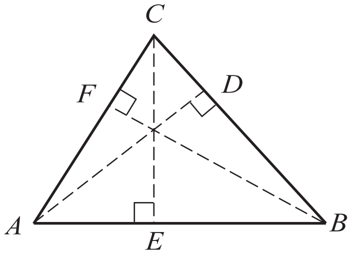


FIGURE 21.1 ***Right triangle construction in acute oblique triangles***

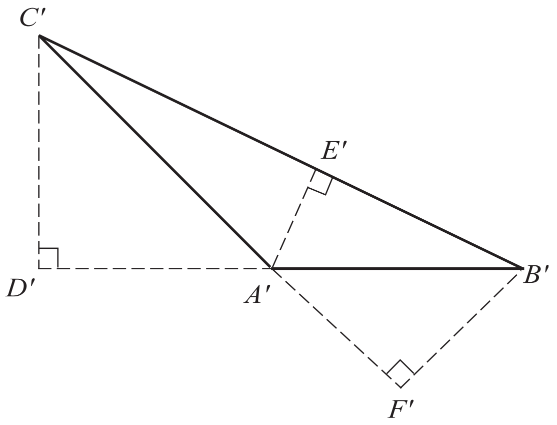


FIGURE 21.2 ***Right triangle construction in obtuse oblique triangles***

In the case of acute oblique triangles typified in Figure 21.1, the altitudes all fall within triangle *ABC* and six right triangles are formed: *ADC* and *ADB*; *BFA* and *BFC*; *CEA* and *CEB*. For obtuse oblique triangles such as shown in Figure 21.2, two of the altitudes, *C′D′* and *B′F′*, fall outside triangle *A′B′C′*, and one altitude, *A′E′*, falls within the triangle. Here, too, six right triangles are formed; *A′F′B′* and *A′D′C′*, *B′D′C′* and *B′E′A′*, *C′E′A′* and *C′F′B.′*

The choice of which right triangle to use in a solution depends on which angles and sides are known. Example 21.1 demonstrates a solution for an acute oblique triangle and Example 21.2 demonstrates a solution for an obtuse oblique triangle.

EXAMPLE 21.1: *Solving an Acute Oblique Triangle by Constructing Right Triangles*

The acute oblique triangle *ABC* shown in Figure 21.3, has $AB = 22.50$ mm, $AC = 19.80$ mm and $m\angle C = 55$. Solve for $m\angle A$, $m\angle B$ and *BC*.

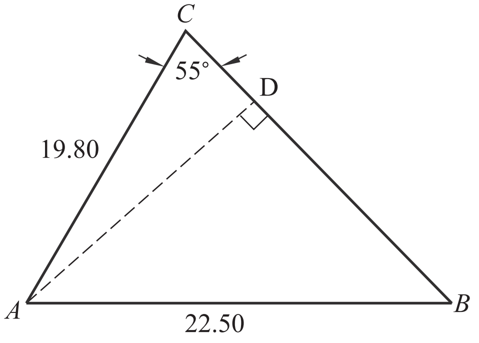


FIGURE 21.3

Solution:

STEP 1: Draw altitude *AD* from angle *A* to side *BC* forming the right triangle *ADC* where angle *C* is known. Solve for side *AD*.

EXAMPLE 21.1: ***Solving an Acute Oblique Triangle by Constructing Right Triangles (Continued)***

$$\sin C = \frac{AD}{AC}$$
$$\begin{aligned} AD &= AC \sin 55° \\ &= (19.80)(0.81915) \\ &= 16.22 \text{ mm} \end{aligned}$$

STEP 2: In right triangle *ADB*, solve for $m\angle B$:

$$\sin B = \frac{AD}{AB} = \frac{16.22}{22.50}$$
$$\sin B = 0.7209$$
$$B = \sin^{-1} 0.7209 = 46.13°$$

STEP 3: Solve for $m\angle A$:

$$m\angle A + m\angle B + m\angle C = 180$$
$$\begin{aligned} m\angle A &= 180 - (m\angle B + m\angle C) \\ &= 180 - (46.13 + 55) \\ &= 180 - 101.13 \\ &= 78.87 \end{aligned}$$

STEP 4: Solve for *DC* in right triangle *ADC*:

$$\cos C = \frac{DC}{AC}$$
$$\begin{aligned} DC &= AC \cos 55° \\ &= (19.8)(0.57358) \\ &= 11.36 \text{ mm} \end{aligned}$$

STEP 5: Solve for *BD* in right triangle *ADB*:

$$\cos B = \frac{BD}{AB}$$
$$\begin{aligned} BD &= AB \cos B = AB \cos 46.13° \\ &= (22.5)(0.6931) \\ &= 15.59 \text{ mm} \end{aligned}$$

STEP 6: Solve for *BC*:

$$\begin{aligned} BC &= BD + DC \\ &= 15.59 + 11.36 \\ &= 26.95 \text{ mm} \end{aligned}$$

EXAMPLE 21.2: ***Solving an Obtuse Oblique Triangle by Constructing Right Triangles***

The obtuse oblique triangle *EFG* shown in Figure 21.4, has *EF* = 38.00 inches, *EG* = 44.50 inches, and $m\angle G = 23$. Solve for $\angle E$, $\angle F$, and *FG*.

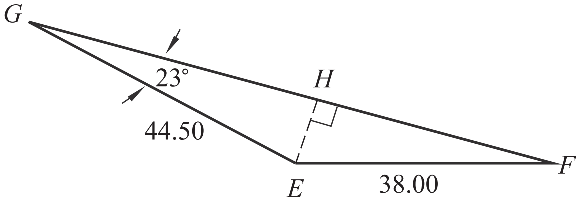


FIGURE 21.4

Solution:

STEP 1: Draw altitude *EH* from angle *E* to side *FG*, forming the right triangle *EHG* where angle *G* is known. Solve for *EH*:

$$\sin\ G = \frac{EH}{EG}$$
$$\begin{aligned} EH &= EG \sin G \\ &= (44.50)(\sin 23°) \\ &= (44.50)(0.3907) \\ &= 17.39 \text{ inches} \end{aligned}$$

STEP 2: In right triangle *EHF*, solve for $\angle F$:

$$\sin F = \frac{EH}{EF}$$
$$\sin F = \frac{17.39}{38.00} = 0.4576$$
$$F = \sin^{-1} 0.4576 = 27.23°$$

STEP 3: In right triangle *EHG*, solve for *GH*:

$$\cos G = \frac{GH}{EG}$$
$$\begin{aligned} GH &= EG \cos G \\ &= (44.50)(\cos 23°) \\ &= (44.50)(0.9205) \\ &= 40.96 \text{ inches} \end{aligned}$$

STEP 4: In right triangle *EHF*, solve for *HF*:

$$\cos F = \frac{HF}{EF}$$

EXAMPLE 21.2: ***Solving an Obtuse Oblique Triangle by Constructing Right Triangles (Continued)***

$$\begin{aligned} HF &= EF\ \cos F \\ &= (38.00)(\cos 27.23°) \\ &= (38.00)(0.8891) \\ &= 33.79 \text{ inches} \end{aligned}$$

STEP 5: Solve for *FG*:

$$\begin{aligned} FG &= HF + GH \\ &= 33.79 + 40.96 \\ &= 74.75 \text{ inches} \end{aligned}$$

STEP 6: Solve for $\angle E$:

$$\begin{aligned} &m\angle E + m\angle F + m\angle G = 180 \\ &m\angle E = 180 - m\angle F - m\angle G \\ &m\angle E = 180 - 27.23 - 23 \\ &\angle E = 129.7° \end{aligned}$$

EXERCISES

21.1 The dimensions in Figure 21.5 are in centimeters. Add auxiliary lines and find using right triangles:

a) $m\angle A$ **b)** AC

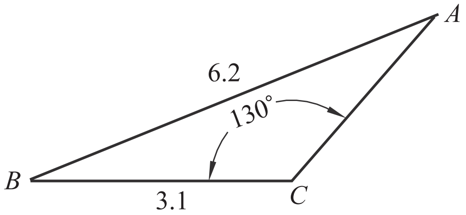


FIGURE 21.5

21.2 The dimensions in Figure 21.6 are in millimeters. Add auxiliary lines and find using right triangles:

a) the length of side *g* **b)** the length of side *e*

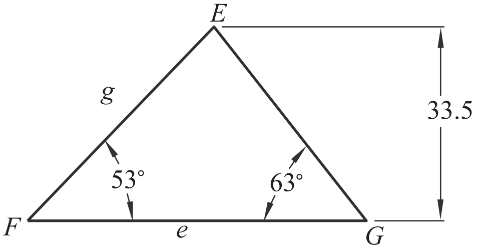


FIGURE 21.6

21.3 The dimensions in Figure 21.7 are in inches. Add auxiliary lines and find using right triangles:

a) the length of the altitude from point *C*

b) *AB*

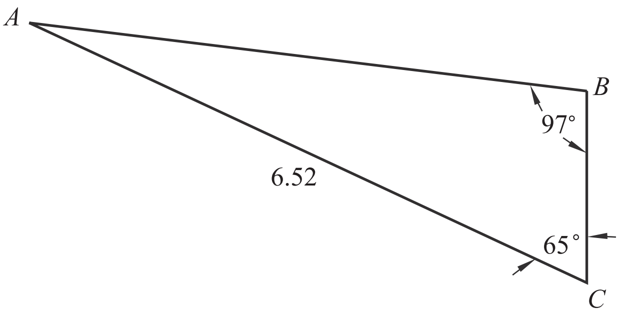


FIGURE 21.7

21.4 Determine the length of radius *R* in Figure 21.8. Dimensions are in inches.

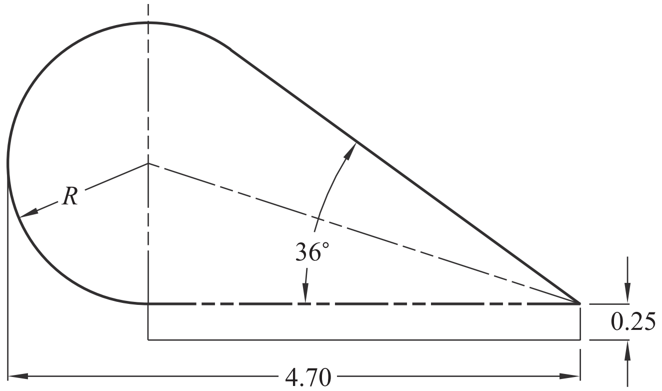


FIGURE 21.8

21.5 Determine the length of radius *R* in Figure 21.9. Dimensions are in inches.

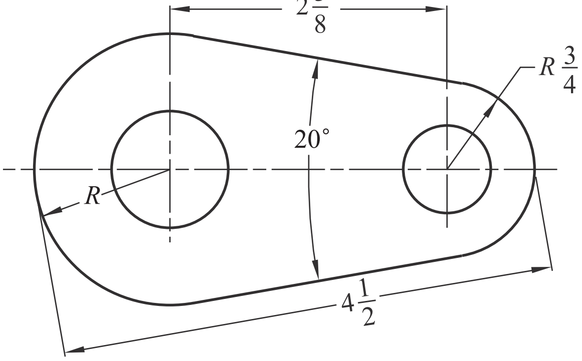


FIGURE 21.9

21.6 The dimensions in Figure 21.10 are in meters. Determine:

a) *AC* **b)** *BC*

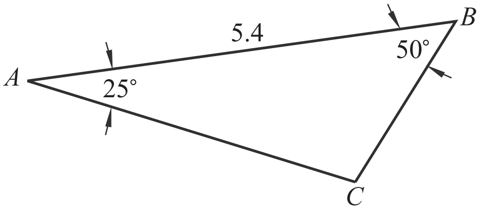


FIGURE 21.10

21.7 The dimensions in Figure 21.11 are in inches. Find dimension *X*.

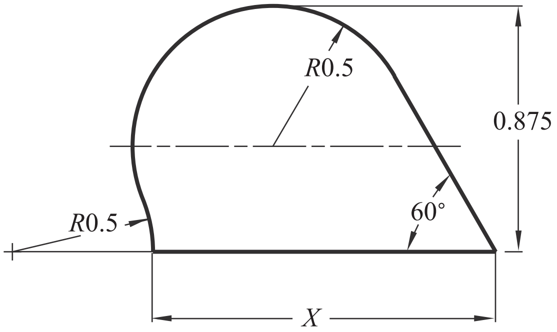


FIGURE 21.11

21.2 Special Laws of Trigonometry

While most trigonometry problems can be solved by constructing right triangles, some special trigonometry laws can also be applied to directly solve oblique triangles when certain data are known. Three of these special laws for oblique triangles are the *law of sines*, the *law of cosines*, and the *cotangent law*. In many applications, much calculation can be avoided by using these laws, and the solutions to the problems are often quicker and easier to handle.

The Law of Sines

If any triangle—right, acute, or obtuse—is examined, it is readily apparent that the longest side always lies opposite the largest angle and the shortest side always lies opposite the smallest angle. This relationship leads to the law of sines, which states that in any triangle, the ratio of the sine of an angle to its opposite side is equal to the same ratio for another angle and its opposite side.

Consider Figure 21.12 in which acute oblique triangle *ABC* is shown. If altitude *h* is drawn from angle *B* to the opposite side *b*, then by definition of the sine function:

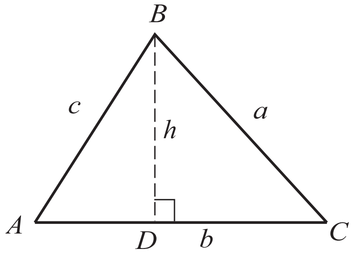


FIGURE 21.12 ***Acute oblique triangle ABC***

$$\sin A = \frac{h}{c} \quad \text{and} \quad \sin C = \frac{h}{a}$$

Therefore, $h = c \sin A$, and $h = a \sin C$.

If we equate these two expressions for h, we find:

$$c \sin A = a \sin C \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

In similar fashion we find:

$$\frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{and} \quad \frac{\sin A}{a} = \frac{\sin B}{b}$$

and we arrive at the law of sines.

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

These relationships may also be inverted; the three possible equations are:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{a}{\sin A} = \frac{c}{\sin C} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

The law of sines is used in solving a triangle where three of four terms in any one equation are known. If an oblique triangle, either acute or obtuse, has two known sides and one angle opposite one of the known sides, or two known angles and a side opposite one of the known angles, the law of sines may be used.

EXAMPLE 21.3: ***Law of Sines***

In triangle ABC shown in Figure 21.13, the following information is given: $c = 12.56$ inches, $a = 10.48$ inches, and angle A is 22.55°. Solve for angle C and side b using the law of sines.

Solution:

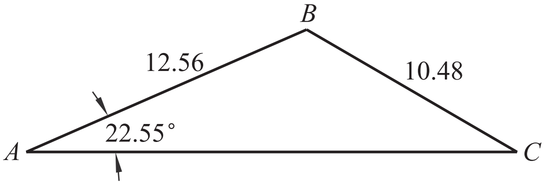


FIGURE 21.13

EXAMPLE 21.3: ***Law of Sines (Continued)***

STEP 1: Observe that $m\angle A$ and its opposite side a are known.

STEP 2: Set up the proportion of the law of sines with the unknown in the numerator of the first ratio and solve for $\angle C$:

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{12.56} = \frac{\sin 22.55°}{10.48}$$

$$\sin C = \frac{(12.56)(\sin 22.55°)}{10.48}$$

$$= \frac{(12.56)(0.3835)}{10.48} = 0.4596$$

$$C = \sin^{-1} 0.4596 = 27.36°$$

STEP 3: In the same way, solve for the length of side b noting that $m\angle B = 180 - 22.55 - 27.36 = 130.09$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$b = \frac{a \sin B}{\sin A}$$

$$= \frac{(10.48)(\sin 130.09°)}{\sin 22.55°}$$

$$= \frac{(10.48)(0.7651)}{0.3835} = 20.91 \text{ inches}$$

The Law of Cosines

The *law of cosines* is useful in calculating the length of an unknown side of an oblique triangle when the length of the other two sides and the size of their included angle is known.

The law of cosines is sometimes called the *generalized Pythagorean theorem*.

Consider the acute oblique triangle shown in Figure 21.14A and the obtuse oblique triangle shown in Figure 21.14B. In both triangles, the following relationships are true:

$$a^2 = h^2 + (BD)^2$$ ***Equation 1***

$$b^2 = h^2 + (AD)^2$$ ***Equation 2***

For the acute oblique triangle, $BD = c - AD$.

For the obtuse oblique triangle, $BD = AD - c$.

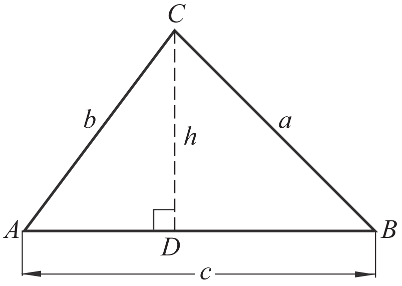


FIGURE 21.14A ***An acute oblique triangle***

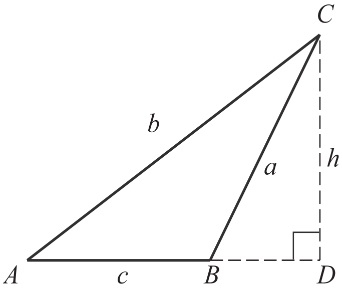


FIGURE 21.14B ***An obtuse oblique triangle***

Substituting either of the two expressions for *BD* into Equation 1 yields:

$$a^2 = h^2 + (c - AD)^2 \quad \text{or} \quad a^2 = h^2 + (AD - c)^2$$

In either case, a^2 works out to:

$$a^2 = h^2 + c^2 - 2c(AD) + (AD)^2$$

Transposing Equation 2 yields:

$$h^2 = b^2 - (AD)^2$$

Substituting this expression for h^2 into the expression for a^2 and rearranging:

$$a^2 = b^2 + c^2 - 2c(AD) + (AD)^2 - (AD)^2$$

or

$$a^2 = b^2 + c^2 - 2c(AD)$$

Examining the figures for either the acute oblique or the obtuse oblique triangles, reveals that $\cos A = \dfrac{AD}{b}$, which means that $AD = b \cos A$. Finally, substituting this expression for *AD*, we can write

$a^2 = b^2 + c^2 - 2bc \cos A$, which is the *law of cosines.*

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Notice that if one of the angles is a right angle, the law of cosines becomes the Pythagorean theorem since cos 90° = 0.

The same development works for sides *b* and *c*, hence:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = c^2 + a^2 - 2ca \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Application of the law of cosines requires careful attention to labeling. Notice that the side whose length appears on the left side of the equation is opposite the angle whose cosine is used on the right side of the equation.

EXAMPLE 21.4A: ***Law of Cosines***

If in the acute oblique triangle shown in Figure 21.14A, $a = 91$ mm, $b = 82.5$ mm, and $\angle C$ is 82.24°. Solve for side c using the law of cosines.

Solution:

STEP 1: State the law with c^2 being the unknown:

$$c^2 = a^2 + b^2 - 2ab \cos C$$

STEP 2: Substitute values given in problem and solve for c:

$$\begin{aligned} c^2 &= (91)^2 + (82.5)^2 - (2)(91)(82.5)(\cos 82.24°) \\ &= 8281 + 6806.25 - (15{,}015)(0.1350) \\ &= (15{,}087.25 - 2{,}027.03) \\ &= 13{,}060.22 \end{aligned}$$

Taking the positive square root gives: $c = 114.28$ mm.

EXAMPLE 21.4B: ***Law of Cosines***

If in the acute oblique triangle shown in Figure 21.14A, $a = 91$ mm, $b = 82.5$ mm, $c = 114.28$ mm and $\angle C$ is 82.24° solve for $\angle A$ using the law of cosines.

Solution:

STEP 1: State the law with cos A as the unknown:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Transposing gives:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

STEP 2: Substitute known values and solve:

$$\begin{aligned} \cos A &= \frac{(82.5)^2 + (114.275)^2 - (91)^2}{2(82.5)(114.28)} \\ &= \frac{(6806.25 + 13{,}058.78 - 8281)}{18{,}856.2} \\ &= \frac{11{,}584.0}{18{,}856.2} = 0.6143 \\ A &= \cos^{-1}(0.6143) = 52.1° \end{aligned}$$

EXAMPLE 21.4C: ***Law of Cosines***

Solve for $\angle B$ in the obtuse oblique triangle shown in Figure 21.14B when $a = 7.85$ inches, $b = 11.00$ inches, and $c = 5.15$ inches.

Solution:

STEP 1: State the law of cosines with cos B as the unknown:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

EXAMPLE 21.4C: ***Law of Cosines (Continued)***

STEP 2: Substitute known values for a, b, and c:

$$\cos B = \frac{(7.85)^2 + (5.15)^2 - (11.00)^2}{2(7.85)(5.15)}$$

$$= \frac{61.6225 + 26.5225 - 121}{80.855}$$

$$= \frac{-32.855}{80.855}$$

$$= -0.4063$$

$$B = \cos^{-1}(-0.4063) = 113.98°$$

Watch for positive and negative signs.

The Cotangent Law

An oblique triangle with two angles and the included side given may be solved directly for the length of the perpendicular from the vertex of the unknown angle to the known side by use of the *cotangent law.*

To develop the formula for the cotangent law, consider acute oblique triangle *ABC* shown in Figure 21.15 and presume that side *b* and the adjacent angles *A* and *C* are given. A similar triangle, *A′B′C′* also shown in Figure 21.15, is constructed with altitude *B′D′* equal to one.

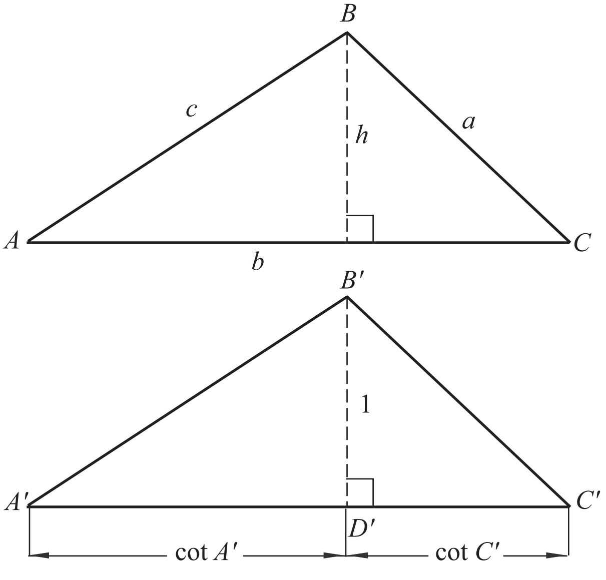


FIGURE 21.15 ***Cotangent for an acute oblique triangle***

In triangle $A'B'D'$, observe that $\cot A' = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{A'D'}{1}$ or $\cot A' = A'D'$

Likewise $\cot C' = \frac{C'D'}{1}$ or $\cot C' = C'D'$,

However, $A'C' = A'D' + D'C'$.

Substituting the values for $A'D'$ and $D'C'$ in the equation above:

$$A'C' = \cot A' + \cot C'$$

Since triangle $A'B'C'$ is similar to triangle ABC by construction, we know that $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, and $\angle C' \cong \angle C$. Therefore, $\cot A' + \cot C' = \cot A + \cot C$ and $A'C' = \cot A + \cot C$.

Also, since ABC and $A'B'C'$ are similar triangles, by construction:

$$\frac{h}{B'D'} = \frac{AC}{A'C'}$$

However, $B'D' = 1$, $AC = b$, and $A'C' = \cot A' + \cot C'$.

Therefore, by substitution: $\dfrac{h}{1} = \dfrac{AC}{A'C'} = \dfrac{b}{A'D' + D'C'} = \dfrac{b}{\cot A' + \cot C'}$

or, $h = \dfrac{b}{\cot A + \cot C}$, which is the cotangent law for acute oblique triangles.

$$h = \frac{b}{\cot A + \cot C}$$

Cotangent Law for Acute Oblique Triangle

In the case of an obtuse oblique triangle, where the desired altitude lies *outside* the triangle as in Figure 21.16, the cotangent formula becomes:

$$h = \frac{b}{\cot A - \cot(180° - C)}$$

Cotangent Law for Obtuse Oblique Triangle

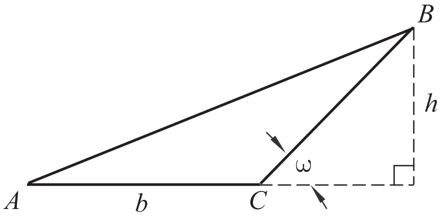


FIGURE 21.16 ***Cotangent law for an obtuse oblique triangle***

EXAMPLE 21.5A: *Cotangent Law*

In Figure 21.17, solve for altitude h given that $b = 1225$ mm, $\angle A$ is 70°, and $\angle C$ is 35°.

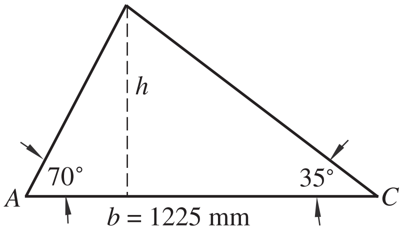


FIGURE 21.17

EXAMPLE 21.5A: ***Cotangent Law (Continued)***

Solution:

Substitute known values into the acute oblique cotangent law formula:

$$\begin{aligned} h &= \frac{b}{\cot A + \cot C} \\ &= \frac{1225}{\cot 70° + \cot 35°} \\ &= \frac{1225}{0.3640 + 1.4281} \\ &= \frac{1225}{1.7921} \\ h &= 683.6 \text{ mm} \end{aligned}$$

EXAMPLE 21.5B: ***Cotangent Law***

In Figure 21.18, solve for the altitude h given that $b = 4.62$ inches, $\angle A$ is 45°, and $\angle C$ is 110°.

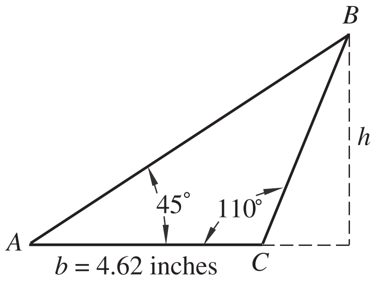


FIGURE 21.18

Solution: Substitute known values into the obtuse oblique cotangent law formula:

$$\begin{aligned} h &= \frac{b}{\cot A - \cot (180° - C)} \\ &= \frac{4.62}{\cot 45° - \cot (180° - 110°)} \\ &= \frac{4.62}{\cot 45° - \cot 70°} \\ &= \frac{4.62}{1.000 - 0.3640} \\ &= \frac{4.62}{0.6360} \\ h &= 7.26 \text{ inches} \end{aligned}$$

EXERCISES

Exercises 21.8–21.15 can be solved using the law of sines.

21.8 An oblique triangle has two sides and two angles known: angle A is 36.42°, angle B is 42.60°, side b is 101.77 mm, and side c is 147.60 mm. Determine angle C and the length of side a.

21.9 In an oblique triangle where side a is 15.75 inches, side b is 21 inches, and angle A is 43.25°, determine angles B and C and the length of side c.

21.10 A vee-shaped cut is made in a steel plate, as shown in Figure 21.19. The angle of the vee is 23.25° and the width of the vee on the surface of the plate is 0.478 inch. The sides of the vee are of equal length. What is their length?

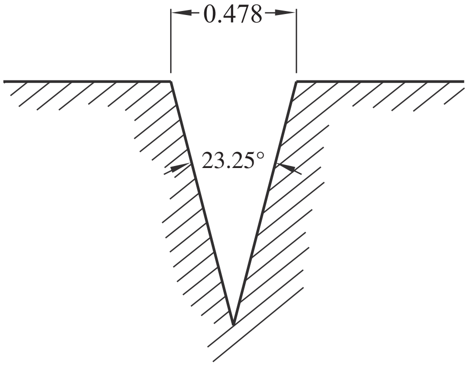


FIGURE 21.19

21.11 An oblique triangle has two angles and a side opposite one of the angles known. If angle A is 55°, angle B is 51°, and side b is 5.64 inches, what is the length of side c?

21.12 The extrusion die shown in Figure 21.20 has 12 points symmetrically placed around a base circle of 2.75 inches. Find the dimension D.

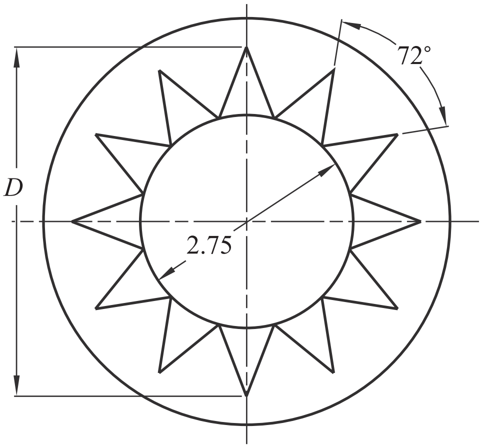


FIGURE 21.20

21.13 Compute angle *A* for the internal cam shown in Figure 21.21. Dimensions are in inches.

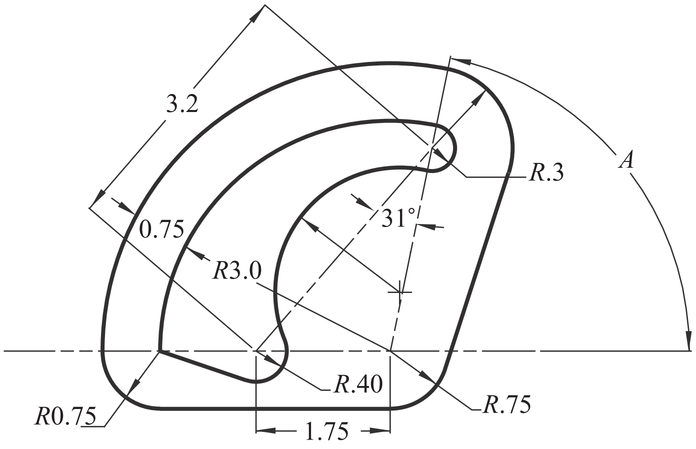


FIGURE 21.21

21.14 Three gears mesh as shown in Figure 21.22. Find the distance between points *A* and *C*. Dimensions are in centimeters.

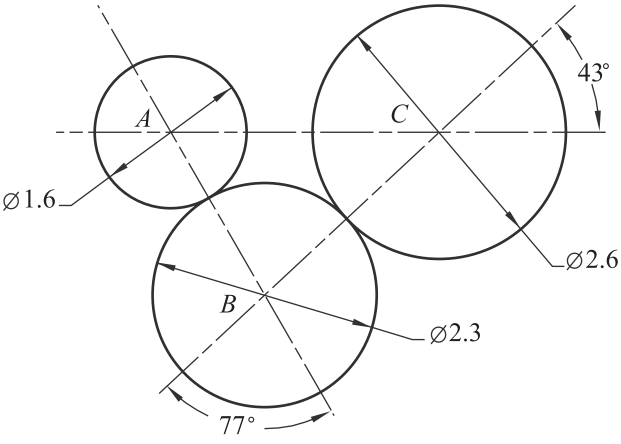


FIGURE 21.22

21.15 Solve for ∠*B* and the lengths of sides *AC* and *BC* for the adjoining oblique triangles shown in Figure 21.23. Dimensions are in inches.

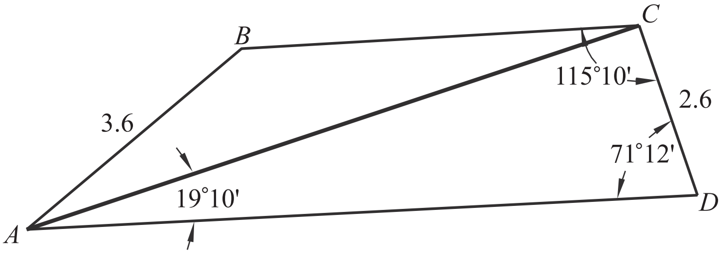


FIGURE 21.23

Exercises 21.16–21.20 can be solved using the law of cosines.

21.16 For the triangle shown in Figure 21.24, solve for:

a) $\angle A$ **b)** $\angle B$

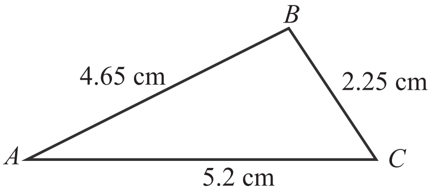


FIGURE 21.24

21.17 For the triangle shown in Figure 21.25, solve for:

a) $\angle A$ **b)** $\angle B$

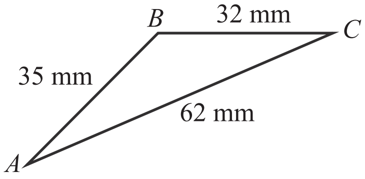


FIGURE 21.25

21.18 Determine the distance *D* in Figure 21.26. Dimensions are in inches.

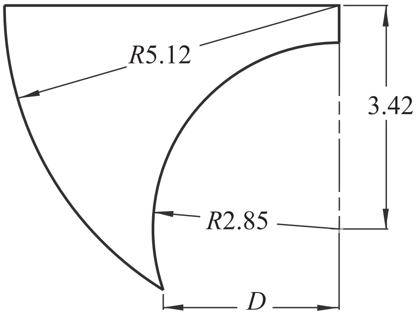


FIGURE 21.26

21.19 Solve for dimension *A* in Figure 21.27. Dimensions are in inches.

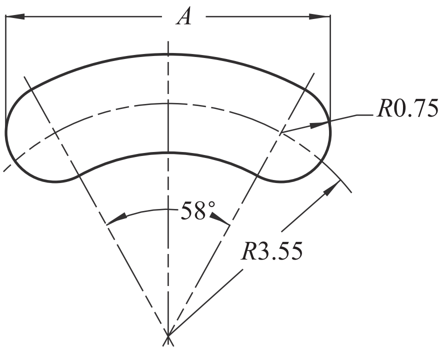


FIGURE 21.27

21.20 Solve for *BD* in Figure 21.28. Dimensions are in meters.

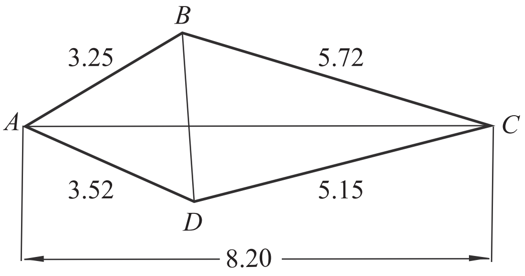


FIGURE 21.28

Exercises 21.21–21.24 can be solved using the cotangent law.

21.21 Determine altitude *h* for the triangle *ABC* shown in Figure 21.29. Dimensions are in millimeters.

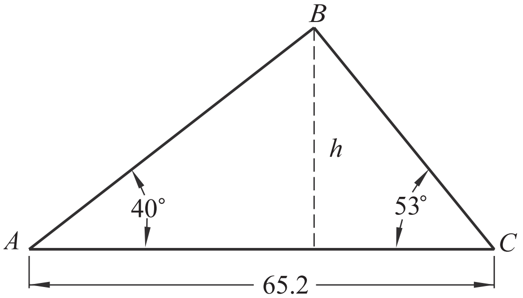


FIGURE 21.29

21.22 Solve for *h* in the triangle shown in Figure 21.30. Dimensions are in millimeters.

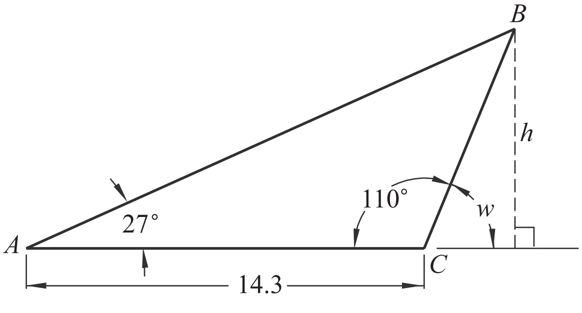


FIGURE 21.30

21.23 Determine the diameter *D* of the gauge shown in Figure 21.31. Dimensions are in inches.

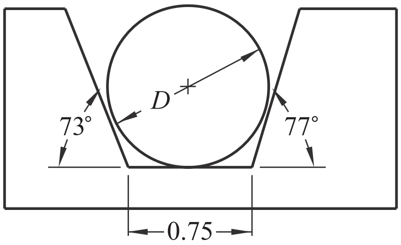


FIGURE 21.31

21.24 Determine distance P in Figure 21.32. Dimensions are in inches.

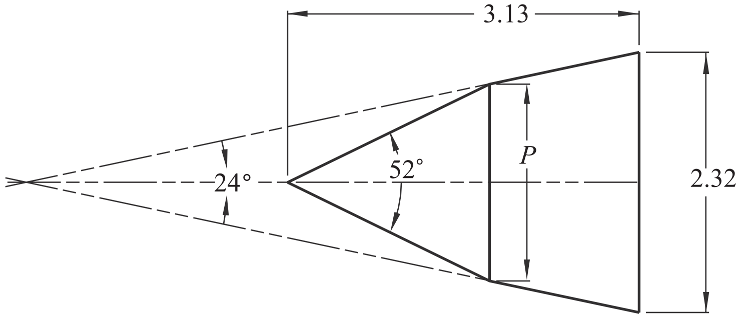


FIGURE 21.32

21.25 Solve for dimensions A and B in Figure 21.33. Dimensions are in inches.

Solve Exercises 21.25 to 21.38 using the law of sines, the law of cosines, or the cotangent law as needed.

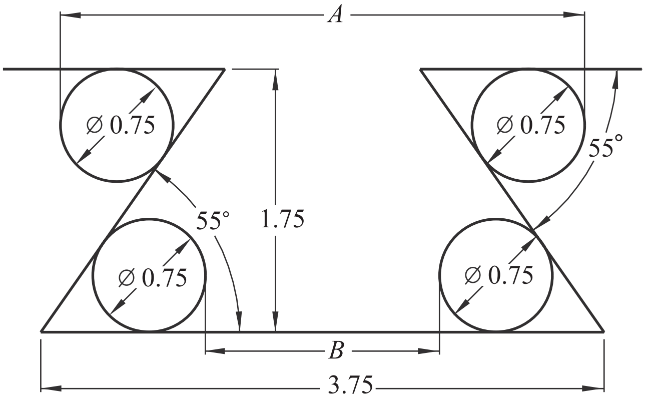


FIGURE 21.33

21.26 In Figure 21.34, the vernier height gauge reading obtained for H_1 is 4.678 inches and for H_2 the reading is 6.122 inches. Calculate the included angle of the taper and the taper per foot of the arbor.

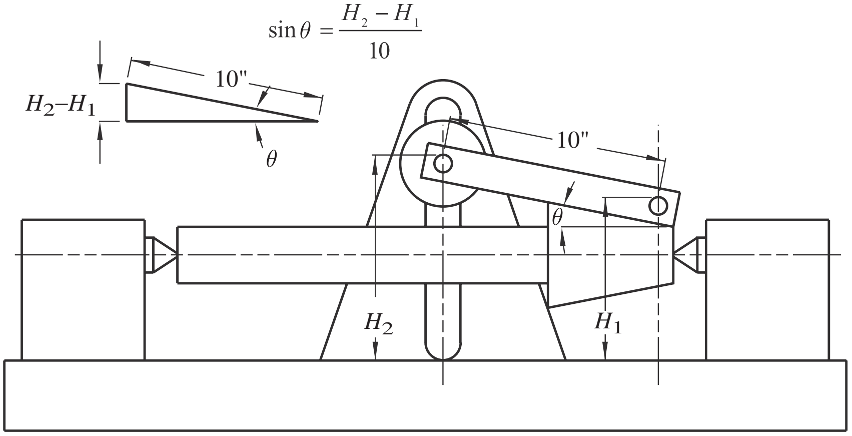


FIGURE 21.34

21.27 Solve for angle α in Figure 21.35. Dimensions are in inches.

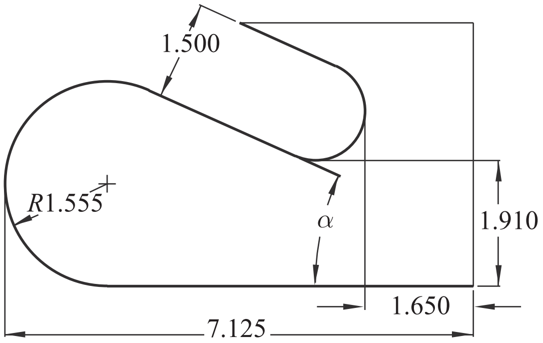


Figure 21.35

21.28 Solve for dimension *X* in Figure 21.36. Dimensions are in inches.

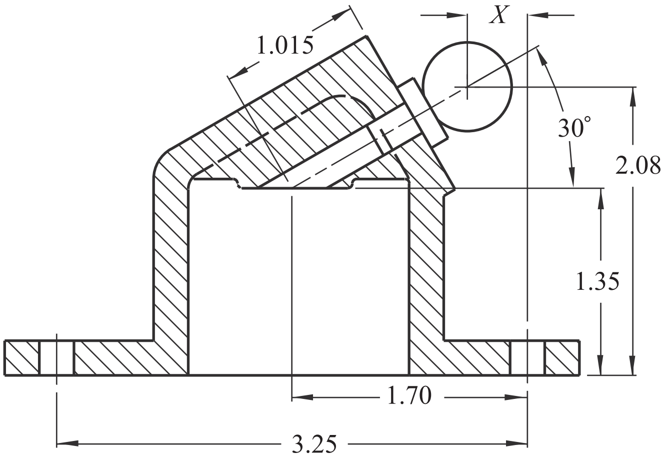


Figure 21.36

21.29 Solve for dimension *X* in Figure 21.37. Dimensions are in inches.

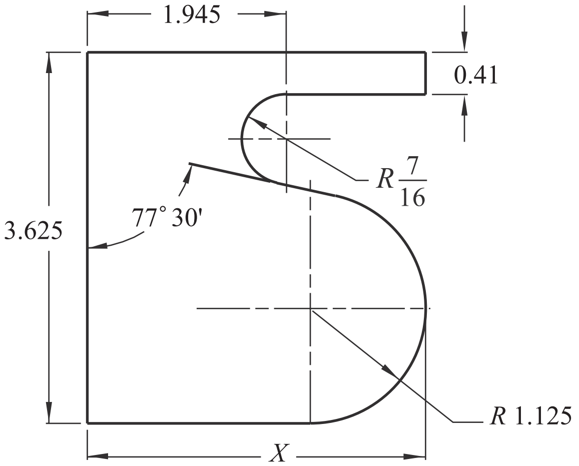


Figure 21.37

21.30 Solve for distance *X* in Figure 21.38. Dimensions are in inches.

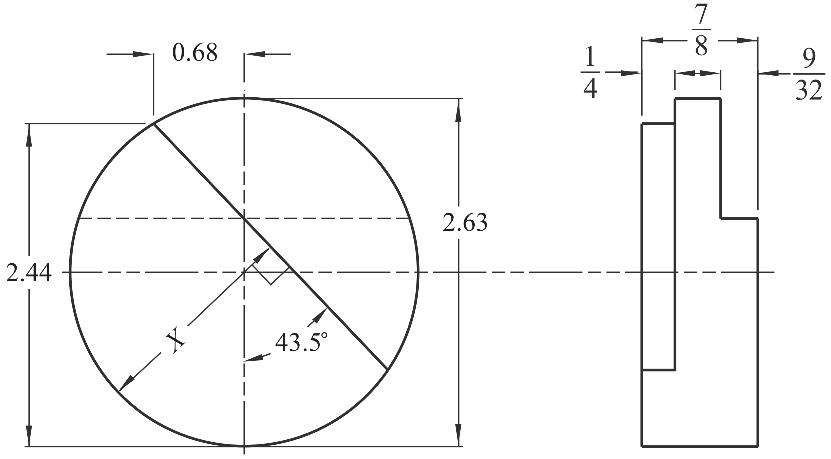


FIGURE 21.38

21.31 Solve for dimension *D* in Figure 21.39. Dimensions are in centimeters.

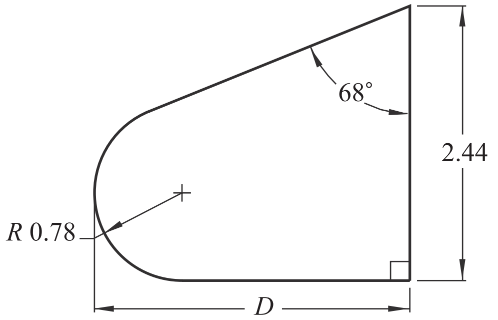


FIGURE 21.39

21.32 Solve for dimension *A* in Figure 21.40. Dimensions are in inches.

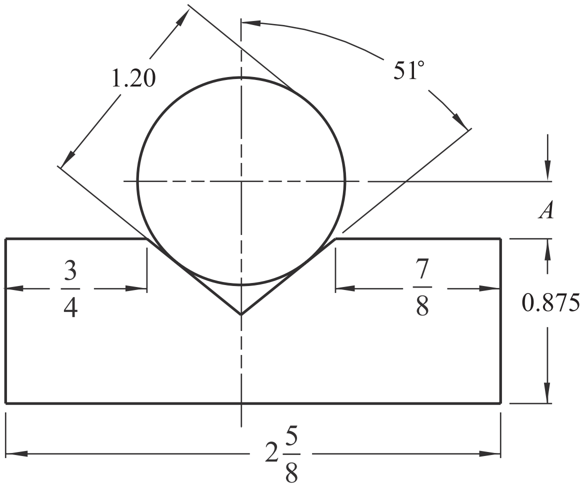


FIGURE 21.40

21.33 Solve for distance *D* in Figure 21.41. Dimensions are in inches.

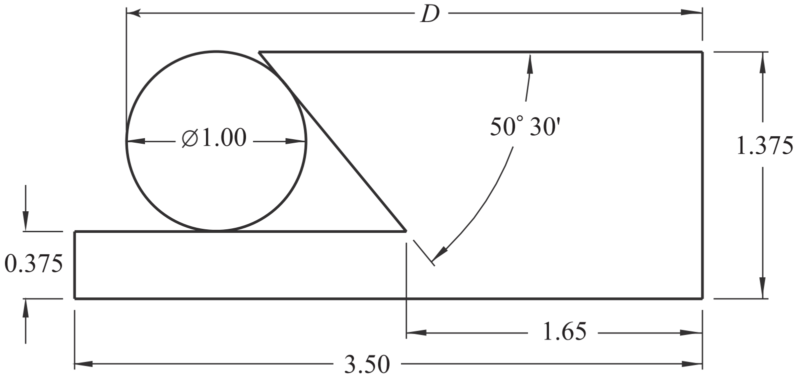


FIGURE 21.41

21.34 Solve for dimension *X* and angle α for the gear shown in Figure 21.42. Dimensions are in inches.

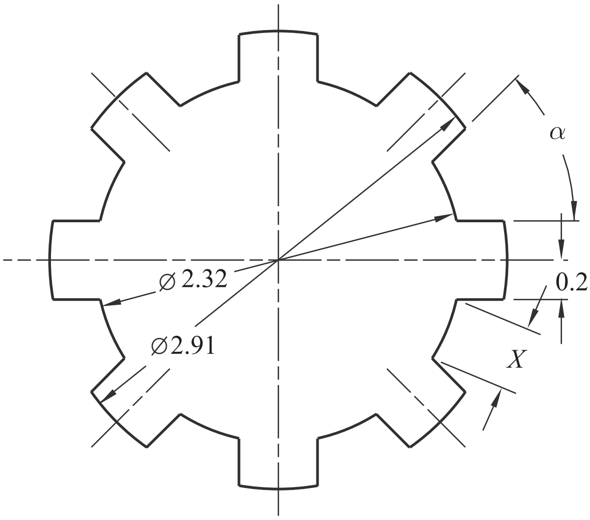


FIGURE 21.42

21.35 Solve for dimension *X* in Figure 21.43. Dimensions are in millimeters.

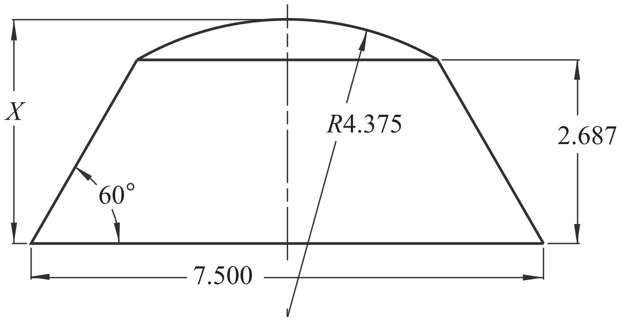


FIGURE 21.43

21.36 Solve for radii *X* and *Y* in Figure 21.44. Dimensions are in inches.

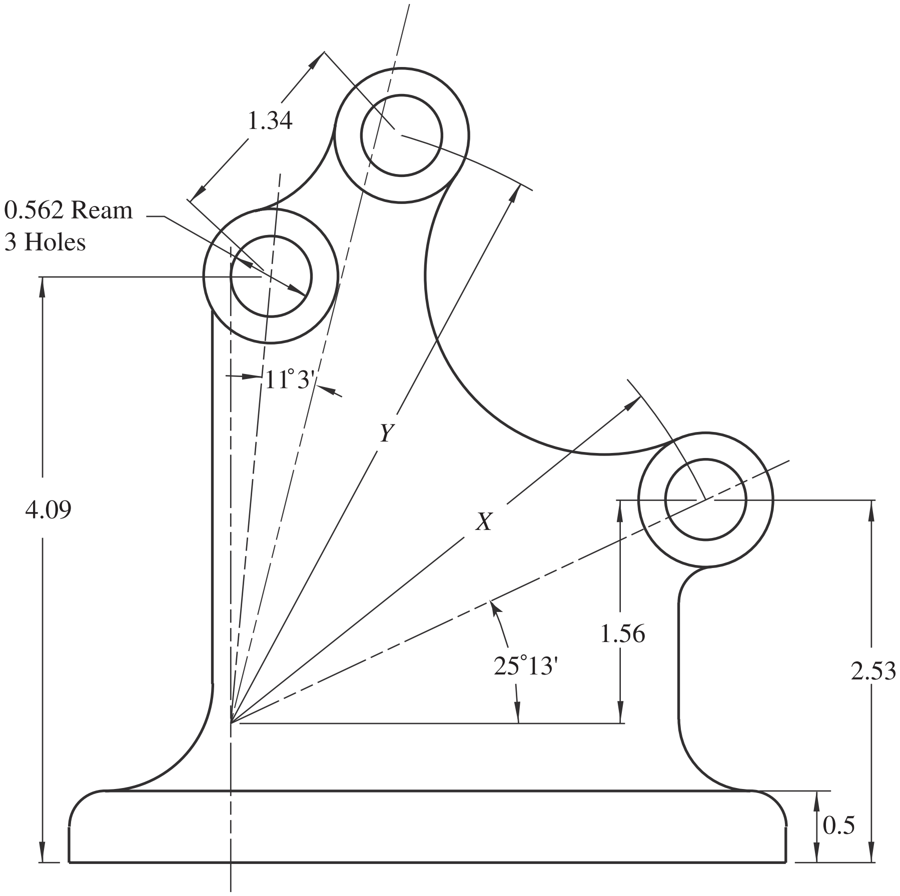


FIGURE 21.44

21.37 Solve for distance *X* in Figure 21.45. Dimensions are in inches.

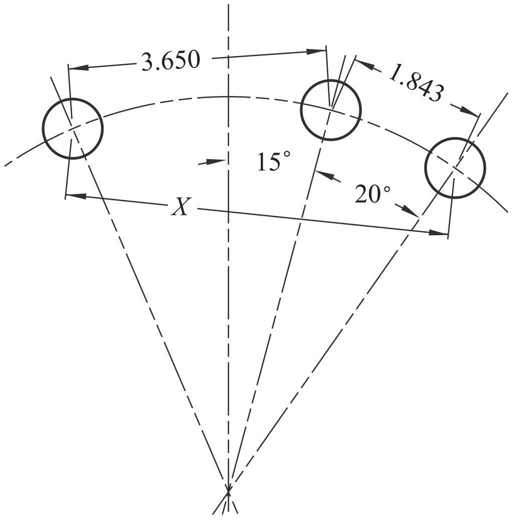


FIGURE 21.45

21.38 Solve for diameter *X* in Figure 21.46. Dimensions are in inches.

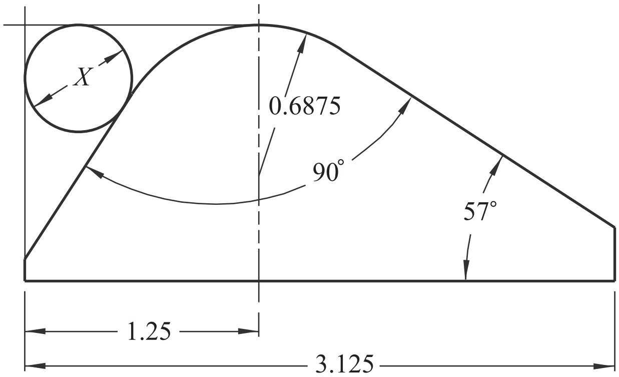


FIGURE 21.46

22

SHOP TRIGONOMETRY

Trigonometry is a most useful tool in the shop and in the drafting room. Many parts that are designed and produced involve angles or angular relationships. Trigonometry is needed to design these parts, to set them up on machine tools, to fabricate jigs and fixtures on which the parts are made, and to measure them after machining. The objective of this chapter is to provide examples and problems of typical shop and design room situations where trigonometry is used.

22.1 Sine Bars and Sine Plates

A typical sine bar and sine plate are shown in Figures 22.1A and 22.1B. These precision tools are used in tool and die shops to make precise angular measurements and to hold workpieces in precise angular relationships on machine tools such as jig borers, jig grinders, and surface grinders. When the angle to be machined or measured has a limit of accuracy of 0.1° (6 minutes) or less, sine bars and sine plates are almost indispensable tools.

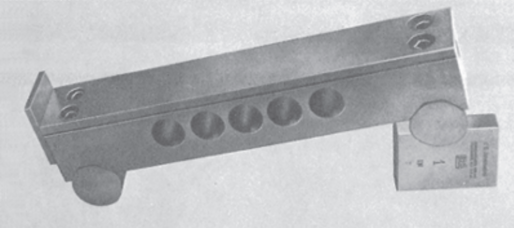


Figure 22.1A *Sine bar*

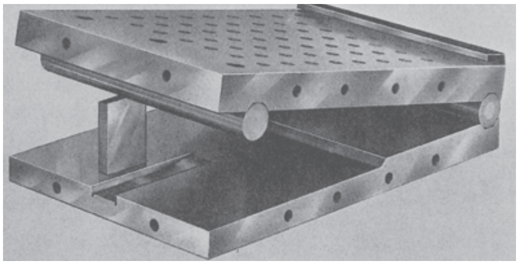


Figure 22.1B *Sine plate*

These tools are precision-ground on all surfaces. As shown in Figure 22.2, two cylindrical rolls having equal diameters are attached to each end of these tools so that they are parallel and held at a precise distance *L* – usually 5, 10, or 20 inches apart in the English system, and 100 or 200 mm apart in the metric system.

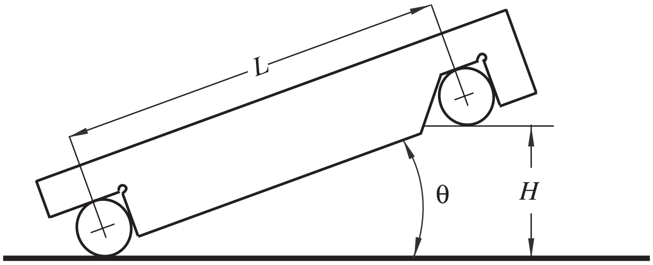


FIGURE 22.2 ***Angular relationship on a sine bar***

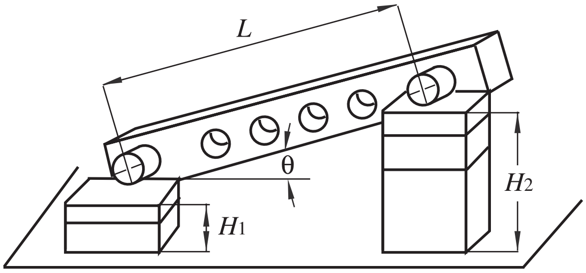
The measure of angle θ is obtained by the distance *L* at which the rolls are held apart in combination with the distance *H* at which one of the rolls is elevated above the reference surface by a stack of precision gauge blocks. Thus, in Figure 22.2 the following simple trigonometric relationship is obtained:

$$\sin\theta = \frac{H}{L} \text{ or } H = L\sin\theta$$

where H = height of gauge blocks stack (inches or mm),

L = distance between rolls (inches or mm).

A somewhat different type of sine bar is shown in Figure 22.3. It has two cylindrical plugs of equal diameter that are pressed into two holes that have been machined a precise distance *L* apart in the sine bar. The two plugs project out from this sine bar enabling precise measurements to be made on the projecting ends.



$$\sin\theta = \frac{H_2 - H_1}{L}$$

FIGURE 22.3 ***Plug-type sine bar supported on gauge blocks***

In practice, these measurements are made over the top of the plugs with a vernier height gauge, usually in combination with a test type dial indicator. This sine bar may also be set at a given angle by resting the bar on the angle of the part to be measured as in Figure 22.4. While in this position, the H_1 and H_2 dimensions are determined by using the vernier height gauge set-up and measuring over the plugs. In either case, the angular relationship depends upon the distance *L* and the difference in the heights H_1 and H_2, as shown by the simple formula:

$$\sin\theta = \frac{H_2 - H_1}{L}$$

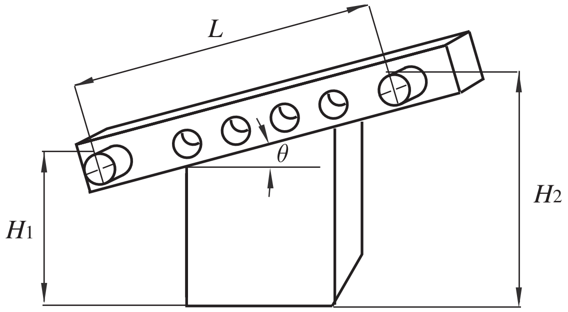


FIGURE 22.4 ***Set-up for measuring the angle of a part with plug type sine bar***

In Figure 22.5, a small (5-inch) sine bar is shown measuring the angle of a precision gauge block scriber attachment. The sine bar is tipped to the required angle by the gauge blocks placed under one of the rolls.

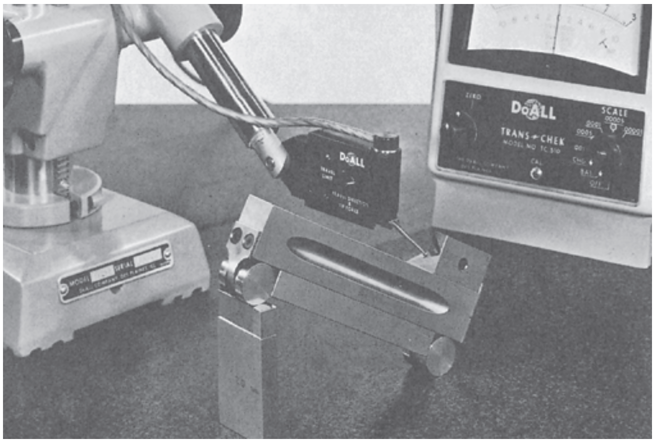


FIGURE 22.5 ***Measuring an angle on a surface plate using a sine bar, gauge blocks, and an electronic height gauge***

EXAMPLE 22.1: ***Using a Sine Bar***

The angle of the scriber point shown in Figure 22.2 is to be 35°. Calculate the height of the gauge block stack required to obtain this angle using a 5-inch sine bar.

Solution: Substituting the given values into the formula $H = L \sin\theta$, we find:

$$\begin{aligned} H &= (5)(\sin 35^\circ) \\ &= (5)(0.5736) \\ &= 2.8679 \text{ inches} \end{aligned}$$

EXAMPLE 22.1A: *Using a Sine Bar*

The 10-inch sine bar shown in Figure 22.6 (along with a blown-up detail of the half-angle of taper) is used to measure the angularity of the taper on the shaft positioned between two bench centers. The difference in height $H_2 - H_1$ over the buttons is measured using a vernier height gauge and a dial test indicator. Calculate the half-angle of the taper when $H_2 = 6.122$ inches and $H_1 = 4.678$ inches.

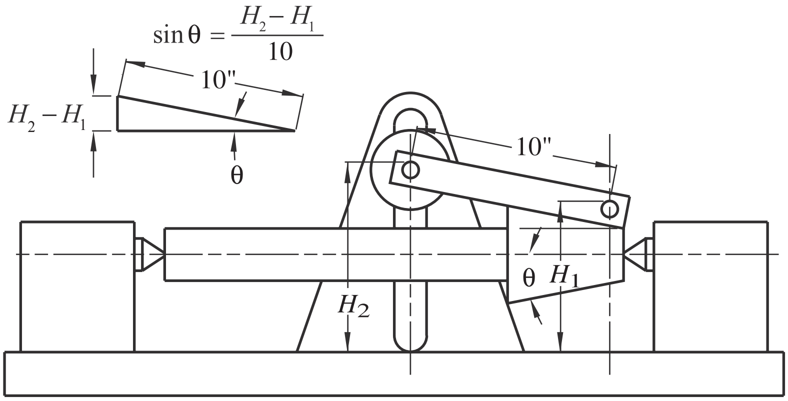


FIGURE 22.6

Solution: Substituting the given values into the formula

$$\sin\theta = \frac{H}{L} = \frac{H_2 - H_1}{10},$$

we find

$$\sin\theta = \frac{6.122 - 4.678}{10} = \frac{1.444}{10} = 0.1444$$

$$\theta = \sin^{-1}(0.1444)$$

$$\text{Half-angle } \theta = 8.30°$$

The included angle of the taper is: $2 \times \theta = 2 \times 8.30° = 16.60°$

EXERCISES

22.1 When a sine bar was in position to measure an angle, the gauge block stack height was 1.99368 inches. What would be the angle if:

a) a 5-inch sine bar were used? **b)** a 10-inch sine bar were used?

22.2 A taper shaft similar to the one shown in Figure 22.6 was found to have dimension $H_2 = 8.326$ inches and $H_1 = 6.168$ inches. Calculate the total included angle of the taper if a 10-inch sine bar was used.

22.3 A toolmaker requires an angle of 39.636°.

a) Using a 5-inch sine plate, what height H of gauge blocks is required?
b) If the stack of gauge blocks were used with a 10-inch sine plate, what would be the angle measurement?

22.4 In checking an angle on a machined casting, a 5-inch sine bar was used that had buttons (rolls) fastened to the side. The lower button was 2.1672 inches above the surface plate and the upper button was 4.8634 inches above the surface plate. What is the angle measurement?

22.5 A 10-inch sine bar with buttons on the side has the following two measurements: lower, 5.6729 inches; upper, 9.2173 inches. What is the measure of the angle?

22.6 The measurement to the buttons of a 10-inch sine bar from the flat were taken as 3.2763 inches and 8.4976 inches. What was the measure of the angle?

22.7 Determine the stack height (gauge blocks to the upper button) of a 5-inch sine bar with two buttons when the lower button is 0.7500 inch above the flat and the angle is 22.373°.

22.8 Calculate the stack height of gauge blocks to set a 10-inch sine plate to an angle of 37.175°.

22.9 The stack height for a 10-inch sine plate used in measuring an angle is 3.5837 inches.

a) What is the angle measurement?
b) If a 5-inch sine plate were used on the same stack, what would be the angle measurement?

22.10 A 10-inch sine bar with buttons is placed on a taper shaft whose included angle of taper is 19.5°. If the short stack measures 1.350 inches, what is the height of the upper button stack?

22.11 A workpiece is placed on a simple sine plate as shown in Figure 22.7. Calculate the stack height H of required gauge blocks to raise the upper plate to an angle of 31.83° when L = 1000 mm.

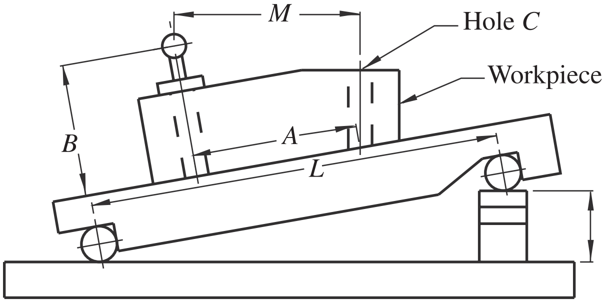


FIGURE 22.7

22.12 In Figure 22.7, solve for M when L = 127 mm, H = 82.672 mm, A = 28.575 mm, and B = 30.162 mm. Give answer to 0.001 mm.

22.13 The L distance on a metric sine bar is 100 mm. Find the angle of a set-up when the gauge block stack equals 14.925 mm.

22.14 An angle of 47.66° is to be set up on a sine bar whose L dimension is 200 mm. Calculate the stack height of the required gauge blocks.

22.2 Hole Circle Spacing

Spacing holes in a circular pattern such as around a flange of a pipe is a common machine shop task. The circle on which the holes are located is called the *hole circle*. To lay out the holes with even spacing, the chordal distance between the holes must be determined as illustrated in Example 22.2.

EXAMPLE 22.2: ***Hole Circle Spacing***

Twenty equally spaced holes are to be placed along a 10-inch diameter hole circle as shown in Figure 22.8. Calculate the chordal distance AB between the adjacent holes.

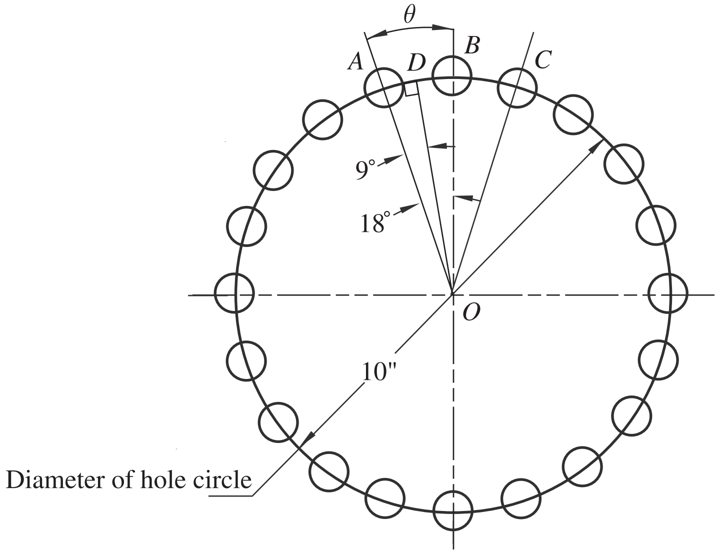


FIGURE 22.8

Solution:

In the unique case of six equally spaced holes, used quite often in industry, the chordal distance is equal to the radius of the hole circle.

STEP 1: Draw radii OA and OB. Calculate $\angle AOB$:

$$\angle AOB = 360° \div \text{number of holes}$$
$$= \frac{360}{20} = 18°$$

STEP 2: Bisect $\angle AOB$ by drawing radius OD:

$$\angle AOD = \frac{18}{2} = 9°$$

STEP 3: In right triangle ADO:

$$\sin AOD = \frac{AD}{AO}$$
$$AD = AO \sin 9°$$

$$\text{but} \quad AO = \frac{1}{2}\text{diameter} = \frac{10}{2} = 5 \text{ inches}$$

EXAMPLE 22.2: ***Hole Circle Spacing (Continued)***

$$\text{so} \quad AD = (5)(0.1564) = 0.7822 \text{ inches}$$

STEP 4: Calculate chordal distance *AB*:

$$AB = 2AD$$
$$= (2)(0.7822) = 1.5643 \text{ inches}$$

An alternate formula for solving chordal distances of equally spaced holes along a known diameter hole circle is:

$$AB = \text{Diameter of hole circle} \times \sin\left[\frac{1}{2}\left(\frac{360°}{\text{number of holes}}\right)\right]$$

$$\text{or} \quad AB = (10)\ \sin\left[\frac{360°}{(2)(20)}\right] = 1.5643 \text{ inches}$$

EXERCISES

22.15 Find the chordal distances between adjacent holes for the number of equally spaced holes and hole circle diameter shown in the accompanying chart:

	Number of Holes	Diameter Hole-Circle	Chordal Distance
a)	5	8-inch	
b)	9	6-inch	
c)	14	12-inch	
d)	17	10-inch	
e)	22	20-inch	
f)	8	7-inch	
g)	4	9-inch	
h)	7	7-inch	
i)	3	3-inch	
j)	5	6-inch	
k)	8	10-inch	
l)	7	9-inch	
m)	11	10-inch	
n)	22	22-inch	
o)	14	9-inch	
p)	9	6-inch	
q)	10	10-inch	
r)	18	12-inch	

22.3 Coordinate Distances

The precise location of holes is a problem that is frequently encountered in tool and die work. The *coordinate dimensioning system* was developed to accurately measure the location of holes. This system is used on drawings for jig-bore work and is designed to be compatible with the coordinate axes of movement available on the jig-boring machine. For jig-boring work, the coordinates are generally referred from the intersections of lines that are tangent to the hole-circle at the *top* and at the *left*, as shown in Figure 22.9. Thus, the horizontal coordinate dimensions are taken from a vertical line passing through the center of the hole at the extreme left and the vertical coordinate dimensions are taken from a horizontal line passing through the center of the hole at the top.

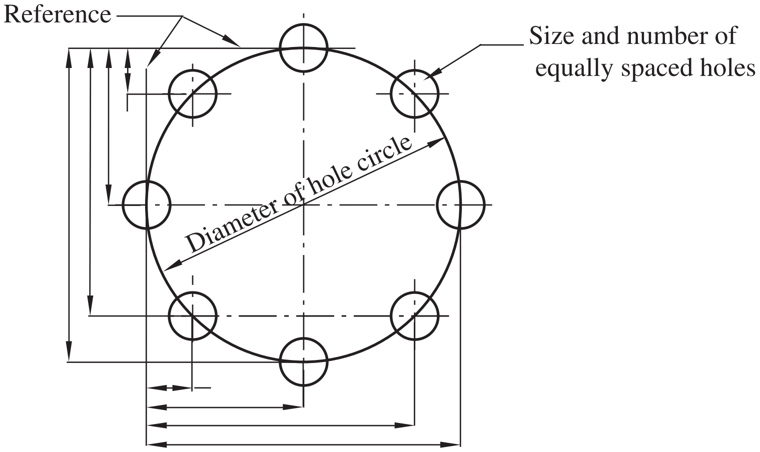


FIGURE 22.9 ***Jig bore coordinates***

This problem is also frequently encountered on numerically controlled machines. In the system used on these machines, the table movements must be compatible with the coordinate axes of the dimensioning system. The base or reference tangents in this system are usually *below* and to the *left* of the hole-circle as shown in Figure 22.10.

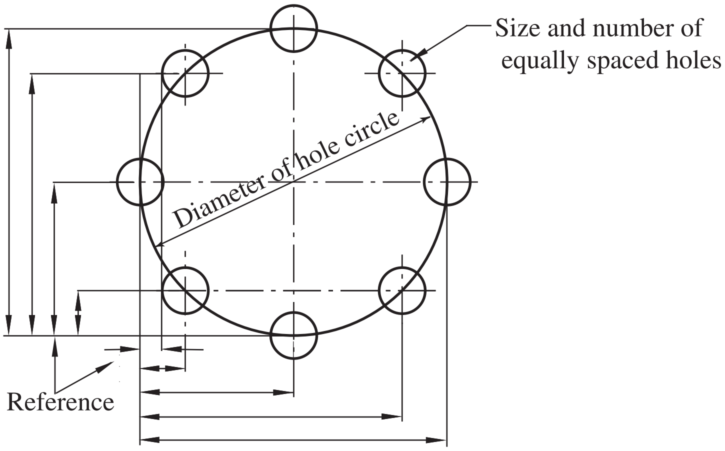


FIGURE 22.10 ***Numerical control coordinates***

EXAMPLE 22.3: ***Jig-Bore Coordinates***

The coordinate distances for eight holes equally spaced around a 10-inch diameter hole circle to be machined on a jig-borer are shown in Figure 22.11A. Show how the coordinate distances were calculated.

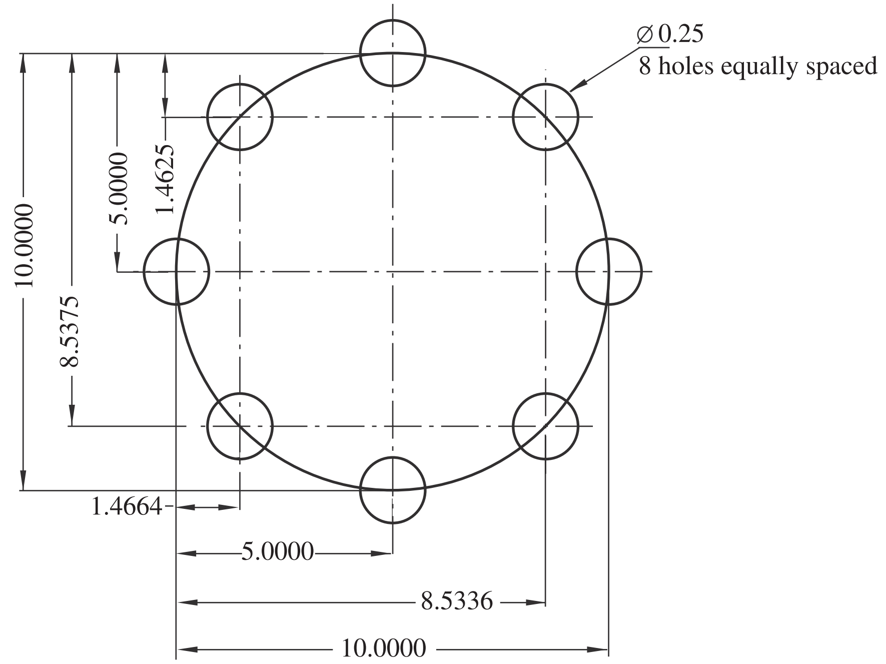


FIGURE 22.11A

Solution:

STEP 1: Redraw Figure 22.11B with related reference lines and central angle about one of the holes, such as hole *A*. In right triangle *OBA*

$$\angle AOB = \frac{360^\circ}{8} = 45^\circ$$

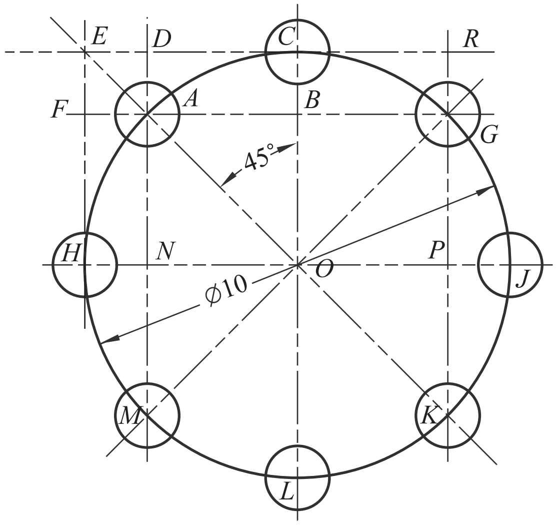


FIGURE 22.11B

EXAMPLE 22.3: *Jig-Bore Coordinates (Continued)*

STEP 2: Calculate the length of *AB*:

$$AB = AO \sin 45°$$

$$\text{where } AO = \frac{1}{2} \text{ diameter} = \frac{10}{2} = 5.0000 \text{ inches}$$

$$AB = (5.0000)(0.70711)$$

$$= 3.5356 \text{ inches}$$

STEP 3: Find the coordinate distances to hole *A*:

Notice that in right triangle *EDA*:

$$ED = AD \quad \text{(isosceles right triangle)}$$

$$EC = \frac{1}{2}\text{diameter} = 5.0000 \text{ inches}$$

$$ED = EC - DC \quad (\text{notice that } DC = AB)$$

Horizontal coordinate: $ED = 5.0000 - 3.5356 = 1.4644$ inches
Vertical coordinate: $EF = ED = AD = 1.4644$ inches

STEP 4: Find the coordinate distances to hole *C*:

In right triangle *ECO*, $EC = OC = EH = \frac{1}{2}\text{diameter} = 5.0000 \text{ inches}$

Horizontal coordinate: $EC = EH = 5.000$ inches

The vertical coordinate in this case is zero.

STEP 5: Find the coordinate distances to hole *G*:

In right triangle *AOG*:

$$BG = AB = 3.5356 \text{ inches}$$

$$FG = BG + FB \quad \left(\text{where } FB = \frac{1}{2}\text{diameter} = 5.0000 \text{ inches}\right)$$

Horizontal coordinate: $FG = 5.0000 + 3.5356 = 8.5356$ inches

Vertical coordinate: $EF = 1.4644$ inches

STEP 6: Find the coordinate distances to hole *J*:

Horizontal coordinate: HJ = diameter = 10.0000 inches

Vertical coordinate: $EH = \frac{1}{2} \text{ diameter} = 5.0000 \text{ inches}$

STEP 7: Find the coordinate distances to hole *H*:

The horizontal coordinate in this case is zero.

Vertical coordinate: $HO = \frac{1}{2} \text{ diameter} = 5.0000 \text{ inches}$

EXAMPLE 22.3: ***Jig-Bore Coordinates (Continued)***

STEP 8: Find the coordinate distances to hole K:

$KP = PG = BG = AB = 3.5356$ inches

Horizontal coordinate: $FG = 5.0000 + 3.5356 = 8.5356$ inches

Vertical coordinate: $PK = 5.0000 + 3.5356 = 8.5356$ inches

STEP 9: Find the coordinate distances to hole L:

Horizontal coordinate: $HO = \frac{1}{2}\text{ diameter} = 5.0000\text{ inches}$

Vertical coordinate: $CL = \text{diameter} = 10.0000$ inches

STEP 10: Find the coordinate distances to hole M:

Horizontal coordinate: $ED = 1.4644$ inches

Vertical coordinate:

$$DM = KP + \frac{1}{2}\text{ diameter} = 3.5356 + 5.000 = 8.5356\text{ inches}$$

EXAMPLE 22.4: ***Numerical Control Coordinates***

The coordinate distances for ten holes equally spaced around an 8-inch diameter hole circle to be machined on a numerically controlled (NC) boring machine are shown in Figure 22.12. Show how the coordinate distances for hole D were calculated.

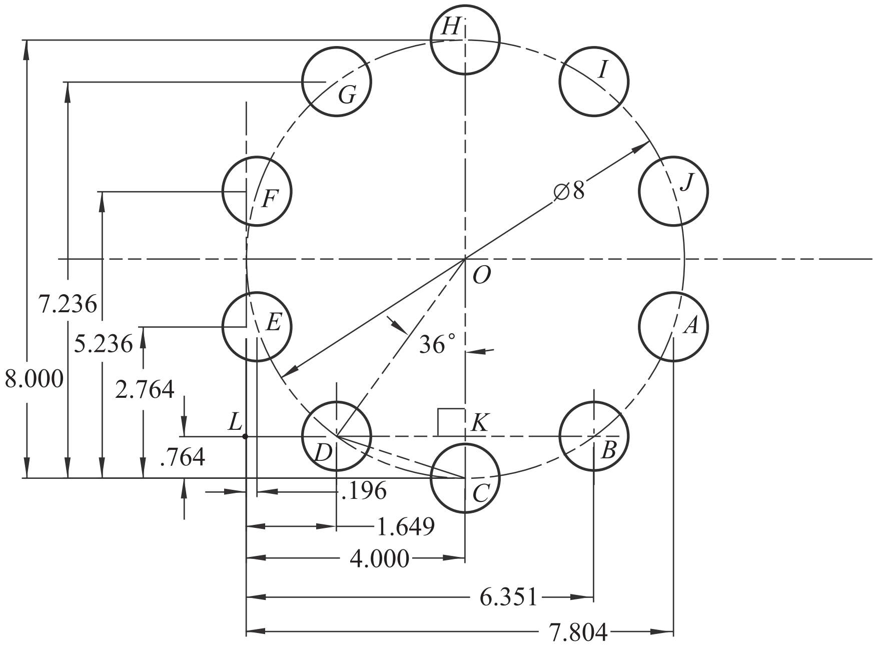


FIGURE 22.12

EXAMPLE 22.4: *Numerical Control Coordinates (Continued)*

Solution:

STEP 1: Select two adjacent holes, such as C and D, and construct related reference lines and the central angle.

STEP 2: In right triangle OKD, solve for length of DK:

$$DK = OD \sin\angle DOC$$

When an odd number of holes is involved, or when the holes are not spaced symmetrically with respect to the reference axes, a separate X, Y-coordinate calculation is necessary for each hole.

$$OD = \frac{1}{2}\text{ diameter} = \frac{8}{2} = 4.0000\text{ inches}$$

$$\angle DOK = \frac{360^\circ}{10} = 36^\circ$$

$$DK = (4.0000)(\sin 36^\circ) = (4.0000)(0.58778)$$

$$= 2.3511\text{ inches}$$

STEP 3: Calculate the coordinate distances to hole D:

Horizontal coordinate:

$$LD = LK - DK = 4.0000 - 2.3511 = 1.6489\text{ inches}$$

Vertical coordinate: In right triangle CKD,

$$\angle KCD = \frac{1}{2}(180^\circ - 36^\circ) = 72^\circ$$

$$CK = \frac{DK}{\tan 72^\circ} = \frac{2.3511}{3.0777} = 0.7639\text{ inch}$$

EXERCISES

22.16 Solve for the jig-bore coordinate distances for the number of equally spaced holes and the hole circle diameters indicated:

a) Five holes on an 8-inch diameter hole circle.
b) Ten holes on a 12-inch diameter hole circle.
c) Six holes on an 8-inch diameter hole circle.
d) Seven holes on a 10-inch diameter hole circle.
e) Twelve holes on a 12-inch diameter hole circle.

22.17 Solve for the numerical control coordinate distances for the number of equally spaced holes indicated for the specified hole circle diameters (one or more holes on a center line):

a) Seven holes on a 10-inch diameter hole circle.
b) Three holes on a 5-inch diameter hole circle.
c) Six holes on an 8-inch diameter hole circle.
d) Twelve holes on a 12-inch diameter hole circle.
e) Ten holes on a 6-inch diameter hole circle.

22.18 The plate shown in Figure 22.13 is to have two holes bored using a boring machine. After hole *A* is bored, the table of the machine is moved in two directions, *X* and *Y*, to locate hole *B*. Find the distances *X* and *Y* when line *AB* and ∠*A* have the following measurements:

a) *AB* = 8.75 inches, ∠*A* = 45° **b)** *AB* = 13.125 inches, ∠*A* = 9°

c) *AB* = 237.45 mm, ∠*A* = 66.90°

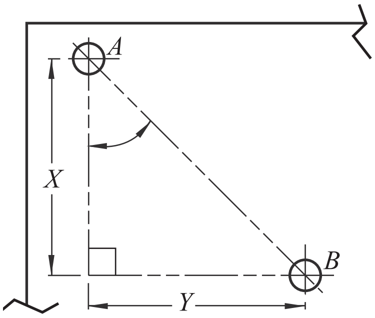


Figure 22.13

22.19 Nine holes are equally spaced around the 7.125-inch diameter bolt circle shown in Figure 22.14. Find the chordal distance between the following holes:

a) *A* to *B* **b)** *A* to *C* **c)** *A* to *D*

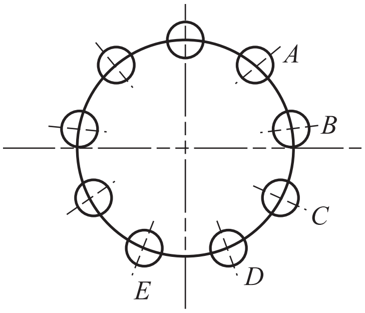


Figure 22.14

22.20 Seven holes are equally spaced on the 114.30 mm bolt circle shown in Figure 22.15. They are to be checked by measuring the distances *AB* and *AC*. Determine these distances.

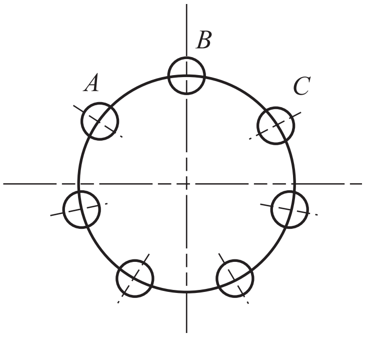


Figure 22.15

22.21 Five equally spaced holes are on the 6.435 inch bolt circle shown in Figure 22.16. Find the indicated chordal distances:

a) from hole *A* to hole *B*

b) from hole *A* to hole *C*

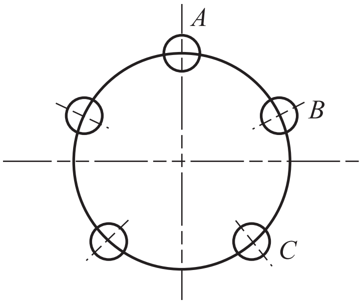


FIGURE 22.16

22.4 Solving Practical Shop Problems

The solutions for the next two examples are based on the use of geometric construction and trigonometric calculations. In general, the solution for linear dimensions can be approached by the construction of lines parallel to the linear dimension to be solved for: lines parallel to lines in the figure of the problem, lines connecting the centers of circles or arcs, and/or lines drawn perpendicular to known or constructed lines. When problems have rolls (circles) shown contacting two lines in a Vee construction, remember that a line drawn from the center of the circle to the vertex of the angle of the Vee bisects that angle. Also radii drawn to the point of tangency are perpendicular to that side of the Vee at the point of tangency.

EXAMPLE 22.5: ***Practical Shop Problem***

The included angle of the Vee-block shown in Figure 22.17 is 90°. Solve for dimension M when *L* = 1.2500 inches and the roll diameter is 1.7500 inches.

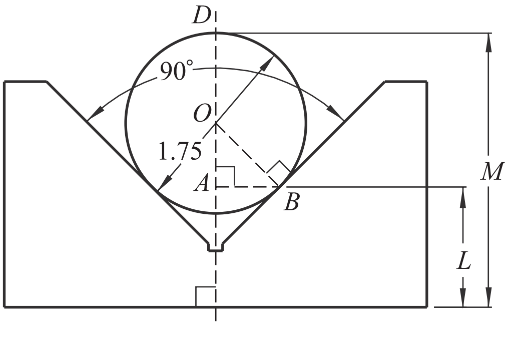


FIGURE 22.17

Solution:

STEP 1: Draw *DA* through center *O* of roll and perpendicular to the base.

STEP 2: Draw *OB* perpendicular to the side of the Vee-block. Draw *AB* parallel to the base.

EXAMPLE 22.5: *Practical Shop Problem (Continued)*

STEP 3: In right triangle *OAB*:

$$\angle OBA = \angle AOB = 45^\circ$$

$$OA = OB \sin 45^\circ$$

$$\text{where } OB = \frac{1}{2} \text{ roll diameter} = \frac{1.7500}{2} = 0.8750 \text{ inch}$$

$$OA = (0.8750)(0.70711)$$

$$= 0.6187 \text{ inch}$$

STEP 4: Dimension $M = DO + OA + L$

$$= 0.8750 + 0.6187 + 1.2500$$

$$= 2.7437 \text{ inches}$$

EXAMPLE 22.5A: *Practical Shop Problem*

The detail drawing shown in Figure 22.18 is being checked for accuracy of angular displacement of the two holes using the tooling ball and pin technique. The top of the ball and the top surface of the pin are measured from a surface plate. Dimension *M* is the difference of the two measurements.

What should dimension *M* be if the angle between the two holes is exactly 20°? Dimensions are in inches.

We deduce m∠ACE from m∠P by similar triangles.

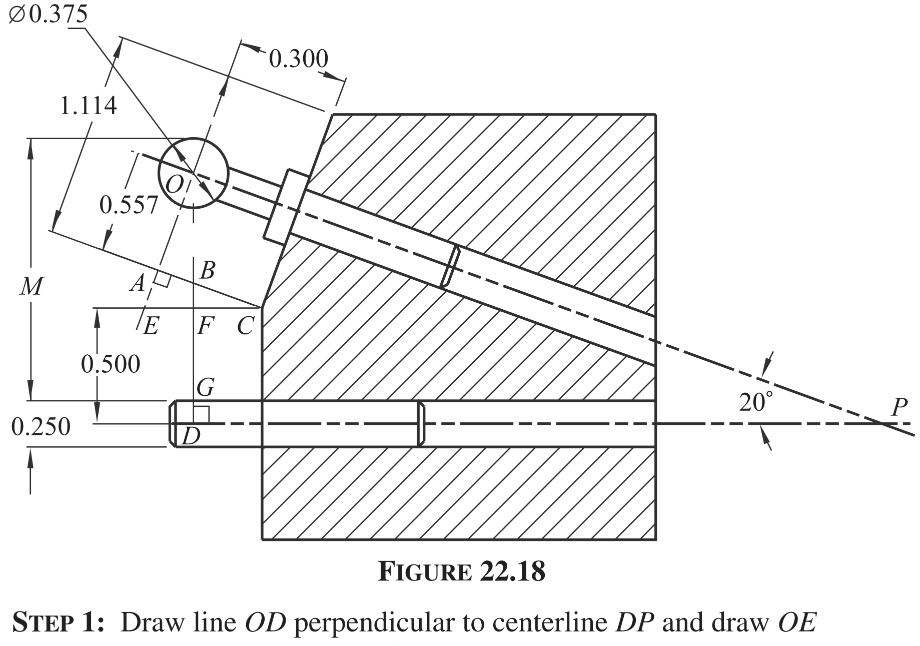


FIGURE 22.18

STEP 1: Draw line *OD* perpendicular to centerline *DP* and draw *OE* perpendicular to centerline OP. In right triangle *EAC*, *AC* = 0.300 inch and $\angle ACE = 20^\circ$. Solve for *AE*:

$$AE = AC \tan 20^\circ$$

$$= (0.300)(0.36397)$$

$$= 0.10919 \text{ inch}$$

EXAMPLE 22.5A: ***Practical Shop Problem (Continued)***

STEP 2: In right triangle OFE: $AO = 0.557$ inch and $\angle AOF = 20°$. Solve for OF:

$$OF = OE\cos 20°$$
$$\text{where } OE = AO + AE = 0.557 + 0.10919$$
$$= 0.66619 \text{ inch}$$
$$\text{therefore, } OF = (0.66619)(0.93969)$$
$$OF = 0.626 \text{ inch}$$

STEP 3: Distance $M = \dfrac{\text{ball diameter}}{2} + OF + FG = \dfrac{0.375}{2} + OF + FG$

where $OF = 0.626$ inch and $FG = 0.500 - \dfrac{0.250}{2} = 0.375$ inch.

Therefore, $M = 0.188 + 0.626 + 0.375 = 1.189$ inches.

EXERCISES

22.22 The included angle of the Vee-block shown in Figure 22.19 is 90°. Solve for dimension M when $L = 2.45$ mm and the roll diameter is 1.75 mm.

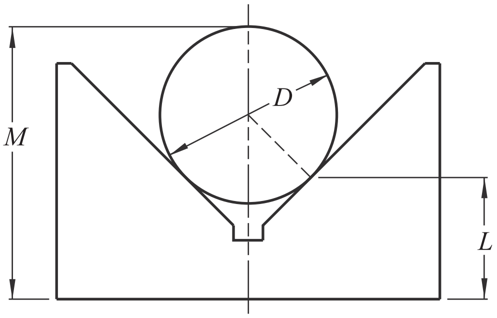


FIGURE 22.19

22.23 In the Vee-block shown in Figure 22.20, the top of the roll is at the same height as the side K. The Vee angle is 90°. Find the diameter of the roll when $L = 1.563$ inches and $K = 2.575$ inches.

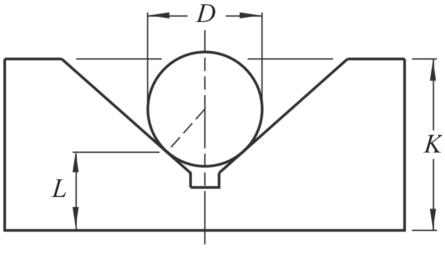


FIGURE 22.20

22.24 In the drill jig shown in Figure 22.21, D_1 is the diameter of the part to be drilled and D_2 is the diameter of the setting roll. Dimension L locates the center of the drill bushing from the locating surface. The Vee angle is 90°. Determine M so that the center of the part will be on the center of the jig bushing. $D_1 = 3.75$ inches, $D_2 = 2.00$ inch, and $L = 4.25$ inches.

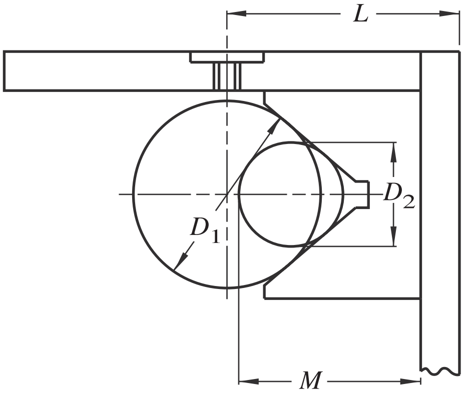


FIGURE 22.21

22.25 Solve for distance A in the Acme thread shown in Figure 22.22. Dimensions are in inches.

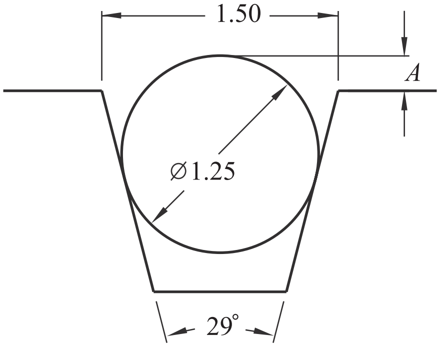


FIGURE 22.22

22.26 Dovetails are common in machine design. The usual method of measurement is to measure across a pair of rolls and compare the measurement to a known dimension such as X in Figure 22.23. Calculate the dimensions X and Y when the angle of the dovetail is 60° and the rolls have 0.75-inch diameters.

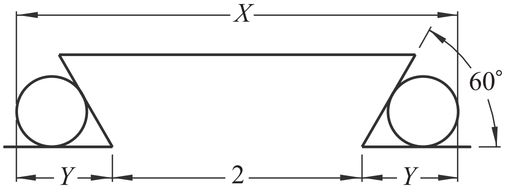


FIGURE 22.23

22.27 Find the values of X and Y in Figure 22.23 if 0.500-inch rolls are used in place of the 0.75-inch rolls.

22.28 Find the value of *X* and *Y* in Figure 22.23 if 0.625-inch rolls are used.

22.29 Solve for distance *M* in Figure 22.24. Dimensions are in millimeters.

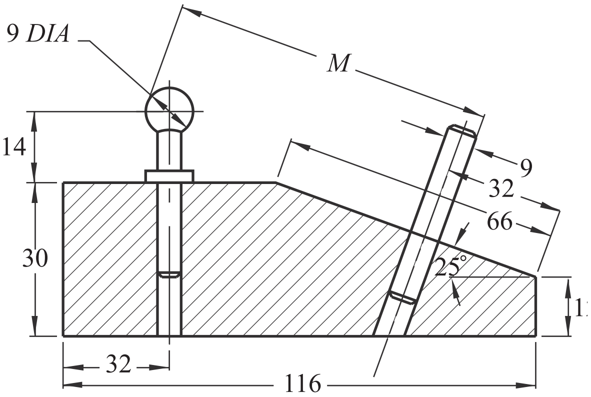


FIGURE 22.24

22.30 Solve for angle β in Figure 22.25. Dimensions are in millimeters.

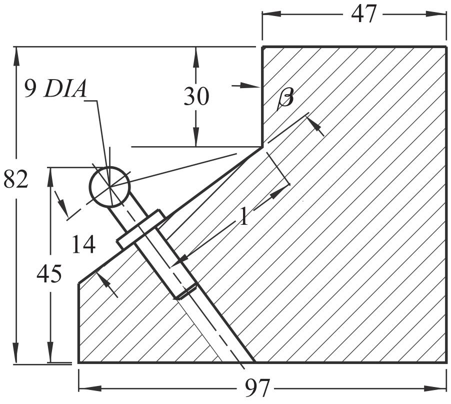


FIGURE 22.25

22.31 Solve for distance *M* in Figure 22.26. Dimensions are in millimeters.

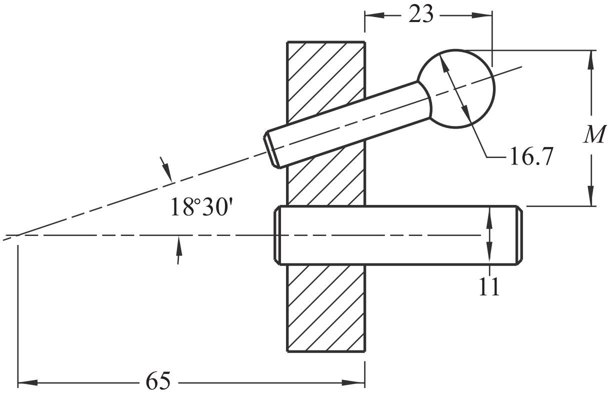


FIGURE 22.26

22.32 Solve for dimension *M* in Figure 22.27. Dimensions are in millimeters.

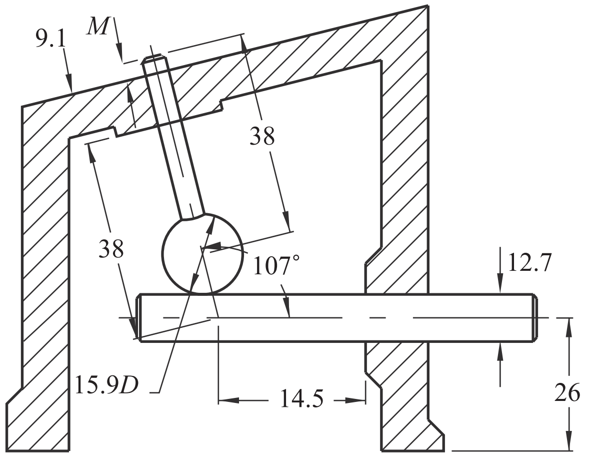


FIGURE 22.27

22.33 Solve for distances *X* and *Y* in Figure 22.28. Dimensions are in inches.

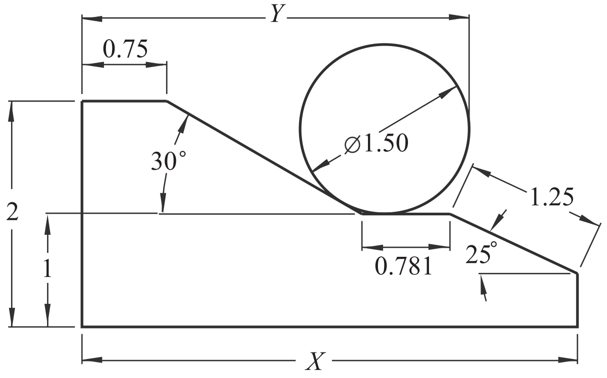


FIGURE 22.28

22.34 Solve for dimension *D* in Figure 22.29. Dimensions are in inches.

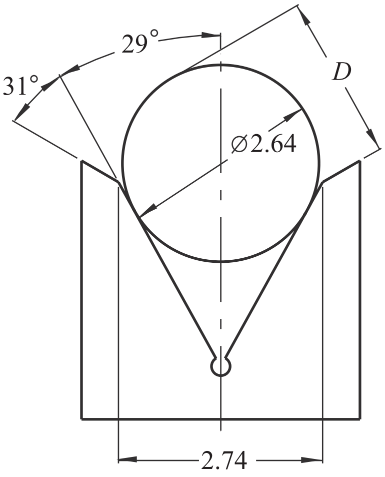


FIGURE 22.29

22.35 Solve for distance *A* in Figure 22.30. Dimensions are in inches.

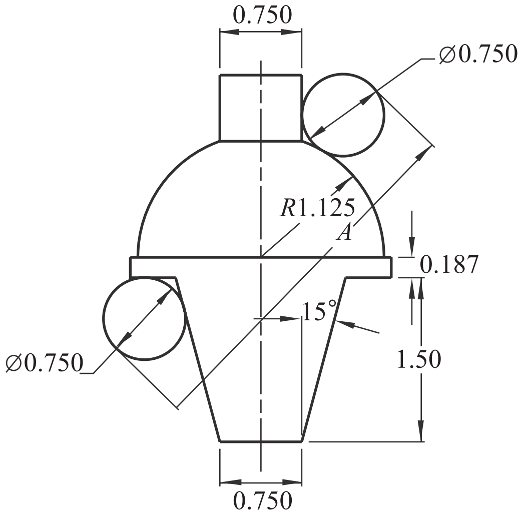


Figure 22.30

22.36 Solve for distance *X* in Figure 22.31. Dimensions are in inches.

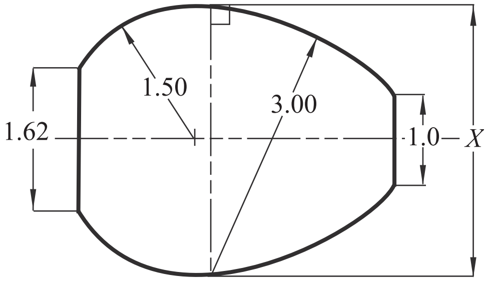


Figure 22.31

22.37 Solve for distances *X* and *Y* in Figure 22.32. Dimensions are in inches.

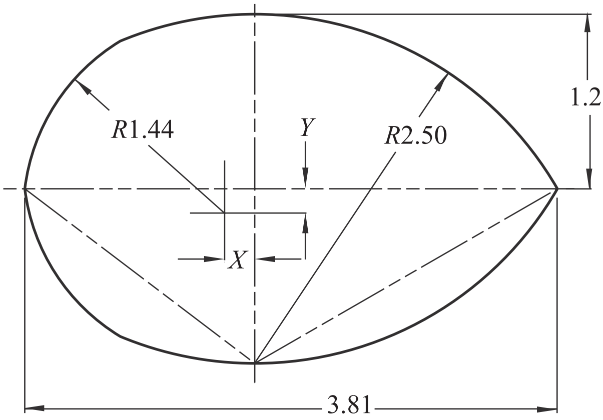


Figure 22.32

22.38 Solve for distance *X* in Figure 22.33. Dimensions are in inches.

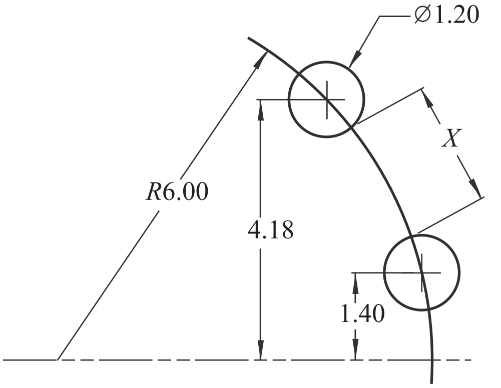


FIGURE 22.33

22.39 Solve for distances *X*, *Z*, and radius *Y* in Figure 22.34. Dimensions are in inches.

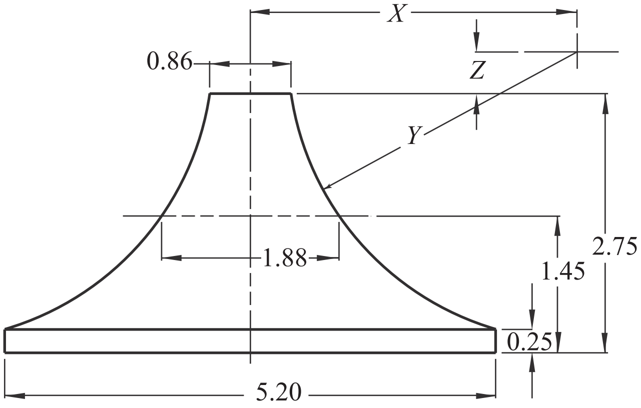


FIGURE 22.34

22.40 Solve for distance *X* in Figure 22.35. Dimensions are in inches.

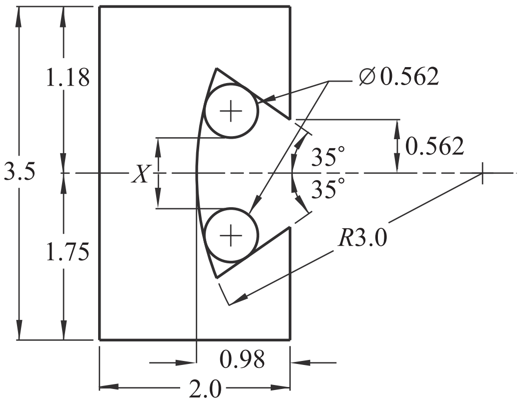


FIGURE 22.35

22.41 Solve for distance X in Figure 22.36. Dimensions are in inches.

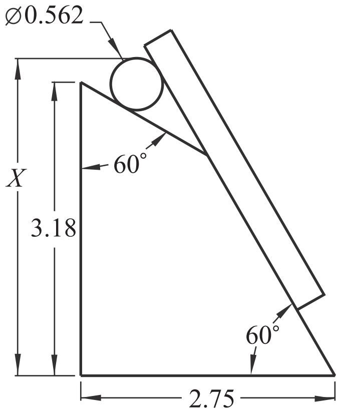


FIGURE 22.36

22.42 Solve for distances M_1 and M_2 when $\angle A$ is 117° and dimension B is 2.264 inches in the typical construction hole and tooling ball set-up shown in Figure 22.37.

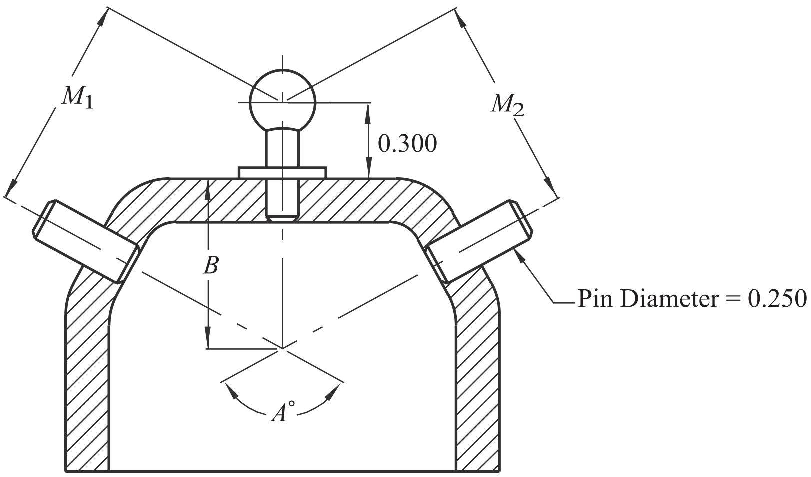


FIGURE 22.37

22.43 In Figure 22.37 solve for M_1 and M_2 when $\angle A$ is 121° and dimension B is 51.275 mm. Answer to 0.01 mm.

22.5 Trigonometric Shop Formulas

Many formulas used in industry contain trigonometric functions. Essentially, these formulas are equations that are solved by methods given in the previous chapters on algebra and trigonometry. To solve many of these problems, the formula is algebraically transposed to place the unknown on the left side of the equal sign and then the known values for the literal terms are substituted. Trigonometric functions become numerical terms.

EXAMPLE 22.6A: *Applying Trigonometric Shop Formulas*

The face of a single-point cutting tool is shown in Figure 22.38. Angle α is the back rake angle and angle β is the side rake angle. The angle ϕ is any angle in the *XY*-plane and ρ is the actual rake angle on the face of the tool at the angle ϕ. There will be one value of the angle ϕ, designated $\phi_{\max}$, where ρ will have a maximum value. These angles can be found by the following formulas:

$$\tan\phi_{\max} = \frac{\tan\beta}{\tan\alpha}$$

$$\tan\rho = \sin\phi\tan\beta + \cos\phi\tan\alpha$$

Find the maximum rake angle when the back rake angle is 5° and the side rake angle is 15°.

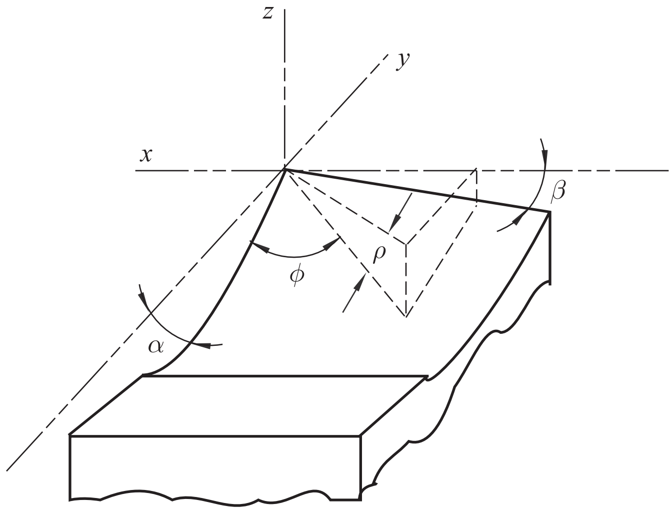


FIGURE 22.38

Solution:

STEP 1: Solving for $\phi_{\max}$:

$$\tan\phi_{\max} = \frac{\tan\beta}{\tan\alpha} = \frac{\tan 15^\circ}{\tan 5^\circ} = \frac{0.26795}{0.08749} = 3.06264$$

$$\phi_{\max} = \tan^{-1}(3.06264) = 71.92^\circ$$

$$= 71.92^\circ$$

EXAMPLE 22.6A: ***Applying Trigonometric Shop Formulas (Continued)***

STEP 2: Solving for angle ρ:

$$\begin{aligned}\tan\rho &= \sin\phi\tan\beta + \cos\phi\tan\alpha\\ &= (\sin 71.92°)(\tan 15°) + (\cos 71.92°)(\tan 5°)\\ &= (0.95061)(0.26795) + (0.31034)(0.08749)\\ &= 0.25472 + 0.02716\\ &= 0.28188\\ \rho &= \tan^{-1}(0.28188)\\ \rho &= 15.74°\end{aligned}$$

EXAMPLE 22.6B: ***Applying Trigonometric Shop Formulas***

The pitch diameter d of a worm gear can be calculated by the following formula:

$$d = \frac{L\cot a}{\pi}$$

where L is the lead of the worm gear and $\angle\alpha$ is the lead angle.

Calculate the pitch diameter of a worm gear having a lead of 0.750 inch and a lead angle equal to 10°.

Solution:

$$d = \frac{L\cot a}{\pi} = \frac{(0.750)(\cot 10°)}{3.1416} = \frac{(0.750)(5.67128)}{3.1416} = 1.354 \text{ inches}$$

EXERCISES

22.44 Calculate the maximum rake angle of a single-point cutting tool when the back rake angle and the side rake angle are both 10°.

22.45 The efficiency E of a worm gear drive can be calculated by the following formula:

$$E = \frac{\cos\phi - f\tan\theta}{\cos\phi + f\ \text{cotan}\ \theta}$$

where ϕ is the normal pressure angle,

θ is the lead angle of the worm,

f is the coefficient of friction.

Calculate the efficiency of a worm gear drive when: $\phi = 20°$, $\theta = 15°$, $f = 0.05$.

22.46 A formula for checking the accuracy of threads by the three-wire method is given in *Machinery's Handbook* as:

$$M = E - T \cot A + W(1 + \csc A)$$

where M is the dimension over the wires (inches, by micrometer),

E is the pitch diameter of thread to be checked (inches),

T is $\frac{1}{2}$ pitch or width of thread at diameter E (inches),

A is $\frac{1}{2}$ included thread angle in axial plane,

W is the diameter of wires used in measurements (inches).

Solve for M when E is 0.450 inch, T is 0.03846 inch, A is 30°, and W is 0.07013 inch.

22.47 Calculate the transverse offset required to cut a circular flute in a $\frac{1}{2}$-inch, 13-threads-per-inch tap using a circular form milling cutter having a radius of 0.150 inch if the tap is to be made with a tangential hook angle of 5°. Use the formula:

$$X = \frac{D}{2}(1 - \sin\phi) + r[1 - \cos(\phi - \alpha)]$$

where X is the tangential offset,

D is the actual outside diameter of the tap (inches),

r is the radius of form milling cutter (inches),

ϕ is $\frac{1}{2}$ of angle between flutes, in this case, 45°,

α is the tangential hook angle.

22.48 The wire diameter for obtaining pitch-line contact at the back of a buttress thread may be determined by the following formula:

$$W = P\left(\frac{\cos a}{1 + \cos A}\right)$$

where P is the pitch of the thread $= \dfrac{1}{\text{number of threads per inch}}$,

A is the included angle of thread,

α is the angle of front face,

W is the diameter of wire (inches).

Solve for the wire diameter W when P is 0.0625 inch, α is 7°, and A is 52°.

22.49 In gearing work it is sometimes necessary to check involute teeth by measuring the distance across two or more teeth. This distance is related to the pressure angle of the gear, the pitch radius, tooth thickness, and the number of teeth on the gear. The checking formula is given as:

$$M = R\cos A\left(\frac{T}{R} + \frac{6.2832\,S}{N} + F\right)$$

where N is the number of teeth on the gear,

R is the pitch radius of gear (inches),

A is the pressure angle of gear,

T is the tooth thickness along pitch circle (inches),

S is the number of tooth spaces within caliper jaws,

F is the factor relating to pressure angle (0.01109 for $14\frac{1}{2}$°).

Determine the distance M over three teeth of a spur gear having a pressure angle of 14.5° with 30 teeth of 6 diametral pitch and pitch radius of 2.5 inches. T for 6 diametral pitch is 0.2618 inch (no allowance for backlash) and F is 0.01109.

A

APPENDIX

A.1 Greek Letters and Standard Abbreviations

The Greek letters are frequently used in mathematical expressions and formulas. The Greek alphabet is given below.

Α	α	Alpha	Η	η	Eta	Ν	ν	Nu	Τ	τ	Tau
Β	β	Beta	Θ	ϑ θ	Theta	Ξ	ξ	Xi	ϒ	υ	Upsilon
Γ	γ	Gamma	Ι	ι	Iota	Ο	ο	Omicron	Φ	ϕ	Phi
Δ	δ	Delta	Κ	κ	Kappa	Π	π	Pi	Χ	χ	Chi
Ε	ε	Epsilon	Λ	λ	Lambda	R	ρ	Rho	Ψ	ψ	Psi
Ζ	ζ	Zeta	Μ	μ	Mu	Σ	σ ς	Sigma	Ω	ω	Omega

A.2 Factors and Prefixes for Decimal Multiples of SI Units

Decimal multiples and sub-multiples of SI units are formed by using the prefixes given in the following table, which represent the numerical factors shown.

Factor by which the unit is multiplied	Prefix	Symbol	Factor by which the unit is multiplied	Prefix	Symbol
10^{12}	tera	T	10^{-2}	centi	c
10^{9}	giga	G	10^{-3}	milli	m
10^{6}	mega	M	10^{-6}	micro	μ
10^{3}	kilo	k	10^{-9}	nano	n
10^{2}	hecto	h	10^{-12}	pico	p
10	deka	da	10^{-15}	femto	f
10^{-1}	deci	d	10^{-18}	atto	a

A.3 Linear Measure Conversion Factors

Metric

1 kilometer (km) =
1000 meters
100,000 centimeters
1,000,000 millimeters
0.539956 nautical mile
0.621371 mile
1093.61 yards
3280.83 feet
39,370.08 inches
1 meter (m) =
10 decimeters
100 centimeters
1000 millimeters
1.09361 yards
3.28084 feet
39.37008 inches
1 decimeter (dm) = **10** centimeters
1 centimeter (cm) =
0.01 meter
10 millimeters
0.0328 foot
0.3937 inch
1 millimeter (mm) =
0.001 meter
0.1 centimeter
1000 microns
0.03937 inch
1 micrometer or *micron* (μ*m*) =
0.000001 meter = one millionth meter
0.0001 centimeter
0.001 millimeter
0.00003937 inch
39.37 micro-inches

US Customary

1 mile (mi) =
0.868976 nautical mile
1760 yards
5280 feet
63,360 inches
1.609344 kilometers
1609.344 meters
160,934.4 centimeters
1,609,344 millimeters
1 yard (yd) =
3 feet
36 inches
0.9144 meter
91.44 centimeter
914.4 millimeter
1 foot (international ft) =
12 inches = $\mathbf{\frac{1}{3}}$ yard
0.3048 meter
30.48 centimeters
304.8 millimeters
1 survey foot =
1.000002 international feet
$\frac{12}{39.37}$ = 0.3048006096012 meter
1 inch (in) =
1000 mils
1,000,000 micro-inches
2.54 centimeters
25.4 millimeters
25,400 microns
1 mil =
0.001 inch
1000 micro-inches
0.0254 millimeter
1 micro-inch (μ*in*) =
0.000001 inch = one millionth inch
0.0254 micrometer (micron)

Surveyors Measure

1 mile = **8** furlongs = **80** chains
1 furlong = **10** chains = **220** yards
1 chain = **4** rods = **22** yards = **66** feet = **100** links
1 rod = **5.5** yards = **16.5** feet = **25** links = 5.0292 meters
1 link = 7.92 inches
1 span = 9 inches
1 hand = 4 inches

Nautical Measure

1 league = 3 nautical miles
1 nautical mile =
1.1508 statute miles
6076.11549 feet
1.8516 kilometers
1 fathom = **2** yards = **6** feet
1 knot = nautical unit of speed =
1 nautical mile per hour
1.1508 statute miles per hour
1.8516 kilometers per hour

Note: Figures in **bold** indicate exact conversion values.

A.4 Square Measure Conversion Factors

Metric System

1 square kilometer (km^2) =
- **100** hectares
- **1,000,000** square meters
- 0.3861 square mile
- 247.1 acres

1 hectare (ha) =
- **0.01** square kilometer
- **100** ares
- **10,000** square meters
- 2.471 acres
- 107,639 square feet

1 are (a) =
- **0.0001** square kilometer
- **100** square meters
- 0.0247 acre
- 1076.4 square feet

1 square meter (m^2) =
- **0.000001** square kilometer
- **100** square decimeters
- **10000** square centimeters
- **1,000,000** square millimeters
- 10.764 square feet
- 1.196 square yards

1 square decimeter (dm^2) =
- **100** square centimeters

1 square centimeter (cm^2) =
- **0.0001** square meter
- **100** square millimeters
- 0.001076 square foot
- 0.155 square inch

1 square millimeter (mm^2) =
- **0.01** square centimeter
- **1,000,000** square microns
- 0.00155 square inch

1 square micrometer (micron) (μm^2) =
- $\mathbf{1 \times 10^{-12}}$ square meter
- **0.000001** square millimeters
- 1×10^{-9} square inch
- 1549.997 square micro-inch

U.S. System

1 square mile (mi^2) =
- **640** acres
- **6400** square chains
- 2.5899 square kilometers

1 acre =
- **10** square chains
- **4840** square yards
- **43,560** square feet
- a square, 208.71 feet on a side
- 0.4046856 hectare
- 40.47 ares
- 4046.856 square meters

1 square chain =
- **16** square rods
- **484** square yards
- **4356** square feet

1 square rod =
- **30.25** square yards
- **272.25** square feet
- **625** square links

1 square yard (yd^2) =
- **9** square feet
- **1296** square inches
- **0.83612736** square meter
- **8361.2736** square centimeter
- **836,127.36** square millimeter

1 square foot (ft^2) =
- 0.111111 square yard
- **144** square inches
- **0.09290304** square meter
- **929.0304** square centimeters
- **92,903.04** square millimeters

1 square inch (in^2) =
- 0.0007716 square yard
- 0.006944 square foot
- 0.00064516 square meter
- **6.4516** square centimeters
- **645.16** square millimeters

1 square mil (mil^2) =
- **0.000001** square inch
- **0.00064516** square millimeter

1 square micro-inch (μin^2) =
- $\mathbf{1 \times 10^{-12}}$ square inch
- **0.00064516** square micrometer (micron)

Note: Figures in **bold** indicate exact conversion values.

A.5 Cubic Measure and Conversion Factors

Metric System

1 cubic meter (m^3) =
1000 cubic decimeters (liters)
1,000,000 cubic centimeters
1.30795 cubic yards
35.314667 cubic feet
61,023.74 cubic inches
264.17205 U.S. gallons
219.96925 British Imperial gallons

1 liter (l) or 1 cubic decimeter (dm^3) =
1 liter = volume of 1 kg water at 39.2°F
0.001 cubic meter
1000 cubic centimeters
10 deciliters
0.03531466 cubic foot
61.023744 cubic inches
0.2642 U.S. gallon
0.21997 British Imperial gallon
1.0566882 U.S. quarts
33.814 U.S. fluid ounces

1 cubic centimeter (cm^3) =
0.001 liter
1000 cubic millimeters
0.061024 cubic inch

1 cubic millimeter = **0.001** cubic centimeters
1 hectoliter (hl) = **100** liters
1 deciliter (dl) = **10** centiliters
1 centiliter (cl) = **10** milliliters

U.S. System

1 cubic yard (yd^3) =
27 cubic feet
201.97403 U.S. gallons
46,656 cubic inches
0.7646 cubic meter

1 cubic foot (ft^3) =
1728 cubic inches
7.4805 U.S. gallons
6.23 British Imperial gallons
0.02831685 cubic meter
28.31685 liters

1 cubic inch (in^3) =
0.55411256 U.S. fluid ounce
16.387064 cubic centimeters

Shipping Measure

For measuring internal capacity of a vessel:
1 register ton = 100 cubic feet
For measurement of cargo:
1 shipping ton =
Approximately 40 cubic feet of merchandise is considered a shipping ton, unless that bulk would weigh more than 2000 pounds, in which case the freight charge may be based upon weight.

40 cubic feet =
32.143 U.S. bushels
31.16 Imperial bushels

British (Imperial) Liquid and Dry Measure

1 British Imperial gallon =
0.1605 cubic foot
277.42 cubic inches
1.2009 U.S. gallon
160 Imperial fluid ounces
4 Imperial quarts
8 Imperial pints
4.54609 liters

1 quart =
2 Imperial pints
8 Imperial gills
40 Imperial fluid ounces
69.354 cubic inches
1.1365225 liters

1 pint =
4 Imperial gills
20 Imperial fluid ounces
34.678 cubic inches
568.26125 milliliters

1 gill =
5 Imperial fluid ounces
8.669 cubic inches
142.07 milliliters

U.S. Liquid Measure

1 U.S. gallon =
0.13368 cubic foot
231 cubic inches
128 U.S. fluid ounces
4 U.S. quarts
8 U.S. pints
0.8327 British Imperial gallon
3.785411784 liters

1 quart =
2 U.S. pints
8 U.S. gills
32 U.S. fluid ounces
57.75 cubic inches
0.9463529 liters

1 pint =
4 U.S. gills
16 U.S fluid ounces
28.875 cubic inches
473.176 milliliters

1 gill =
1/2 cup = **4** U.S. fluid ounces
7.21875 cubic inches
118.29 milliliters

Note: Figures in **bold** indicate exact conversion values.

British (Imperial) Liquid and Dry Measure

1 British Imperial fluid ounce =
1.733871 cubic inch
$\frac{1}{160}$ British Imperial gallon
28.41306 milliliters
1 British Imperial bushel =
8 Imperial gallons = 1.284 cubic feet
2219.36 cubic inches

Apothecaries' Fluid Measure

1 U.S. fluid ounce =
1.8046875 cubic inch
$\frac{1}{128}$ U.S. gallon
8 drachms
0.02957353 liter
29.57353 milliliters
1 fluid drachm = **60** minims

U.S. Dry Measure

1 bushel (U.S. or Winchester struck bushel) =
1.2445 cubic feet
2150.42 cubic inches
a cylinder 18.5 inches dia., 8 inches deep
a cylinder 47.0 cm dia., 20.3 cm deep
1 bushel = **4** pecks = **32** quarts = **64** pints
1 peck = **8** quarts = **16** pints
1 dry quart = **2** pints =
67.200625 cubic inches
1.101221 liters
1 heaped bushel = $1\frac{1}{4}$ struck bushel
1 cubic foot = 0.8036 struck bushel

Old Liquid Measure

1 barrel (bbl) = $31\frac{1}{2}$ gallons
1 hogshead = 2 barrels = 63 gallons
1 pipe or *butt* = 2 hogsheads = 4 barrels = 126 gallons
1 tierce = 42 gallons
1 puncheon = 2 tierces = 84 gallons
1 tun = 2 pipes = 3 puncheons

Other Cubic Measure

The following are used for wood and masonry:
1 cord of wood = 4 × 4 × 8 feet = 128 cubic feet
1 perch of masonry =
$16\frac{1}{2} \times 1\frac{1}{2} \times 1$ foot = $24\frac{3}{4}$ cubic feet

Barrel Measure

1 drum =
55 U.S. gallon
7.3524 cubic feet
208.19765 liters

1 petroleum barrel (bo) =
42 U.S. gallons
5.614583 cubic feet
158.98729 liters

Note: Figures in **bold** indicate exact conversion values.

A.6 Circular and Angular Measure Conversion Factors

One degree at the equator =
60 nautical miles
69.047 statute miles
111.098 kilometers
One minute at the equator =
1 nautical mile
1.1508 statute miles
1.8516 kilometers

360 degrees at the equator =
circumference angle at equator
21,600 nautical miles
24,856.8 statute miles
39,995.4 kilometers

circumference angle of circle =
360 degrees = 2π radian = 6.283185 radian
1 quadrant = 90 degrees = π/2 radian = 1.570796 radian
1 radian = 57.2957795 degrees

1 degree (°) = 60 minutes = 3600 seconds = π/180 radian = 0.017453 radian
1 minute (′) = 60 seconds = 0.016667 degrees = 0.000291 radian
$\pi = 3.141592654$

A.7 Mass and Weight Conversion Factors

Metric System

1 metric ton (t) =
- **1000** kilograms
- 2204.6223 pounds
- 0.9842 gross or long ton (of 2240 pounds)
- 0.9072 net or short ton (of 2000 pounds)

1 kilogram (kg) =
- **1000** grams = **10** hectograms
- 2.2046 pounds
- 35.274 ounces avoirdupois

1 hectogram (hg) = **10** dekagrams
1 dekagram (dag) = **10** grams
1 gram (g) =
- **10** decigrams
- 0.0022046 pound
- 0.03215 ounce Troy
- 0.03527 ounce avoirdupois
- 15.432 grains

1 decigram (dg) = **10** centigrams
1 centigram (cg) = **10** milligrams

Avoirdupois or Commercial Weight

1 gross or long ton =
- **2240** pounds
- 1.016 metric ton
- 1016 kilograms

1 net or short ton = **2000** pounds
1 pound = **16** ounces
- **7000** grains
- **0.45359237** kilogram
- 453.6 grams

1 ounce =
- **1⁄16** pound
- 16 drachms
- **437.5** grains
- 28.3495 grams
- 0.2780139 newton

1 grain Avoirdupois =
- **1** grain apothecaries' weight =
- **1** grain Troy weight
- 0.064799 gram

Troy Weight

Used for Weighing Gold and Silver

1 pound Troy =
- 12 ounces Troy = 5760 grains
- **144⁄175** avoirdupois pound

1 ounce Troy =
- 20 pennyweights = 480 grains
- 31.103 grams

1 pennyweight = 24 grains
1 grain Troy =
- **1** grain avoirdupois
- **1** grain apothecaries' weight
- 0.0648 gram

1 carat (used in weighing diamonds) =
- 3.086 grains
- **200** milligrams = **1⁄5** gram

1 gold karat = **1⁄24** proportion pure gold

Apothecaries' Weight

1 pound = 12 ounces = 5760 grains
1 ounce =
- 8 drachms = 480 grains
- 31.103 grams

1 drachm = 3 scruples = 60 grains
1 scruple = 20 grains

Old Weight Measures

Measures for weight seldom used in the United States:

1 gross or long ton = 20 hundred-weights
1 hundred-weight = 4 quarters = 112 pounds
1 quarter = 28 pounds
1 stone = 14 pounds
1 quintal = 100 pounds

Note: Figures in **bold** indicate exact conversion values.

A.8 Pressure and Stress Conversion Factors

1 kilogram per sq. millimeter (kg_f/mm^2) =
1422.32 pounds per square inch

1 kilogram per sq. centimeter (kg_f/cm^2) =
14.223 pounds per square inch

1 bar =
1,000,000 dynes per square centimeter
1000 millibars
100 kilopascals
750.06168 torr
1.0197162 kilogram force per square centimeter
14.50377 pounds per square inch
29.529983 inches of mercury at 0°C
10,197.162 mm water at 4°C
33.455256 feet of water at 4°C

1 millibar =
100,000 dynes per square centimeter
100 pascal

1 torr =
760 millimeters mercury
$\frac{1}{760}$ atmosphere
133.224 pascal
1.333224 millibar

1 pound per square inch =
144 pounds per square foot
0.068 atmosphere
2.042 inches of mercury at 62°F
27.7 inches of water at 62°F
2.31 feet of water at 62°F
0.0703 kilogram per square centimeter
6.894757 kilopascals
6894.757 pascal

1 atmosphere =
30 inches of mercury at 62°F
14.7 pounds per square inch
2116.3 pounds per square foot
33.95 feet of water at 62°F

1 foot of water at 62°F =
62.355 pounds per square foot
0.433 pound per square inch

1 inch of mercury at 62°F =
1.132 foot of water
13.58 inches of water
0.491 pound per square inch

1 inch of water =
0.0735559 inch mercury at 0°C
1.8683205 torr
0.5780367 ounce force per square inch
0.0024583 atmosphere

Note: Figures in **bold** indicate exact conversion values.

A.9 Energy Conversion Factors

1 horsepower-hour =
0.746 kilowatt-hour
1,980,000 foot-pounds
2545 Btu (British thermal units)
2.64 pounds of water evaporated at 212°F
17 pounds of water raised from 62° to 212°F

1 kilowatt-hour =
1000 watt-hours
1.34 horsepower-hour
2,655,200 foot-pounds
3,600,000 joules
3415 Btu
3.54 pounds of water evaporated at 212°F
22.8 pounds of water raised from 62° to 212°F

A.10 Power Conversion Factors

1 horsepower =
746 watts
0.746 kilowatt
33,000 foot-pounds/minute
550 foot-pounds/second
2545 Btu/hour
42.4 Btu/minute
0.71 Btu/second
2.64 pounds of water evaporated per hour at 212°F

1 kilowatt =
1000 watts
1.34 horsepower
2,654,200 foot-pounds/hour
44,200 foot-pounds/minute
737 foot-pounds/second
3415 Btu/hour
57 Btu/minute
0.95 Btu/second
3.54 pounds of water evaporated per hour at 212°F

1 watt =
1 joule/second
0.00134 horsepower
0.001 kilowatt
3.42 Btu/hour
44.22 foot-pounds/minute
0.74 foot-pound/second
0.0035 pound of water evaporated per hour at 212°F

A.11 Heat Conversion Factors

1 Btu (British thermal unit) =
1052 watt-seconds
778 foot-pounds
0.252 kilogram-calorie
0.000292 kilowatt-hour
0.000393, horsepower-hour
0.00104 pound of water evaporated at 212°F
1 kilogram calorie = 3.968 Btu

1 foot-pound =
1.36 joules
0.000000377 kilowatt-hour
0.00129 Btu
0.0000005 horsepower-hour
1 kilogram-meter =
7.233 foot-pounds

1 joule =
1 watt-second
0.00000078 kilowatt-hour
0.00095 Btu
0.74 foot-pound

A.12 Temperature Conversion Formulas

To Convert	To	Use Formula	To Convert	To	Use Formula
Celsius, t_C	K, t_K	$t_K = t_C + 273.15$	Kelvin, t_K	°C, t_C	$t_C = t_K - 273.15$
	°F, t_F	$t_F = \frac{9}{5} t_C + 32$		°F, t_F	$t_F = \frac{9}{5} t_K - 459.67$
	R, t_R	$t_R = \frac{9}{5}(t_C + 273.15)$		R, t_R	$t_R = \frac{9}{5} \times t_K$
Fahrenheit, t_F	K, t_K	$t_K = \frac{5}{9}(t_F + 459.67)$	Rankine, t_R	K, t_K	$t_K = \frac{5}{9} \times t_R$
	°C, t_C	$t_C = \frac{5}{9}(t_F - 32)$		°C, t_C	$t_C = \frac{5}{9} \times t_R - 273.15$
	R, t_R	$t_R = t_F + 459.67$		°F, t_F	$t_F = t_R - 459.67$

A.13 Gage Block Sets — Inch Sizes

Table 1. Gage Block Sets—Inch Sizes *Federal Specification GGG-G-15C*

Set Number 1 (81 Blocks)										
First Series: 0.0001 Inch Increments (9 Blocks)										
.1001	.1002	.1003	.1004	.1005	.1006	.1007	.1008	.1009		
Second Series: 0.001 Inch Increments (49 Blocks)										
.101	.102	.103	.104	.105	.106	.107	.108	.109	.110	
.111	.112	.113	.114	.115	.116	.117	.118	.119	.120	
.121	.122	.123	.124	.125	.126	.127	.128	.129	.130	
.131	.132	.133	.134	.135	.136	.137	.138	.139	.140	
.141	.142	.143	.144	.145	.146	.147	.148	.149		
Third Series: 0.050 Inch Increments (19 Blocks)										
.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	
.550	.600	.650	.700	.750	.800	.850	.900	.950		
Fourth Series: 1.000 Inch Increments (4 Blocks)										
1.000	2.000	3.000	4.000							
Set Number 5 (21 Blocks)										
First Series: 0.0001 Inch Increments (9 Blocks)										
.0101	.0102	.0103	.0104	.0105	.0106	.0107	.0108	.0109		
Second Series: 0.001 Inch Increments (11 Blocks)										
.010	.011	.012	.013	.014	.015	.016	.017	.018	.019	.020
One Block 0.01005 Inch										
Set Number 6 (28 Blocks)										
First Series: 0.0001 Inch Increments (9 Blocks)										
.0201	.0202	.0203	.0204	.0205	.0206	.0207	.0208	.0209		
Second Series: 0.001 Inch Increments (9 Blocks)										
.021	.022	.023	.024	.025	.026	.027	.028	.029		
Third Series: 0.010 Inch Increments (9 Blocks)										
.010	.020	.030	.040	.050	.060	.070	.080	.090		
One Block 0.02005 Inch										
Long Gage Block Set Number 7 (8 Blocks)										
Whole Inch Series (8 Blocks)										
5	6	7	8	10	12	16	20			
Set Number 8 (36 Blocks)										
First Series: 0.0001 Inch Increments (9 Blocks)										
.1001	.1002	.1003	.1004	.1005	.1006	.1007	.1008	.1009		
Second Series: 0.001 Inch Increments (11 Blocks)										
.100	.101	.102	.103	.104	.105	.106	.107	.108	.109	.110
Third Series: 0.010 Inch Increments (8 Blocks)										
.120	.130	.140	.150	.160	.170	.180	.190			
Fourth Series: 0.100 Inch Increments (4 Blocks)										
.200	.300	.400	.500							
Whole Inch Series (3 Blocks)										
1	2	4								
One Block 0.050 Inch										
Set Number 9 (20 Blocks)										
First Series: 0.0001 Inch Increments (9 Blocks)										
.0501	.0502	.0503	.0504	.0505	.0506	.0507	.0508	.0509		
Second Series: 0.001 Inch Increments (10 Blocks)										
.050	.051	.052	.053	.054	.055	.056	.057	.058	.059	
One Block 0.05005 Inch										

Set number 4 is not shown, and the Specification does not list a set 2 or 3.
Arranged here in incremental series for convenience of use.

A.14 Gage Block Sets — Metric Sizes

Table 2. Gage Block Sets—Metric Sizes ***Federal Specification GGG-G-15C***

Set Number 1M (45 Blocks)									
First Series: 0.001 Millimeter Increments (9 Blocks)									
1.001	1.002	1.003	1.004	1.005	1.006	1.007	1.008	1.009	
Second Series: 0.01 Millimeter Increments (9 Blocks)									
1.01	1.02	1.03	1.04	1.05	1.06	1.07	1.08	1.09	
Third Series: 0.10 Millimeter Increments (9 Blocks)									
1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	
Fourth Series: 1.0 Millimeter Increments (9 Blocks)									
1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	
Fifth Series: 10 Millimeter Increments (9 Blocks)									
10	20	30	40	50	60	70	80	90	
Set Number 2M (88 Blocks)									
First Series: 0.001 Millimeter Increments (9 Blocks)									
1.001	1.002	1.003	1.004	1.005	1.006	1.007	1.008	1.009	
Second Series: 0.01 Millimeter Increments (49 Blocks)									
1.01	1.02	1.03	1.04	1.05	1.06	1.07	1.08	1.09	1.10
1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.20
1.21	1.22	1.23	1.24	1.25	1.26	1.27	1.28	1.29	1.30
1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.40
1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	
Third Series: 0.50 Millimeter Increments (19 Blocks)									
0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	
Fourth Series: 10 Millimeter Increments (10 Blocks)									
10	20	30	40	50	60	70	80	90	100
One Block 1.0005 mm									
Set Number 4M (45 Blocks)									
First Series: 0.001 Millimeter Increments (9 Blocks)									
2.001	2.002	2.003	2.004	2.005	2.006	2.007	2.008	2.009	
Second Series: 0.01 Millimeter Increments (9 Blocks)									
2.01	2.02	2.03	2.04	2.05	2.06	2.07	2.08	2.09	
Third Series: 0.10 Millimeter Increments (9 Blocks)									
2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	
Fourth Series: 1 Millimeter Increments (9 Blocks)									
1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	
Fifth Series: 10 Millimeter Increments (9 Blocks)									
10	20	30	40	50	60	70	80	90	
Long Gage Block Set Number 8M (8 Blocks)									
Whole Millimeter Series (8 Blocks)									
125	150	175	200	250	300	400	500		

Set numbers 3M, 5M, 6M, and 7M are not listed.

Arranged here in incremental series for convenience of use.

Note: Gage blocks measuring 1.09 millimeters and under in set number 1M, blocks measuring 1.5 millimeters and under in set number 2M, and block measuring 1.0 millimeter in set number 4M are not available in tolerance grade 0.5.

B

ANSWERS TO SELECTED EXERCISES

B.1 Chapter 1 Exercises

1.1

a) $>$
b) $\leq$
c) $\sqrt{\ }$
d) $|\ |$
e) α
f) β
g) $\perp$
h) $\measuredangle$

1.2

a) III
b) XVIII
c) CLX
d) CXXXIII
e) DXXIV
f) MI
g) MMV
h) MMMMCCLXII

1.3

a) 4
b) 24
c) 251
d) 414
e) 906
f) 333
g) 706
h) 2017

1.4

a) Sixty-seven thousand
b) Three million eight hundred eighty thousand one hundred thirty-one
c) Two hundred five thousand nine
d) Twenty-seven thousand three hundred ninety-three
e) Seven million one hundred twenty-three thousand two hundred twenty-six
f) Eighteen million three hundred twenty-three thousand five hundred sixteen
g) One hundred four million three hundred sixty-two thousand four hundred fifty-six
h) Nine hundred thirty-two million four hundred eighteen thousand two hundred seven

1.5

a) Reflexive
b) Symmetric
c) Transitive
d) Multiplicative identity

1.5

e) Reflexive
f) Additive identity
g) Commutative property of addition
h) Multiplicative identity
i) Associative property of multiplication
j) Distributive property (commutative property of multiplication)
k) Commutative property of multiplication
l) Distributive property of multiplication over addition

1.6

a) $A \cup B = \{\text{dog, cat, bird, bear, dolphin, goat, fish}\}$
b) $A \cap B = \{\ \}$ or $\varnothing$

1.7

a) $\{0, 3, 10, 12, 23, 99, 131\}$
b) $\{3\}$

1.8

a) False $\left(\frac{1}{3}\right)$
b) False $(-1 \in Z \cap -1 \notin N)$
c) True
d) True
e) False (e.g. π)
f) True

1.9

a) 12
b) 42
c) 42
d) 18
e) 15
f) 16
g) 12
h) 32

1.10

a) 16
b) 10
c) 12
d) 19
e) 14
f) 8
g) 2
h) 6

1.11

a) 108
b) 1536
c) 3
d) 1056
e) 4
f) 2
g) 3
h) 1008
i) 600
j) 3

1.12

a) 30
b) $537\frac{5}{17}$
c) $273\frac{7}{16}$
d) $30\frac{10}{33}$
e) $2100\frac{1}{2}$
f) $71\frac{8}{19}$

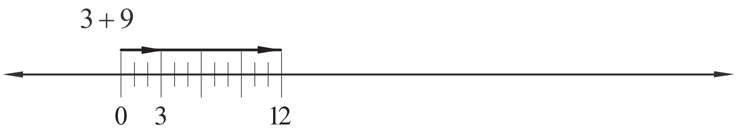
1.13

a) 22
b) 62
c) 43
d) 23
e) 22
f) 13
g) 27
h) 25
i) 34
j) 63

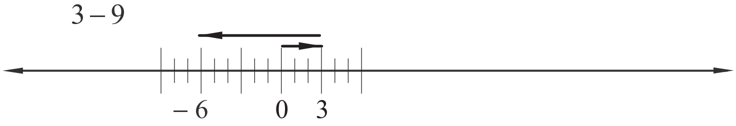
B.2 Chapter 2 Exercises

2.1

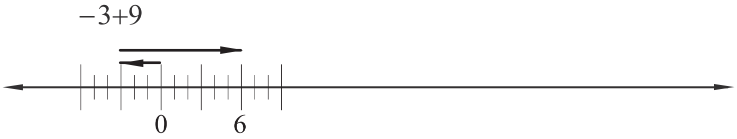
a)



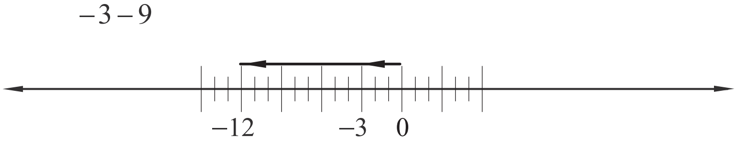
b)



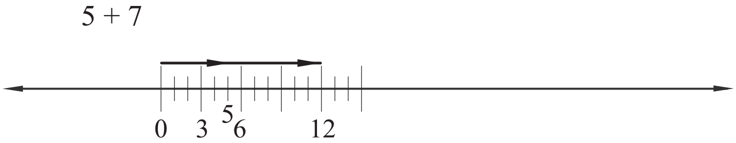
c)



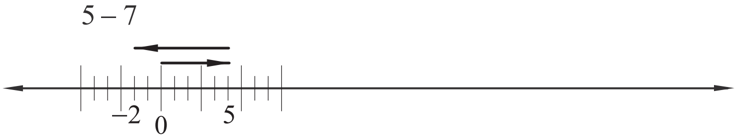
d)



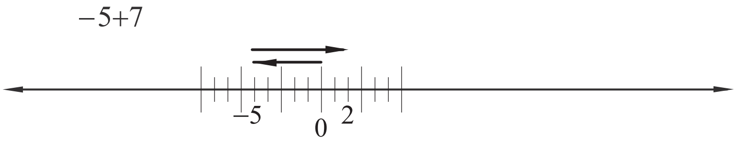
e)



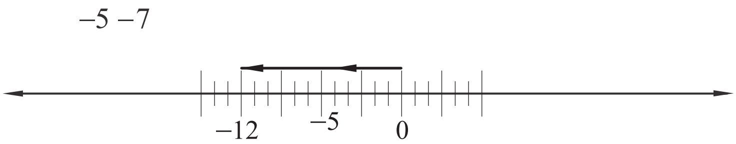
f)



g)



h)



2.2

a) 4 **b)** –6 **c)** –8 **d)** 10
e) 0 **f)** 3 **g)** –3 **h)** –7

2.3

a) 4 **b)** 6 **c)** 8 **d)** 10
e) 0 **f)** 3 **g)** 3 **h)** 7

2.4

a) 0 **b)** 0 **c)** 2 **d)** –2
e) 28 **f)** –8 **g)** –21 **h)** 1

2.5

a) 30 **b)** 15 **c)** 42 **d)** 5
e) 3 **f)** 14 **g)** –30 **h)** –75
i) –10 **j)** 0

2.6

a) 108 **b)** –108 **c)** –108 **d)** –9
e) 12 **f)** 90 **g)** –90 **h)** 90
i) –15 **j)** 6 **k)** –30 **l)** 30
m) –30 **n)** 10 **o)** 3

2.7

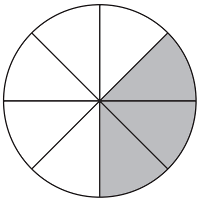
a) 6 **b)** 14 **c)** 32 **d)** –30
e) –144 **f)** 44 **g)** 25 **h)** 33
i) 0 **j)** 19 **k)** 13 **l)** –22

B.3 Chapter 3 Exercises

3.1

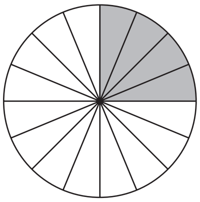
a)

$\frac{3}{8}$



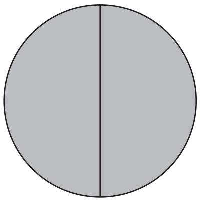
b)

$\frac{4}{16}$



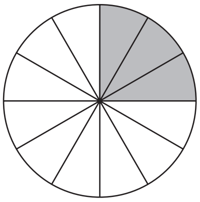
c)

$\frac{2}{2}$



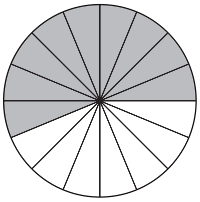
d)

$\frac{3}{12}$



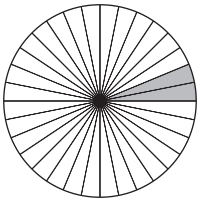
e)

$\frac{9}{16}$



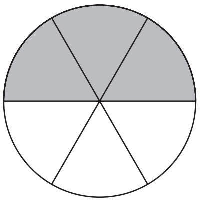
f)

$\frac{2}{32}$



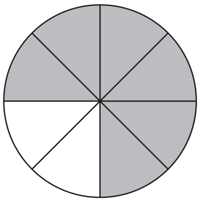
g)

$\frac{3}{6}$



h)

$\frac{6}{8}$



3.2

a) Improper **b)** Proper **c)** Mixed **d)** Proper
e) Improper **f)** Proper **g)** Mixed **h)** Improper

3.3

a) $2\frac{2}{5}$ **b)** $7\frac{3}{5}$ **c)** $12\frac{1}{3}$ **d)** $40\frac{1}{8}$

e) $12\frac{1}{8}$ **f)** $19\frac{1}{3}$ **g)** 4

3.4

a) $\frac{13}{2}$ **b)** $\frac{11}{4}$ **c)** $\frac{79}{8}$

d) $\frac{153}{12}$ **e)** $\frac{129}{5}$ **f)** $\frac{258}{13}$

3.5

a) 8 **b)** 10 **c)** 18 **d)** 44

3.6

a) 6 **b)** 18 **c)** 48 **d)** 32

3.7 40
3.8 68
3.9 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
3.10 An even number is $2 \times n$, so 2 is a factor.
3.11 3 does not divide 803. 3 does divide 1704

3.12

a) 2, 37 **b)** 2, 3, 11 **c)** 3, 11 **d)** 2, 3, 7
e) 2, 3, 7 **f)** 5, 11 **g)** 7, 13 **h)** 3, 23

3.13

a) $\frac{1}{5}$ **b)** $\frac{13}{15}$ **c)** $\frac{21}{23}$ **d)** $\frac{4}{5}$

e) $\frac{7}{8}$ **f)** $\frac{3}{4}$ **g)** $\frac{1}{2}$ **h)** $\frac{7}{8}$

i) $19\frac{1}{3}$ **j)** $5\frac{1}{5}$ **k)** $7\frac{1}{2}$ **l)** $9\frac{2}{9}$

3.14

a) 16 **b)** 126 **c)** 40 **d)** 30 **e)** 72 **f)** 72

3.15

a) 4 **b)** $\frac{5}{8}$ **c)** $-\frac{9}{32}$ **d)** $\frac{5}{16}$

e) $1\frac{5}{9}$ **f)** $\frac{27}{64}$ **g)** $\frac{11}{24}$ **h)** $\frac{5}{8}$

i) $\frac{59}{60}$ **j)** $\frac{23}{36}$ **k)** $1\frac{5}{12}$ **l)** $\frac{23}{168}$

3.15

m) $\frac{44}{45}$ **n)** $\frac{13}{60}$ **o)** $\frac{17}{60}$ **p)** $\frac{2}{21}$

q) $\frac{13}{16}$

3.16 $\frac{9}{16}$ inch **3.17** $3\frac{1}{2}$ inches **3.18** $5\frac{11}{16}$ inches

3.19 $1\frac{13}{24}$ lb **3.20** $1\frac{1}{4}$ inches **3.21** $2\frac{9}{16}$ inches

3.22
$A = \frac{23}{32}$ inch
$B = 1\frac{11}{16}$ inches
$C = 2\frac{3}{8}$ inches
$D = 3\frac{5}{8}$ inches
$E = 5$ inches
$F = 5\frac{11}{16}$ inches

3.23 $2\frac{5}{8}$ inches

3.24 Horizontal
$A = \frac{7}{16}$ inch
$B = 1\frac{3}{32}$ inches
$C = 2$ inches
$D = 2\frac{13}{16}$ inches
Vertical
$E = \frac{7}{16}$ inch
$F = \frac{31}{32}$ inch
$G = 1\frac{15}{32}$ inches
$H = 2\frac{13}{32}$ inches

3.25

a) $5\frac{5}{16}$ **b)** $10\frac{17}{24}$ **c)** $6\frac{27}{32}$ **d)** $9\frac{3}{8}$

e) $8\frac{13}{24}$ **f)** $3\frac{113}{120}$ **g)** $10\frac{29}{30}$ **h)** $26\frac{7}{18}$

3.26 $4\frac{1}{8}$ inches **3.27** $3\frac{3}{4}$ inches long, $1\frac{1}{16}$ inches wide

3.28 $1\frac{3}{8}$ inches wide, $4\frac{7}{32}$ inches long **3.29** $\frac{55}{64}$ inch

3.30 $A = 1\frac{1}{8}$ inches, $B = \frac{27}{32}$ inch, $C = \frac{13}{16}$ inch, $D = 4\frac{13}{32}$ inches

3.31 60 feet $3\frac{3}{8}$ inches

3.32 1373′ 2″

3.33

a) $\frac{1}{12}$ **b)** $\frac{1}{8}$ **c)** $\frac{2}{3}$ **d)** 6 **e)** $\frac{3}{4}$

f) $\frac{3}{8}$ **g)** $\frac{2}{3}$ **h)** $\frac{10}{27}$ **i)** $1\frac{1}{2}$ **j)** 4

3.34

a) $\frac{4}{21}$ **b)** $\frac{1}{3}$ **c)** $\frac{1}{2}$ **d)** $\frac{1}{12}$

e) $\frac{80}{429}$ **f)** 4 **g)** $\frac{1}{6}$ **h)** $\frac{1}{30}$

i) 7 **j)** $\frac{64}{175}$

3.35

a) 1 **b)** $\frac{1}{5}$ **c)** $\frac{49}{72}$ **d)** $\frac{125}{378}$ **e)** 3

f) $\frac{1}{80}$ **g)** $\frac{47}{125}$ **h)** $\frac{9}{56}$ **i)** $\frac{2}{3}$ **j)** 24

3.36 $\frac{11}{80}$

3.37 $93\frac{1}{3}$ lb tin
$74\frac{2}{3}$ lb copper
56 lb zinc

3.38 135 fittings

3.39 9 holes

3.40 236 discs

3.41 $\frac{3}{32}$ inch

3.42 1440 Blocks

3.43

a) $9\frac{1}{6}$ **b)** $12\frac{927}{1328}$ **c)** $485\frac{19}{24}$ **d)** $17\frac{23}{32}$

e) $561\frac{1}{20}$ **f)** $3\frac{41}{88}$ **g)** $11\frac{4}{19}$ **h)** $4\frac{37}{896}$

i) $70\frac{64}{77}$ **j)** $51\frac{171}{320}$ **k)** 44 **l)** $23\frac{49}{160}$

3.43

m) $\frac{1}{8}$ **n)** $10\frac{5}{32}$ **o)** $2\frac{35}{108}$ **p)** $28\frac{17}{38}$
q) $4\frac{32}{489}$

3.44 $116\frac{2}{3}$ lb **3.45** $24\frac{25}{32}$ inches **3.46** $15\frac{2}{3}$ inches
3.47 $84\frac{7}{32}$ gallons **3.48** 23 nipples **3.49** $13.65
3.50 $8119\frac{1}{5}$ lb **3.51** $2077\frac{3}{4}$ lb

3.52

a) $1\frac{1}{6}$ **b)** $\frac{3}{4}$ **c)** $\frac{4}{5}$ **d)** $\frac{3}{8}$
e) $1\frac{4}{9}$ **f)** $1\frac{1}{9}$ **g)** $6\frac{2}{3}$ **h)** $\frac{45}{128}$
i) $21\frac{2}{3}$ **j)** $\frac{31}{16}$ or $1\frac{15}{16}$ **k)** $\frac{8}{5}$ or $1\frac{3}{5}$ **l)** $\frac{43}{91}$

3.53 $\frac{5}{8}$inch

B.4 Chapter 4 Exercises

4.1
a) Three and forty-six hundredths
b) Zero and six hundred ninety-three ten thousandths
c) Four hundred twenty-five and three hundred fifty-six thousandths
d) Fifty-three and nine hundred sixty-three thousandths
e) Two hundred thirty-seven and two thousand five hundred sixty-two ten thousandths

4.2
a) 200.011 **b)** 216.0001 **c)** 525.0319 **d)** 3.000015 **e)** 0.080

4.3
a) 0.375, 0.25, 0.04, 0.2, 0.04, 5.046875
b) 0.75, 0.625, 0.1875, 0.25, 0.34375, 0.875, 0.328125
c) 0.333…, 0.666…, 0.777…, 0.230769, 0.230769230769…, 0.46875, 0.390625, 0.578125, 0.59375

4.4

a) 3.875 **b)** 6.234375 **c)** 2.171875 **d)** 9.53125
e) 1.203125 **f)** 6.375 **g)** 0.046875 **h)** 12.625

4.5

a) $\frac{4}{5}$ **b)** $\frac{1}{25}$ **c)** $\frac{3}{8}$ **d)** $1\frac{27}{100}$
e) $15\frac{7}{1000}$ **f)** $6\frac{83}{200}$ **g)** $\frac{9}{100}$ **h)** $121\frac{7}{8}$
i) $\frac{9}{2000}$ **j)** $8\frac{3}{50}$ **k)** $2\frac{3}{4}$ **l)** $18\frac{1}{125}$

4.6

a) $\frac{1}{64}$ **b)** $6\frac{7}{16}$ **c)** $2\frac{55}{64}$ **d)** $\frac{13}{16}$
e) $3\frac{3}{4}$ **f)** $\frac{1}{8}$ **g)** $5\frac{31}{32}$ **h)** $\frac{13}{32}$
i) $\frac{301}{1}$ **j)** $77\frac{9}{250}$ **k)** $3\frac{333}{1000}$ **l)** $\frac{127}{40000}$

4.7

a) 1.645 **b)** 1.227 **c)** 1.801 **d)** 4.6225

4.8

a) 2.561 **b)** 0.9766

4.9

a) 5.7637 **b)** 0.712 **c)** 0.34375 **d)** 3
e) 8.875 **f)** 6.3501 **g)** 8.5898 **h)** 0.25
i) 11.7155 **j)** 1.40625 **k)** 7.8541666… **l)** $\frac{11}{32}$ or 0.34375

4.10

a) 2.250 **b)** 3.750 **c)** 1.500 **d)** 1.500 **e)** 3.5 **f)** 3.5

4.11

a) 4.50 **b)** 5.25 **c)** 6.00 **d)** 1.25

4.12

a) 1.75 **b)** 2.500 **c)** 3.250 **d)** 4.25

4.13

a) 539.08 **b)** 116.798 **c)** 130.515

4.14

a) 19.645 **b)** 20.597 **c)** 12.875 **d)** 34.155

4.15

a) 0.3761 **b)** 5.51 **c)** 9.9725 **d)** 1.25

4.16

a) 2.96
b) 9.07

4.17

a) 286.1875
b) 155.7573864
c) 2.7209226
d) 0.018840352
e) 0.02242656
f) 3104.9724
g) 8.63306233
h) 0.0000003304
i) 0.000003015
j) 4.823758
k) 0.00703501
l) 59.21
m) 2.9375
n) 586.0512

4.18

a) 60655.96195
b) 35844.66
c) 6.08125
d) 72746000000
e) 119160
f) 2.635
g) 4944.9048
h) 4.597402597
i) 16
j) 9.673076923
k) 19411

4.19

a) 4.94636612
b) 15.17965093

4.20

a) 21.11429582
b) 17.000125

4.21

a) 32.2
b) 993
c) 0.002
d) 15.26
e) 3260
f) 1.3973
g) 635.22
h) 900
i) 18.336
j) 0

4.22

a) 0.4675, 0.467, 0.47
b) 58, 57.9, 57.895
c) 5.56, 5.6, 6
d) 0.0059387
e) 0.005953
f) 4568
g) 4600
h) 45.7
i) 83
j) 0.778

4.23

a) 0.969 inch
b) 0.992 inch

4.24

a) 855.5
b) 868.47

4.25

a) 87.0
b) 2256.4

4.26

a) 4200
b) 80
c) 583620

4.27 Precision means that there is a high probability of consistency. Accuracy is a measure of variance from the actual value.

4.28 Calibration error of measuring device, or operator error.

4.29 For 1.125 inches: use 1 inch and 0.125 inch gage blocks. For 0.2188 inch: use 0.1002 inch, 0.1006 inch, and 0.018 inch gage blocks.

4.30 For 2.365 inches: use 2 inches, 0.350 inch, and 0.015 inch gage blocks.

4.31 Depth = 0.0475 inch

4.32 0.11 inch

B.5 Chapter 5 Exercises

5.1

a) 75% **b)** 12.90% **c)** 97.06% **d)** 6.25%
e) 29.23% **f)** 38.89% **g)** 17.31% **h)** 18.18%

5.2

a) 0.68 **b)** 0.75 **c)** 1.25 **d)** 0.91
e) 0.02666… **f)** 0.0381818… **g)** 0.43 **h)** 2.33

5.3

a) $\frac{1}{2}$ **b)** $5\frac{5}{16}$ **c)** $\frac{147}{625}$ **d)** $\frac{33}{100}$
e) $\frac{169}{400}$ **f)** $\frac{19}{300}$ **g)** $1\frac{1}{10}$ **h)** $\frac{1299}{10000}$

5.4

a) 7.25% **b)** 115% **c)** 0.25%
d) 18.75% **e)** 1025% **f)** 0.1352%

5.5 9% **5.6** 288 board feet **5.7** 87.5% **5.8** 2760

5.9 $3375 **5.10** 2786.28 square feet **5.11** 1204 additional blocks

5.12

a) $3510 **b)** $4212

5.13 $226666.67 **5.14** 12% **5.15** $313

5.16 $2, 6\frac{6}{7}$ **5.17** $23\frac{17}{21}\%$ or 23.81%

5.18 14.608 inches minimum, 14.643 inches maximum
5.19 0.005 inch
5.20 7.225 lb tin, 0.85 lb antimony, 0.34 lb copper, 0.085 lb lead
5.21 21.12 lb tin, 39 lb lead
5.22 3798.075 lb iron, 383.59 lb nickel
5.23 199 lb water
5.24 308.755... cubic feet $H_2O \approx$ 309 cubic feet

5.25

a) $150.00 **b)** $509.60 **c)** $1968.75 **d)** $618.75
e) $543.38 **f)** $236.00 **g)** $127.50 **h)** $119.00

5.26

a) $600 **b)** $800 **c)** $187.50 **d)** $1250
e) $100 **f)** $1175 **g)** $452.75 **h)** $245.38

5.27

a) 40% **b)** 50% **c)** 75% **d)** 20%
e) 17.5% **f)** 30% **g)** 2.5% **h)** 5%

5.28

a) \$90,000 **b)** \$105,636 **c)** \$120,000 **d)** \$125,457
e) \$150,000 **f)** \$174,381 **g)** \$180,000 **h)** \$189,162

5.29 \$3200, \$5440, \$6732, \$12,790.80 **5.30** \$38.75

5.31 12% discount **5.32** 20% **5.33** 40% **5.34** 30.77% **5.35** \$28

B.6 Chapter 6 Exercises

6.1

a) Yes **b)** Exponent: 8; Base: 2; Power: 256

6.2 10000

6.3

a) –4 **b)** 4 **c)** –8 **d)** 243
e) 15625 **f)** 17 **g)** –8 **h)** –1
i) –1 **j)** 0 **k)** $\frac{1}{2}$ **l)** –1
m) –1 **n)** 1 **o)** 64 **p)** 1,000,000,000
q) $\frac{32}{25}$

6.4 True; False; True; False

6.5 A negative × a negative gives a positive result.
An odd exponent gives $x^{2n+1} = x^{2n} \cdot x =$ positive × negative.

6.6

a) 6 **b)** 1 **c)** 793 **d)** 1
e) –14 **f)** 1 **g)** 0 **h)** Undefined

6.7

a) $\frac{1}{6}$ **b)** $\frac{1}{4}$ **c)** $\frac{-1}{2744}$ **d)** $\frac{1}{2744}$
e) $\frac{1}{8}$ **f)** $\frac{-1}{9}$ **g)** Undefined **h)** $\frac{1}{1024}$

6.8

a) 1 **b)** 128 **c)** 5 **d)** $\frac{1}{256}$ **e)** 11^{12}
f) $\frac{1}{2401}$ **g)** 537824 **h)** $\frac{-1}{32768}$ **i)** $\frac{1}{2744}$

6.9

a) $\frac{1}{2}$
b) $\frac{3^2}{2^2}$
c) $\frac{3^3}{2^3}$
d) $\frac{1}{2^4}$
e) 16
f) 4^3
g) $\frac{3^2}{4^2}$
h) 5^5
i) $\frac{1}{3^4}$
j) 4^1
k) $\frac{-14}{4^1}$
l) $\frac{-5^2}{4^1}$

6.10

a) $\frac{1}{2}$
b) $\frac{9}{4}$ or $2\frac{1}{4}$
c) $\frac{27}{8}$ or $3\frac{3}{8}$
d) $\frac{1}{16}$
e) $\frac{1}{81}$
f) 16
g) 16
h) 64
i) 3125
j) $\frac{9}{16}$
k) 4
l) 125

6.11

a) 8
b) 5
c) ≈ 6.245
d) ≈ 18.581
e) ≈ 0.7987
f) ≈ 7.0071
g) ≈ 6.708
h) 9
i) ≈ 24.8998
j) ≈ 2.7568
k) $\frac{3}{8}$
l) $\frac{1}{2}$
m) ≈ 15.3297
n) 38
o) ≈ 1.8028
p) ≈ 6.0249
q) $\frac{2}{5}$
r) $\frac{4}{7}$

6.12

a) 3
b) ≈ 3.3636
c) 4
d) ≈ 2.6265
e) -4
f) ≈ -6.3096
g) ≈ 3.4731
h) ≈ 3.1896
i) -6
j) ≈ 2.8477
k) ≈ -4.0996
l) 2

6.13

a) $64^{\frac{1}{2}}$
b) $25^{\frac{1}{2}}$
c) $39^{\frac{1}{2}}$
d) $27^{\frac{1}{3}}$
e) $256^{\frac{1}{4}}$
f) $128^{\frac{1}{4}}$
g) $64^{\frac{1}{6}}$
h) $125^{\frac{1}{5}}$
i) $(-216)^{\frac{1}{3}}$
j) $(-64)^{\frac{1}{3}}$
k) $(-10000)^{\frac{1}{5}}$
l) $505.36^{\frac{1}{5}}$
m) $32.45^{\frac{1}{3}}$
n) $(-68.9)^{\frac{1}{3}}$
o) $4325^{\frac{1}{8}}$

6.14

a) 4
b) 125
c) -4
d) $\frac{1}{4}$

6.14

e) 1,000,000 **f)** –243 **g)** $\frac{-1}{243}$ **h)** ≈ 0.3536

6.15

a) 1.0×10^{4} **b)** 1×10^{-4} **c)** 1.73×10^{5}
d) -2.35×10^{2} **e)** 1.5583×10^{-1} **f)** 1.031023×10^{6}
g) 1.445×10^{-3} **h)** -2.23589×10^{-1}

6.16

a) 3.3712×10^{28} **b)** -2.820×10^{9} **c)** -1.95833×10^{11}
d) 1.075×10^{19} **e)** 5.871×10^{15} **f)** 6.048464×10^{4}
g) $5.232235294\times10^{-21}$ **h)** 1.626315789×10^{3}

6.17

a) 5 **b)** 2.658964843 **c)** 3.897021956 **d)** 0
e) 3 **f)** 0.3010299957 **g)** –0.2041199827 **h)** –0.4814860601

6.18

a) 1,000,000 **b)** 0.5956621435 **c)** 1.258925412
d) 0.316227766 **e)** 4.2169 **f)** 10
g) 0.1 or $\frac{1}{10}$ **h)** $\frac{1}{100000} = 0.00001$

6.19

a) –1.108662625 **b)** 11.51292546 **c)** 6.12249281
d) 8.973224663 **e)** 0 **f)** 6.907755279
g) 0.6931471806 **h)** –0.4700036292

6.20

a) 403.4287935 **b)** 0.7985162188 **c)** 1.105170918
d) 0.6065306597 **e)** 1.868245957 **f)** 2.718281828
g) 0.3678794412 **h)** 0.006737947

B.7 Chapter 7 Exercises

7.1 The English system of measurements is based on inches, ounces, seconds, and gallons. There is no unifying factor between units and it is used mostly in the United States.

The S.I. (International System) is based on meters, grams, liters, seconds. Most units are the base units raised to various powers of 10. Computations are simplified in this system.

7.2 Converting between units in the English system requires memorizing many facts, whereas, in the metric system, it usually involves moving only the decimal place. Both are equally useful as far as precision and accuracy are concerned.

7.3

a) Mega- **b)** Milli- **c)** Micro- **d)** Giga-

7.4 A derived unit uses basic units to define itself. For example, force is measured in Newtons, which are derived from the base units of kilogram, meter, and second. Fundamental units are based on physical constants.

7.5 Meter, liter, kilogram, second, ampere, Kelvin, mole, candela

7.6 A Newton is derived from kilogram, meter, and second. Cubic feet is derived from foot.

7.7

a) 201.168 meters **b)** 20116.8 centimeters **c)** 201168 millimeters

7.8

a) 68.2625 millimeters **b)** 0.00004241635 mile **c)** 0.0746527778 yard

7.9

a) 89.69375 millimeters **b)** 102.39375 millimeters
c) 68.2625 millimeters **d)** 151.60625 millimeters
e) 136.525 millimeters **f)** 38.89375 millimeters
g) 261.9375 millimeters **h)** 23.8125 millimeters
i) 163.5125 millimeters **j)** 57.15 millimeters
k) 73.81875 millimeters **l)** 2.38125 millimeters
m) 17.4625 millimeters

7.10

a) 0.625 inch **b)** 185435.2362 inches **c)** 48.58267717 inches
d) 0.3365748031 inch **e)** 0.9687480315 inch **f)** 0.00017952 inch
g) 1.804566929 inches **h)** 0.2695551181 inch

7.11 825 feet **7.12** 8 pieces and 18.4 centimeters scrap

7.13 126 inches **7.14** 210 inches **7.15** $166\frac{2}{3}$ yards
7.16 17 pieces **7.17** 0.15675 kilometer **7.18** 6,115,000 millimeters
7.19 5.32 meters **7.20** 452000 meters **7.21** 3 pieces
7.22 1760 yards **7.23** 434.34 centimeters **7.24** 11.5824 meters

7.25 1600.2 centimeters **7.26** 870 pieces **7.27** 4800 acres
7.28 4047 square meters **7.29** 161.458 square feet **7.30** 1038 loads
7.31 44465.54 cubic centimeters **7.32** 12 board feet

7.33 $666\frac{2}{3}$ or 666.67 board feet

7.34

a) $\frac{2}{3}$ or 0.67 board feet **b)** 1 board feet

7.34

c) 1.33 board feet or $1\frac{1}{3}$ board feet **d)** 2 board feet

7.35 $198.29
7.36 40 board feet per plank, 20,000 board feet total
7.37 426666.67 board feet
7.38 16 board feet/post, 5728 board feet total

7.39

a) 6.3 quarts **b)** 171.429 liters **c)** 0.1333… kl
d) 240 liters **e)** 105 pints

7.40 112.5 miles, 180 kilometers **7.41** 3.81 meters
7.42 9354.8 square millimeters **7.43** 3464.7 meters
7.44 4455.38 feet **7.45** 9984.03 meters
7.46 185.325 acres **7.47** 4620 inches, 117.35 meters
7.48 16390 cubic millimeters **7.49** 28.125 miles, 88 miles per hour
7.50 12.325 liters **7.51** 14.16 cubic meters
7.52 5.77 liters **7.53** 488.10 cubic inches
7.54 5.58 lb/gallon **7.55** 128000 acres, 200 sections
7.56 13.58 inches long, 0.98 inches diameter
7.57 139.7 millimeter × 11.43 millimeter × 2.006 meters

7.58 4.2 gallons **7.59** 48° 51′ 26.28″ **7.60** 35.4256°
7.61 47° 11′ 30″ **7.62** 49° 36′ 51″ **7.63** 72° 35′ 0.24″
7.64 65.3797° **7.65** 65° 4′ 11″ **7.66** 3° 43′ 49″
7.67 46° 0′ 20″ **7.68** 102.9806° or 102° 58′ 50.16″

7.69

a) $1\frac{97}{180}$ turns clockwise **b)** 2.8904 turns counterclockwise
c) 1.019 turns counterclockwise **d)** 2.3903 turns clockwise
e) 1.34 turns clockwise

7.70

a) 0.401 radians **b)** 2.321 radians **c)** 1.138 radians
d) 0.017 radians **e)** 1.309 radians **f)** 1.782 radians
g) 0.340 radians **h)** 6.496 radians

7.71

a) 57.296° **b)** 114.592° **c)** 171.887° **d)** 121.123°
e) 306.475° **f)** 360° **g)** 270° **h)** 225°

7.72

a) 90.857 kg **b)** 81.13 ounces **c)** 1053.8 lb
d) 3.54375 lb **e)** 2.293 tons **f)** 12700.717 kg

7.73 89.8 kg **7.74** 239,655,314 gallons **7.75** $253.53

7.76 Absolute 0 is the temperature at which all matter ceases to move

7.78

a) 10°C **b)** 176.67°C **c)** –18.4°F **d)** 685.15 K **e)** –54.15°C
f) –40°C **g)** 230°F **h)** –40°F **i)** 716.67 K **j)** –49.4 K

7.79 37°C

7.80

a) 6,047,904 calories **b)** 0.4165581 kilowatt hour **c)** 23884.6 calories
d) $\approx 341180.48\dfrac{\text{Btu}}{\text{hour}}$ **e)** 236.95 Btu **f)** 28.1376 kilowatt

7.81 $3.52\dfrac{\text{kg}}{\text{cm}^2}$

7.82 198.04 psi **7.83** 254.7 psia **7.84** 350 feet

7.85 psig = 1.45 psig
kPa gage = 9.99 kPa gauge
psia = 16.15 psia
kPa absolute = 111.32 kPa absolute

7.86 735 psig, 5066.25 kPa gage **7.87** 0.6927 psi

7.88 393 in H_2O, 28.88 in Hg **7.89** 0.0963 microstrain

7.90 4.92×10^{-6} or 4.92 microstrain **7.91** $0.0167\ \dfrac{\text{inch}}{\text{inch}}$

7.92 4.56×10^{-5} or 45.6 microstrain **7.93** $0.004\ \dfrac{\text{cm}}{\text{cm}}$ or 4000 microstrain

B.8 Chapter 8 Exercises

8.1

a) 4, –3, 2 **b)** –2, 1, –4, 7 **c)** –0.1, –2.3, 4 **d)** $1, \dfrac{1}{3}, \dfrac{1}{2}$
e) $\dfrac{1}{2}, -2, 3$ **f)** $\dfrac{1}{4}, -\dfrac{2}{3}, \dfrac{1}{5}$ **g)** 1, 1, –1 **h)** –3.7, 2, 7.2, –3

8.2

a) 1 **b)** 9 **c)** 17

8.3

a) $1\dfrac{5}{6}$ **b)** $-3\dfrac{1}{2}$ **c)** 2

8.5

a) $6\frac{2}{3}$

b) $\frac{1}{3}$

c) $-3\frac{1}{2}$

8.6

a) $6ab$

b) $6ab$

c) x^2z^2

d) x^4y^2z

e) $6bce-9bcg+6bc$

f) $15ab-30bd$

g) z^3

h) x^4y^6

i) $3b$

j) $18xyz$

k) $15abc$

l) ab^2c

m) $12xy-8xz$

n) $6x$

o) $3a+3b-3c$

p) a^2b^2

q) $-4x+4y-4z$

r) $2x$

s) $12b$

t) $2abd$

u) $27a^3b^3$

v) $12xyz-36yz+24xz$

w) x^2-2x

x) $5b+5c-a$

y) w^2

z) $-w^3$

8.7

a) d^{10}

b) $-c^{12}$

c) $w^{.25}$

d) 5

e) x^5y

f) $-8t^2$

g) $\frac{c^{\frac{5}{3}}}{d^{\frac{5}{2}}}$

h) x^6

i) a^3

j) $-5a$

k) 1

l) 1

m) $-6ab$

n) $p^{\frac{18}{5}}d^{\frac{1}{2}}q$

o) $\frac{1}{k}$

p) $\frac{25a^4x^3}{27b^6y^9}$

q) $8x^2y^2$

r) $21y^3$

s) $2a^2b^2$

t) $-16a^2c$

8.8

a) $\sqrt{w},\ \sqrt[3]{z^2},\ \sqrt[3]{a^4},\ \sqrt[5]{q^2},\ \frac{1}{\sqrt{x}},\ \sqrt{p^5},\ \frac{1}{\sqrt[3]{r^2}}$

b) $\sqrt{x-y},\ \sqrt[4]{x+y},\ \sqrt[5]{z^6},\ \sqrt[3]{y^2},\ \sqrt{x^3}$

c) $4\sqrt[3]{p^2},\ \sqrt[3]{16p^2},\ 5\sqrt{2x-3y},\ 5\sqrt{w^3},\ 2\sqrt{x^7}$

d) $-3\sqrt{x},\ \sqrt{-4z},\ -8\sqrt[3]{a},\ 8\sqrt[3]{b^2},\ \sqrt[3]{64n^2}$

e) $3\sqrt{b},\ \sqrt{3b},\ \sqrt{x^2-y^2},\ \sqrt[3]{-8y},\ \sqrt[4]{8w^3}$

8.9

a) $\sqrt[4]{x^3}$

b) $\sqrt[4]{w^9}$

c) $\sqrt[6]{z^{25}}$

8.10

a) $\sqrt[12]{z^5}$ **b)** $\sqrt[4]{x}$ **c)** $\sqrt[4]{q}$

8.11

a) $\frac{3x\sqrt{11}}{11}$ **b)** $\frac{5x^3\sqrt{2x}}{2}$ **c)** $\frac{3x^3\sqrt{5}}{10}$ **d)** $\frac{x\sqrt{y}}{y}$ **e)** $\frac{-x\sqrt{3}}{3}$

f) $\frac{-\sqrt{10x}}{10x}$ **g)** $\frac{x\sqrt{2x}}{8}$ **h)** $\frac{3\sqrt{2x}}{8x}$ **i)** $\frac{\sqrt{x}}{2}$ **j)** $\frac{3\sqrt{x}}{4x}$

8.12

a) $4b-5a$ **b)** $5x+5z-y$ **c)** $7h+k-f$
d) 0 **e)** $3a+7b+3c$ **f)** $7x+4xy+2y$
g) $4a+bc+6c$ **h)** $4xy+yz+2x$ **i)** $10p+3q+5pq$
j) $4w+2b$ **k)** $-2a+3b$ **l)** $3x+7y$
m) $10B+8F+4C+BF$ **n)** $4xyz+2xy+6x+5y$ **o)** $2A+3B+3AB$
p) $9yw+6y+3w$ **q)** $7xy-5y+16x$ **r)** $3r+t+5w$
s) $8x+y$ **t)** $-2x+7y+z$ **u)** $2p+3q$
v) $3x+4xy+y$ **w)** $-2x-4$ **x)** $-2a-6c-2b$
y) $12x^2+2\sqrt{y}$ **z)** $3w^3-3w^2+w$

B.9 Chapter 9 Exercises

9.1

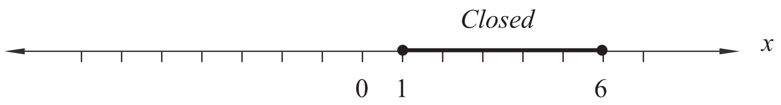
a) 11 **b)** 4 **c)** 10 **d)** 7 **e)** 9
f) 12 **g)** –15 **h)** –30 **i)** 38 **j)** 10
k) 125 **l)** 4 **m)** 5 **n)** 12 **o)** 4
p) 25 **q)** 61 **r)** $3\frac{7}{9}$ **s)** 9 **t)** 625
u) 67 **v)** 1

9.2

a) $\frac{7}{9}$ **b)** $2\frac{13}{99}$

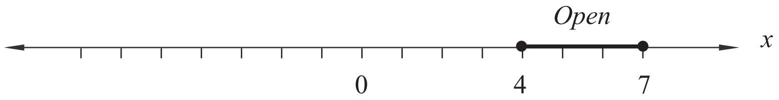
9.3

a)

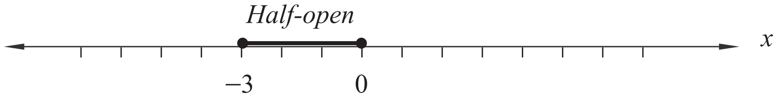


9.3

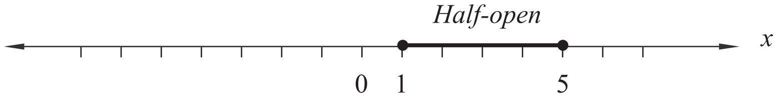
b)



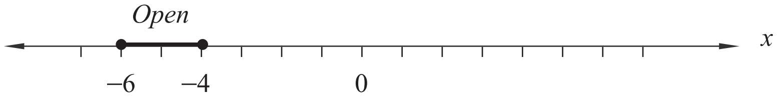
c)



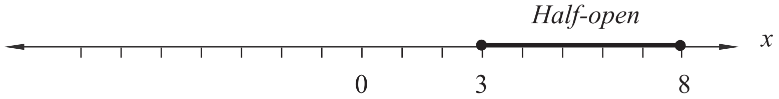
d)



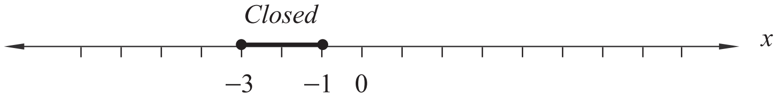
e)



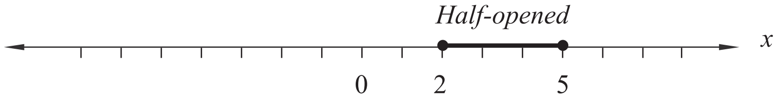
f)



g)



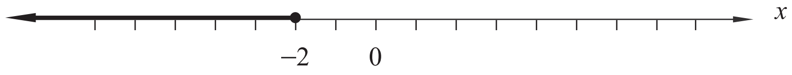
h)



9.4

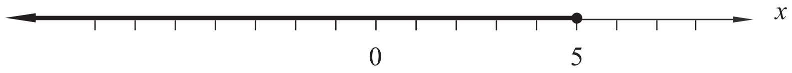
a)

$(-\infty,\ -2)$



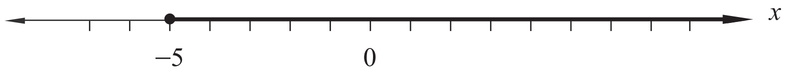
b)

$(-\infty,\ 5]$



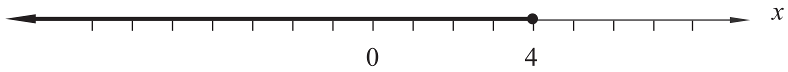
c)

$(-5,\ +\infty)$



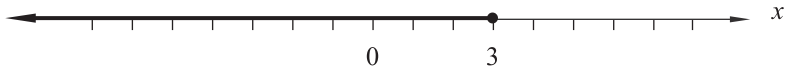
d)

$(-\infty,\ 4)$



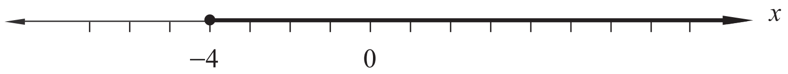
e)

$(-\infty,\ 3]$



f)

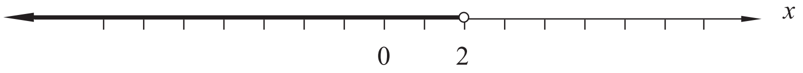
$[-4,\ +\infty)$



9.4

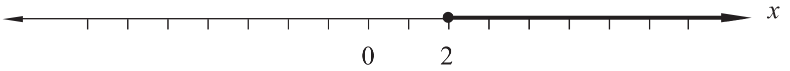
g)

$(-\infty, 2)$



h)

$[2, \infty)$



B.10 Chapter 10 Exercises

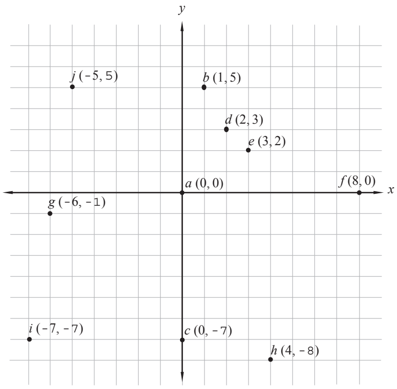
10.1

a) $x = -5,\ y = 5$ **b)** $x = 4,\ y = 20$ **c)** $x = \sqrt{3},\ y = -7$

d) $x = -9,\ y = -13$ **e)** $x = 380,\ y = -74$

10.2 A = (0, 0), B = (6, 3), C = (4, –3), D = (0, –4), E = (–5, 0)

10.3



10.4

a) Quadrant I **b)** Quadrant II **c)** Quadrant III **d)** Quadrant IV

e) y-axis **f)** Quadrant IV **g)** Quadrant IV **h)** x-axis

i) Quadrant III **j)** Quadrant I **k)** Origin **l)** Quadrant III

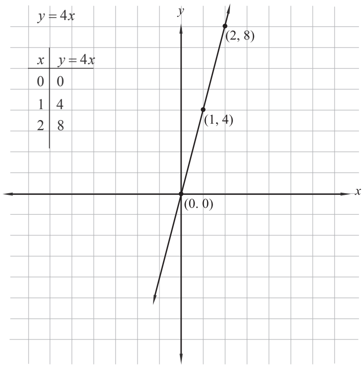
m) Quadrant IV **n)** Quadrant III **o)** Quadrant I

10.5

a) Not a line **b)** Is a line **c)** Not a line **d)** Not a line

10.6

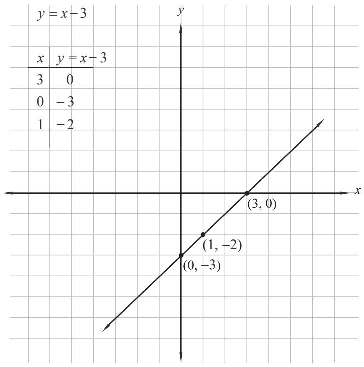
a)



$y = 4x$

x	$y = 4x$
0	0
1	4
2	8

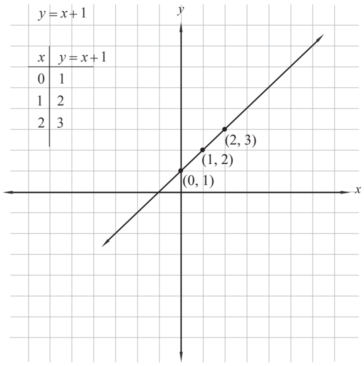
b)



$y = x - 3$

x	$y = x - 3$
3	0
0	−3
1	−2

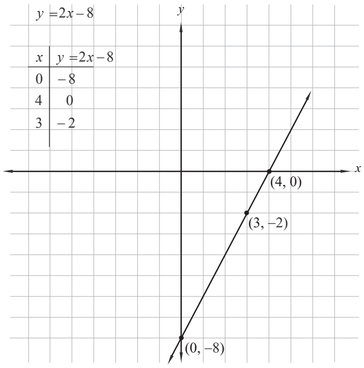
c)



$y = x + 1$

x	$y = x + 1$
0	1
1	2
2	3

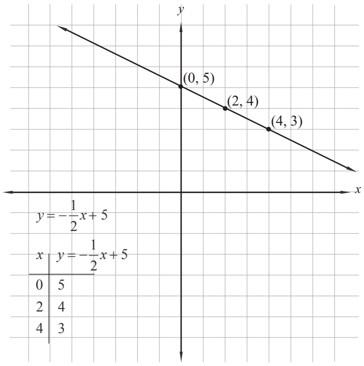
d)



$y = 2x - 8$

x	$y = 2x - 8$
0	−8
4	0
3	−2

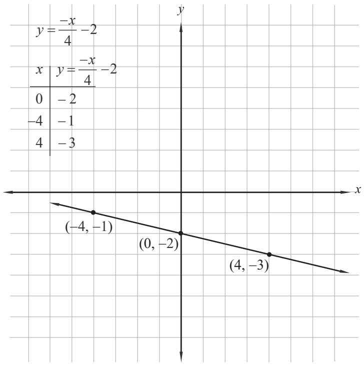
e)



$y = -\frac{1}{2}x + 5$

x	$y = -\frac{1}{2}x + 5$
0	5
2	4
4	3

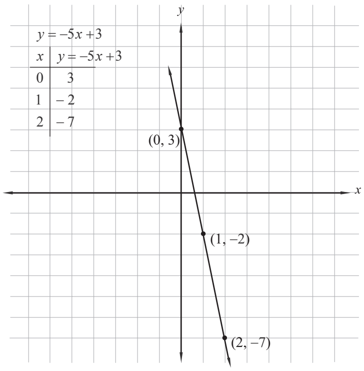
f)



$y = \frac{-x}{4} - 2$

x	$y = \frac{-x}{4} - 2$
0	−2
−4	−1
4	−3

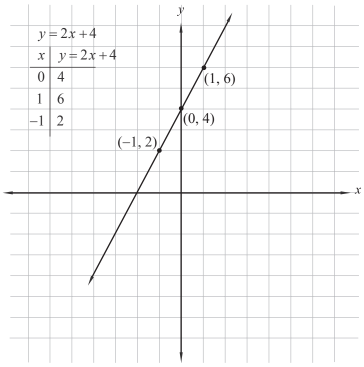
g)



$y = -5x + 3$

x	$y = -5x + 3$
0	3
1	−2
2	−7

h)

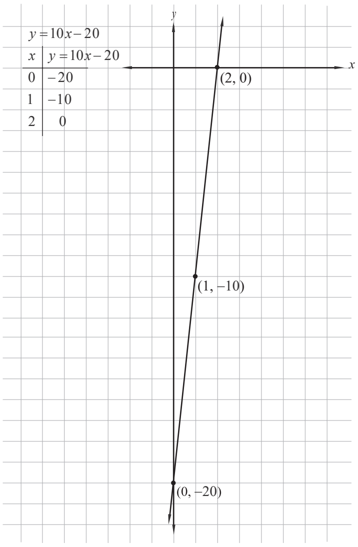


$y = 2x + 4$

x	$y = 2x + 4$
0	4
1	6
−1	2

10.6

i)



10.7

a) Yes **b)** No **c)** Yes **d)** No **e)** Yes

10.8

a) 2 **b)** $\frac{1}{2}$ **c)** 2 **d)** 0 **e)** $\dfrac{y_1 - y_2}{x_1 - x_2}$

10.9

a) Rise **b)** Fall **c)** Rise **d)** Neither **e)** Fall

10.10

a) $\text{slope } = \frac{2}{3},\ y\text{-intercept} = (0,\ -5)$

b) $\text{slope } = -10,\ y\text{-intercept} = \left(0,\ \frac{9}{5}\right)$

c) $\text{slope} = 4,\ y\text{-intercept} = (0,\ -17)$

d) $\text{slope} = -3,\ y\text{-intercept} = (0,\ 16)$

e) $\text{slope} = 8,\ y\text{-intercept} = (0,\ -22)$

f) $\text{slope} = 1,\ y\text{-intercept} = (0,\ 0)$

g) $\text{slope} = -1,\ y\text{-intercept} = (0,\ 14)$

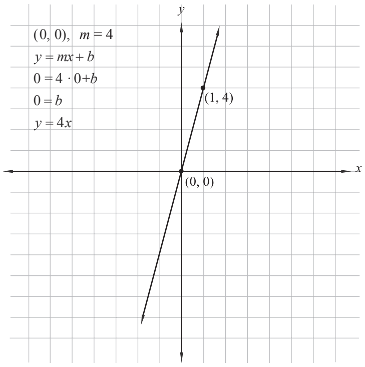
h) $\text{slope} = 0,\ y\text{-intercept} = (0,\ 2)$

i) $\text{slope} = -11,\ y\text{-intercept} = (0,\ 36)$

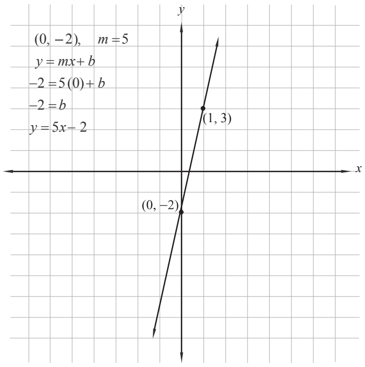
j) $\text{slope} = -\frac{2}{9},\ y\text{-intercept} = (0,\ 1)$

10.11

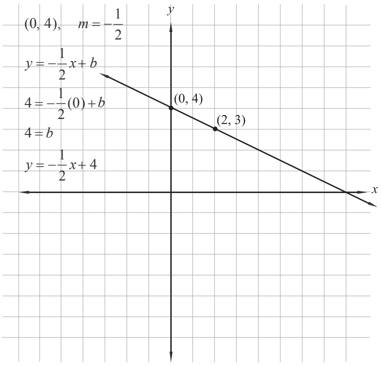
a) $y = 4x$



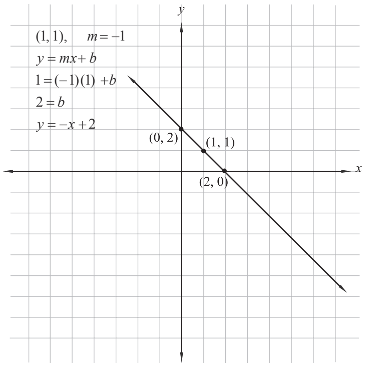
b) $y = 5x - 2$



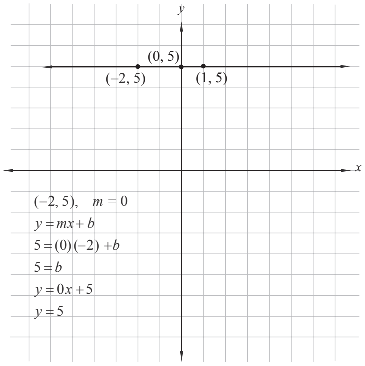
c) $y = -\frac{1}{2}x + 4$



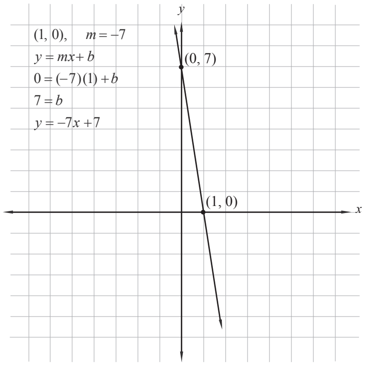
d) $y = -1x + 2$



e) $y = 5$

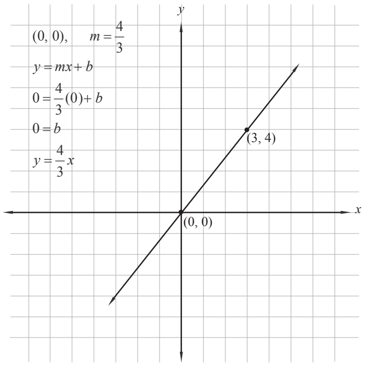


f) $y = 7x + 7$

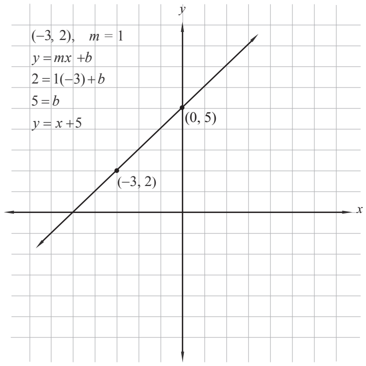


10.11

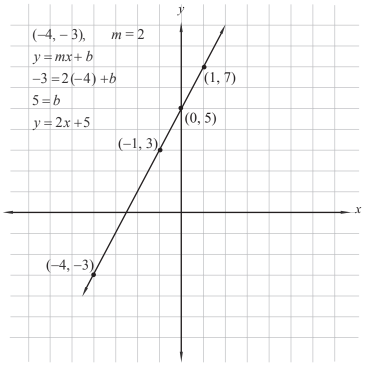
g) $y=\dfrac{4}{3}x$



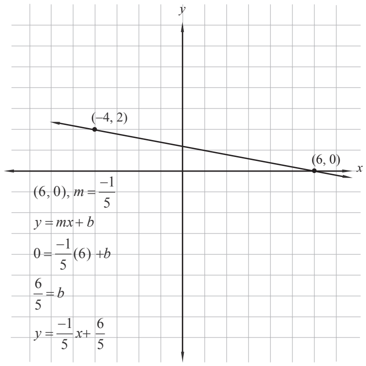
h) $y=x+5$



i) $y=2x-11$



j) $y=-\dfrac{1}{5}x+\dfrac{6}{5}$



10.12

a) $y=16x-4\ (0,\ -4)\ \left(\dfrac{1}{4},\ 0\right)$

b) $y=x-9\ (0,\ -9),\ (9,0)$

c) $y=-\dfrac{5}{2}x+5\ (0,5)\ (2,0)$

d) $y=4x-3\ (0,\ -3)\ (1,1)$

e) $y=6x-4\ (0,\ -4)\ \left(\dfrac{2}{3},\ 0\right)$

f) $y=-x+4\ (0,4)\ (4,0)$

g) $y=-2x+3\ (0,3)\ \left(\dfrac{3}{2},\ 0\right)$

h) $y=7x+17\ (0,\ 17)\ \left(-\dfrac{17}{7},\ 0\right)$

i) $y=-\dfrac{3}{7}x+\dfrac{2}{7}\ \left(0,\ \dfrac{2}{7}\right)\ \left(\dfrac{2}{3},\ 0\right)$

j) $y=-\dfrac{2}{3}x-\dfrac{1}{6}\ \left(0,\ -\dfrac{1}{6}\right)\ \left(-\dfrac{1}{4},\ 0\right)$

10.13

a) $y = \frac{6}{11}x - \frac{19}{11}$ **b)** $y = -\frac{3}{2}x + \frac{1}{2}$

c) $x = 4$ **d)** $y = 4$

e) $y = 3x + 10$ **f)** $y = -\frac{4}{13}x - \frac{11}{13}$

B.11 Chapter 11 Exercises

11.1

a) $a = \frac{1}{5x}$ **b)** $x = \frac{8}{7}y$ **c)** $a = \frac{12}{bc}$

d) $T = 3L^2$ **e)** $E = \frac{3j^2 + 13}{5}$ **f)** $v = \sqrt{\frac{2k}{m}}$

g) $v = \frac{12 - 2j^2}{5}$ **h)** $a = 5b^2$ **i)** $k = \frac{Fr^2}{QQ^1}$

j) $P = 4H + 2$ **k)** $L = \frac{gT^2}{4\pi^2}$ **l)** $x = \frac{y\sqrt{11}}{2}$

m) $x = 36y^2 + 1$ **n)** $t = \frac{v_0 - v}{a}$ **o)** $I = \sqrt{\frac{P}{R}}$

p) $V_0 = \frac{VP_2T_1}{P_1T_2}$ **q)** $h = \frac{V}{\left(2\pi r^2 + 2\pi r\right)}$ **r)** $P_2 = \frac{P_1V_1}{V_2}$

s) $a = \frac{\sqrt{26}}{6}$ **t)** $T = \frac{PV}{nR}$ **u)** $h = \frac{V}{\pi r^2}$

11.2 **a)** 220Ω **b)** 146.667Ω

11.3 **a)** 800 watts **b)** 10Ω **c)** 9.6 Ω

11.4 **a)** 1320 watts **b)** 9.09 amps

11.5 11.735Ω

11.6 **a)** 342.05Ω **b)** 599.97Ω
11.7 **a)** 36575 volts **b)** 282.8 amps
11.8 **a)** 37.08 ihp **b)** 148.31 psi
11.9 **a)** 15.47 bhp **b)** 84.0 lb
11.10 **a)** 34.2719 hp **b)** 364.7330 ft-lb
11.11 **a)** 1.309 hp **b)** 3.38 in-lb
11.12 **a)** 0.45 hp **b)** 10000 lb
11.13 **a)** 500 hp **b)** 6.7 kw

11.14 **a)** 1749 hp **b)** 1960 gpm
11.15 **a)** 124.355 hp **b)** 10.19 psi
11.16 **a)** 0.0558 inch **b)** 2 tpi
11.17 **a)** 3.1429 **b)** $P = 20$
11.18 **a)** 3.8 inches **b)** 1.5 inches
11.19 **a)** 5.839 lb **b)** 0.4496 inch
11.20 **a)** 5592.17 inches **b)** 12.44 inches
11.21 **a)** 217.33 inches **b)** 211.13 inches

11.22
a) 0.0722 inch **b)** 0.0722 inch **c)** 0.1443 inch

11.23 **a)** 1.503 hpc **b)** 6.24 ft/min
11.24 **a)** 1.51 hpm **b)** 0.86

11.25
a) 763.9437 rpm **b)** 166.8971 fpm **c)** 0.413 inch

11.26 **a)** 0.5512 inch **b)** 8 threads per inch
11.27 **a)** 64.38 fpm **b)** 0.8621
11.28 **a)** 81.9966 rpm **b)** 78.5398 fpm
11.29 **a)** 35 inches/minute **b)** 250 rpm
11.30 **a)** 59.38 lb **b)** 1600 lb
11.31 **a)** 6773.33 psi **b)** 2.7093 inches
11.32 **a)** 5.3016 cubic inches/second **b)** 0.8333 square inch
11.33 **a)** 0.0095 feet or 0.1145 inch **b)** 4583.33 psi
11.34 **a)** 6.571 feet/second **b)** 0.2067 feet

11.35 5 inches/second

11.36 **a)** 54.9 gpm **b)** 18.2149 feet/second

11.37 3000 psi **11.38** Yes **11.39** 0.874 inch

B.12 Chapter 12 Exercises

12.1 $\frac{2}{1}$ or 2 to 1 **12.2** $\frac{1}{4}$ or 1 to 4 **12.3** $\frac{16}{1}$ or 16 to 1

12.4 $\frac{2}{1}$ or 2 to 1 **12.5** $\frac{9}{1}$ or 9 to 1 **12.6** $\frac{1}{4}$ or 1 to 4

12.7 $\frac{1}{3}$ or 1 to 3 **12.8** $\frac{1}{9}$ or 1 to 9 **12.9** $\frac{5}{1}$ or 5 to 1

12.10 $\frac{24}{1}$ or 24 to 1 **12.11** $\frac{3}{2}$ or 3 to 2 **12.12** $\frac{6}{1}$ or 6 to 1

12.13 $\frac{5}{2}$ or 5 to 2 **12.14** $\frac{1}{5280}$ or 1 to 5280
12.15 $\frac{16}{125}$ or 16 to 125 **12.16** $\frac{5}{22}$ or 5 to 22

12.17 60 inches **12.18** $30 **12.19** 10 inches **12.20** $\frac{5}{1}$ or 5 to 1
12.21 2 **12.22** 3 **12.23** 80 **12.24** $28.80
12.25 2 hours

12.26 $26\frac{2}{3}$ mpg **12.27** $13\frac{10}{11}$ revolutions

12.28 25 teeth **12.29** 3.7753 turns **12.30** 90 teeth
12.31 45 teeth **12.32** 290 lb **12.33** 1079.5 rpm
12.34 8 hours **12.35** 6.4 to 1 **12.36** 70.83 lb
12.37 52.80 feet **12.38** 8 **12.39** 6
12.40 10 **12.41** $\frac{1}{3}$ **12.42** 625
12.43 40 feet **12.44** 240 cubic inches **12.45** 100 motors
12.46 4 boxcars

12.47 $6\frac{2}{3}$ hours or 6 hours 40 minutes

12.48 1680 lb **12.49** 10 hours **12.50** 500 cars
12.51 $2648.65 **12.52** 3.36 days
12.53 12 gallons red paint, 16 gallons yellow paint

12.54 133.33 lb copper, 80.00 lb lead, 26.67 lb tin
12.55 68.97 lb solder, 28.97 lb lead
12.56 0.95 lb molybdenum, 11.4 lb chrome, 0.95 lb carbon, 81.7 lb iron
12.57 14.29 oz Base metal
12.58 303.24 lb tin, 631.76 lb copper
12.59 81.78 lb copper, 61.33 lb lead, 40.89 lb tin
12.60 216 lb lead
12.61 21.42 lb tin, 2.55 lb antimony, 1.275 lb copper, 0.255 lb lead

12.62 $\frac{1}{16}$ inch per inch **12.63** 0.0267 inch per inch
12.64 0.3204 inch per foot **12.65** 1.6582 inches
12.66 1.175 inches per foot **12.67** 0.750 inch per inch

12.68 13.25 feet or $13\frac{1}{4}$ feet or 13′3″ or 159 inches

12.69 0.166 feet or 1.9946 inches
12.70 1.1010 inches
12.71 0.100 inch per inch
12.72 1.0625 inches
12.73 0.8064 inch
12.74 0.0008 inch
12.75 0.0066 inch per inch
12.76 0.055 foot per foot
12.77 0.25 inch offset
12.78 0.1628 inch
12.80 0.24 inch per ft
12.81 0.073 inch per inch
12.82 0.250 inch
12.83 1.07 inches
12.84 0.673 inch
12.85 0.290 inch
12.86 Taper = 0.071 inch per inch, offset = 0.230 inch
12.87 0.0625 inch per inch

12.88 4.55 inches
12.89 $A = 1.194$ inches

12.90 **a)** $\frac{1}{2}$ **b)** 4 **c)** 8
12.91 **a)** 50 **b)** 2500 **c)** 648
12.92 **a)** 21 **b)** 56 **c)** 189
12.93 **a)** $\frac{1}{3}$ **b)** $\frac{1}{5}$ **c)** $\frac{3}{2}$
12.94 **a)** 20.25 **b)** 4.374 **c)** 68.3438
12.95 **a)** 27 **b)** $\frac{75}{2}$ or $37\frac{1}{2}$ **c)** $60\frac{3}{4}$ or 60.75

12.96 50 mm
12.97 6m/sec^2

12.98

a) $k = 9\times10^9$ **b)** $\frac{N-m^2}{\text{Coulomb}^2}$ **c)** 2.25×10^9 Newtons

B.13 Chapter 13 Exercises

13.1

a) $-48u^4y^3w^3$
b) $72p^2q^4r^3s^4$
c) $-27a^3b^3$
d) $144a^6b^7c$
e) $30x^2y^5z^4$
f) $132a^2bc^2d^3e^3$
g) $6a+6b$
h) $12xy-8xz$
i) $6bce-9bcg+6bc$
j) $15ab-30bd$
k) $12xyz-36yz+24xz$
l) $2b^2-3b-14$
m) $21p^2-5pq-6q^2$
n) $5x^2+13x-6$

13.2

a) $2x^3-2x^2$
b) $4a^3+20a^2+16a$

13.2

c) $42b^3+12b^2$
d) $24a^3+38a^2+31a+12$
e) $14x^3-8x^2-2x-4$
f) $a^4+a^2b^2+b^4$
g) $z^5-5z^3+9z^2-45$
h) $3abc+3abd+3abe$
i) a^3-a^2-2a+8
j) $-6x^3+5x^2-10x+3$
k) $-4x^3+13x^2-11x+2$
l) $2x^5+2x^3y+x^3+3x^2y^2-2x^2+3y^3+xy-2y$
m) $5x^3+15x^2$
n) $w^3+10w^2+22w-15$
o) $-6x^3+15x^2-6x-3$
p) $6x^2-6$

13.3

a) $4a^2+8ab+4b^2$
b) $z^3-9z^2+27z-27$
c) $-2x^2-12x-18$
d) $-z^3+9z^2-27z+27$
e) x^2+4x+3
f) x^2-2x-3
g) x^2+2x-3
h) x^2-4x+3

13.4

a) $4x$
b) $17t$
c) 5
d) x^4y
e) $21p$
f) $-21w$
g) $3w^2h^3$
h) $2\sqrt{y}$
i) $3z^4$
j) $-8x^2z$

13.5

a) $12(x-y)$
b) $3(x-5y)$
c) $-5(a^2-4b^2+3c^2)$
d) $4x(3x^2-1)$ or $-4x(1-3x^2)$
e) $z^3(z^2-1)$
f) $6(x^4-2x^3+x^2-3x-5)$
g) $(y+3)(3+5)$ or $8(y+3)$
h) $3(x-1)^3(a+3a^2(x-1)^2-2b(x-1))$
i) $8(x-1)^3$
j) $-9a^2b^4(2ab+4-3ab^6)$
k) $(a+1)^3(3-2(a+1)^2)$
l) $-2c(10c-21)$
m) $7x^3y^2z(3x^3y^2z^7+9z^3-y^3)$
n) $-2(x^3-7x^2+16x-4)$
o) $-56bc(b-2)$

13.6

a) $(x-4)(x-2)$
b) $(x+4)(x+3)$

13.6

c) $(x+3)(x-2)$

d) $\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$

e) Prime

f) $(4x-3)(4x-1)$

g) $2(x+3)(x-3)$

h) $(x+3)(x+1)$

i) $(5x+1)(x-1)$

j) $(3x+2)(x+1)$

k) $(3x+2)(x-1)$

l) $(x+4)(x-2)$

m) $6\left(x+\frac{1}{9}\right)\left(x+\frac{1}{8}\right)$

n) Prime

o) $(12x+1)(12x+1)$

p) $(x+6)(x+6)$

q) $(3x-1)(x+5)$

r) $(x+4)(x-3)$

s) $(3x-2)(x+2)$

t) $(x-9)(x-6)$

u) Prime

v) $(x+7)(x+6)$

w) $(5x-1)^2$

x) $(x-8)(x+7)$

y) $(2x+3)(x-1)$

z) $4\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$

13.7

a) $(x+8)(x-8)$

b) $\left(x+\sqrt{19}\right)\left(x-\sqrt{19}\right)$

c) $\left(x-4\sqrt{2}\right)\left(x+4\sqrt{2}\right)$

d) $(y-3x)(y+3x)$

e) $(b+2)(b-2)$

f) $(4x-y)(4x+y)$

g) $(5x-6y)(5x+6y)$

h) $\left(5x\sqrt{2}-7\right)\left(5x\sqrt{2}+7\right)$

i) $(11z-1)(11z+1)$

j) $\left(x\sqrt{5}-1\right)\left(x\sqrt{5}+1\right)$

k) $\left(2x\sqrt{3}-5\sqrt{3}\right)\left(2x\sqrt{3}+5\sqrt{3}\right)$

l) $5\left(2x-y\sqrt{3}\right)\left(2x+y\sqrt{3}\right)$

m) $8\left(x\sqrt{3}-2\right)\left(x\sqrt{3}+2\right)$

n) $\left(z-4\sqrt{3}\right)\left(z+4\sqrt{3}\right)$

o) $\left(3x\sqrt{3}+4\sqrt{2}\right)\left(3x\sqrt{3}-4\sqrt{2}\right)$

13.8

a) $\left(x+i\sqrt{7}\right)\left(x-i\sqrt{7}\right)$

b) $\left(x+2i\sqrt{10}\right)\left(x-2i\sqrt{10}\right)$

c) $(x+2yi)(x-2yi)$

d) $\left(2x\sqrt{5}+i\right)\left(2x\sqrt{5}-i\right)$

13.8

e) $\left(3x\sqrt{5}+8yi\right)\left(3x\sqrt{5}-8yi\right)$

f) $\left(x\sqrt{3}+i\right)\left(x\sqrt{3}-i\right)$

g) $(x+6yi)(x-6yi)$

h) $\left(x\sqrt{15}+i\sqrt{2}\right)\left(x\sqrt{15}-i\sqrt{2}\right)$

i) $(7a-6bi)(7a+6bi)$

j) $\left(2x+9i\sqrt{2}\right)\left(2x-9i\sqrt{2}\right)$

k) $(w+5i)(w-5i)$

l) $(x\sqrt{3}+1)(x\sqrt{3}-1)$

m) $\left(a+4bi\sqrt{2}\right)\left(a-4bi\sqrt{2}\right)$

n) $4(x+2i)(x-2i)$

o) $6\left(x+2yi\sqrt{3}\right)\left(x-2yi\sqrt{3}\right)$

13.9

a) $-5a$

b) 5

c) $-8t^2$

d) $-16a^2c$

e) $2a^2b^2$

f) $-6ab$

g) $21y^3$

h) x^2

i) $-4a$

j) $a+3c$

k) $1+rt$

l) $3x^2-1$

m) $8x^2y^2$

n) $-x^2+7x-10$

o) $d+2$

p) $x+2$

q) $5ab-1$

r) $3y+4$

13.10

a) $\dfrac{7y^2}{8}$

b) $\dfrac{wyz}{2}$

c) $\dfrac{x}{x+3}$

d) $\dfrac{12x^3-4x^2-3x+1}{16x^2-1}$

e) $\dfrac{2(x-2)}{5(x+2)}$ or $\dfrac{2x-4}{5x+10}$

f) $\dfrac{20+63x}{12x^2}$

g) $\dfrac{6y+8x}{x^2y^2}$

h) $\dfrac{x(8x^2+30x+33)}{(x+2)(x-2)(x+3)(x+3)}$

i) $(x+3)^2$ or x^2+6x+9

j) $\dfrac{7x-1}{(x+2)(x-3)}$ or $\dfrac{7x-1}{x^2-x-6}$

k) $\dfrac{19x+18}{6x^2}$

l) $\dfrac{9axy+16bx-12ay}{12x^2y^2}$

m) $\dfrac{(x+2)(5x-2)}{x}$

n) $\dfrac{3\left(x^2+5\right)}{5(x+5)}$ or $\dfrac{3x^2+15}{5x+25}$

o) $\dfrac{6x+12}{x^2-9}$ or $\dfrac{6(x+2)}{(x-3)(x+3)}$

13.10

p) $\dfrac{2x}{(x+3)^2(x+2)}$ or $\dfrac{2x}{x^3+8x^2+21x+18}$

q) $\dfrac{3x^2+5x-7}{(x-4)(x+1)(x-1)}$

r) $\dfrac{13x-4}{6x^2}$

13.11

a) $2w$

b) $\dfrac{3x}{y}$

c) $\dfrac{x^3-2x^2}{x^2+3x+2}$ or $\dfrac{x^2(x-2)}{(x+1)(x+2)}$

d) $\dfrac{1}{x-2}$

e) $\dfrac{x(x+1)}{(x+3)^2}$ or $\dfrac{x^2+x}{x^2+6x+9}$

f) $\dfrac{5}{3}$

B.14 Chapter 14 Exercises

14.1

a) $x=\pm 3\sqrt{11}$

b) $x=\pm 8$

c) $\pm 4\sqrt{2}$

d) $x=\pm 3i\sqrt{5}$

e) $x=-\dfrac{1}{4}$ and $\dfrac{1}{3}$

f) $x=\pm\sqrt{74}$

g) $x=\pm 4$

h) $x=\pm 4i$

i) $x=\pm 9i$

j) $x=3$ and -4

k) $x=-3$ and -1

l) $x=7$ and -8

14.2

a) $x=-1$ or $x=-\dfrac{2}{3}$

b) $x=6$ or 7

c) $x=-2$ and $\dfrac{2}{3}$

d) $x=\dfrac{1}{5}$ and $\dfrac{3}{2}$

e) $x=3$ and -18

f) $x=3$ and 1

g) $x=3$ and -4

h) $x=\dfrac{2}{3}$ and -2

i) $x=3$ and 1

j) $x=\dfrac{2}{3}$ and -1

k) $x=12$ and -2

14.3

a) $-3\pm\sqrt{10}$

b) $-1\pm\sqrt{5}$

c) $2\pm\sqrt{5}$

d) $6\pm 2\sqrt{7}$

e) $\dfrac{3}{2}\pm\dfrac{\sqrt{13}}{2}$

f) $-\dfrac{5}{2}\pm\dfrac{\sqrt{5}}{2}$

14.3

g) $\frac{-1}{2}\pm\frac{\sqrt{17}}{2}$

h) $x=\frac{1}{2},\ x=-2.5$

i) $\frac{7}{2}\pm\frac{\sqrt{47}}{2}$

j) $-\frac{1}{2}\pm\frac{\sqrt{153}}{18}$

k) $\pm\sqrt{74}$

l) $-8\pm 2\sqrt{23}$

m) ± 4

n) $-3,\ \frac{1}{2}$

o) $-\frac{3}{2}\pm\frac{\sqrt{5}}{2}$

14.4

a) $2\pm\sqrt{2}$

b) $-18,\ 3$

c) $3,\ 3$

d) $-\frac{1}{2}\pm\frac{i\sqrt{11}}{2}$

e) $-\frac{3}{4}\pm\frac{\sqrt{41}}{4}$

f) $1,\ 3$

g) $1,\ 3$

h) $\frac{3}{2},\ \frac{1}{5}$

i) $-2.5,\ -2.5$

j) -8

k) $\frac{2}{3}\pm\frac{\sqrt{10}}{3}$

l) $4,\ \frac{1}{5}$

m) $\frac{4}{3},\ -1$

n) $7,\ 6$

o) $\frac{3}{2}\pm\frac{\sqrt{5}}{2}$

14.5

a) $-\frac{1}{4}\pm\frac{\sqrt{17}}{4}$

b) $-\frac{3}{2}\ (\text{twice})$

c) $-2,\ \frac{2}{3}$

d) $2\pm 2\sqrt{2}$

e) $\frac{1}{3}\ (\text{twice})$

f) $-1\pm\frac{\sqrt{5}}{2}$

g) $-\frac{1}{144}\pm\frac{i\sqrt{287}}{144}$

h) $-1,\ \frac{2}{3}$

i) $\pm 4i$

j) $\pm 9i$

k) $1,\ -\frac{3}{4}$

l) $\frac{1}{3},\ -\frac{1}{4}$

m) $3,\ -4$

n) $-3,\ -1$

o) $7,\ -8$

14.6

a) $12,\ -2$

b) $\frac{2}{3},\ -1$

c) $\frac{4}{3}\pm\frac{2\sqrt{13}}{3}$

d) $3,\ -4$

e) $-2,\ \frac{2}{3}$

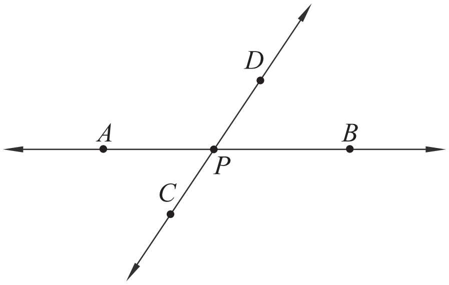
f) $12,\ -2$

g) $1,\ -\frac{1}{2}$

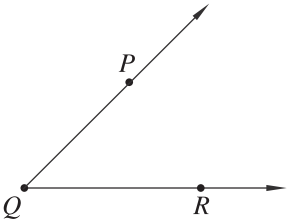
B.15 Chapter 15 Exercises

15.1

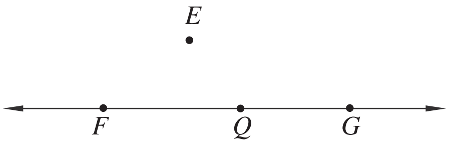
a)



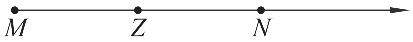
d)



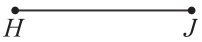
b)



c)



e)



15.2 No **15.3** 3

15.4 Intersect at one point; Intersect at all points (collinear); Not intersect at any points

15.5 $\angle BAC$, $\angle CAB$

15.6

a) Acute **b)** Neither **c)** Acute **d)** Obtuse
e) Neither **f)** Obtuse **g)** Obtuse **h)** Acute

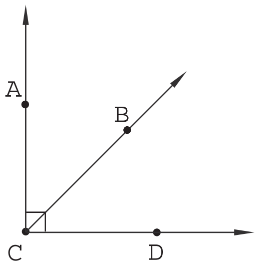
15.7

a) $48°$ **b)** $29°$ **c)** $68°$ **d)** $71°$

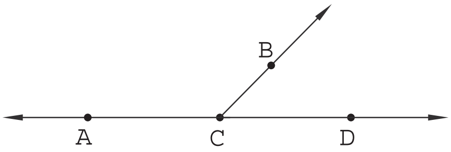
15.8

a) $145°$ **b)** $70°$ **c)** $91°$ **d)** $147°$

15.9



15.10



15.11

a) $\angle A, \angle C, \angle F, \angle H$ **b)** $\angle B, \angle D, \angle E, \angle G$ **c)** $\angle A \;\&\; \angle H$, $\angle C \;\&\; \angle F$

15.11

d) $\angle B \;\&\; \angle G$

$\angle D \;\&\; \angle E$

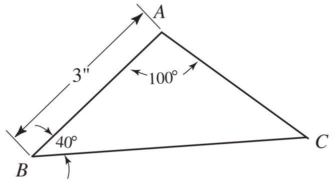
e) $\angle F \;\&\; \angle B$

$\angle A \;\&\; \angle E$

$\angle C \;\&\; \angle G$

$\angle D \;\&\; \angle H$

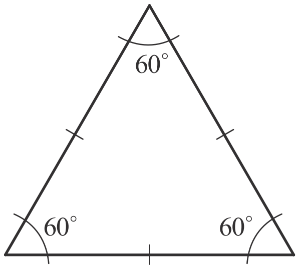
15.12



15.13 Obtuse triangle; Isosceles triangle

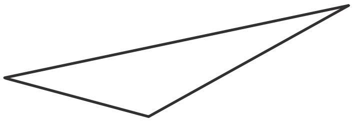
15.14 Regular hexagon

15.15



Equilateral triangle

15.16



Scalene triangle

15.17 Quadrilateral; Parallelogram; Rectangle; Rhombus; Square

15.18 Yes **15.19** Yes

15.20 No, consider triangle with angles: 59º, 61º, 60º

15.21 There are three angles and they add to exactly 180º, two 90º angles = 180º.

15.22 Tetrahedron: 4 faces; Cube: 6 faces; Octahedron: 8 faces; Dodecahedron: 12 faces; Icosahedron: 20 faces

15.23 Triangular prism and pentagonal prism

15.24 Triangular pyramid and pentagonal pyramid

B.16 Chapter 16 Exercises

16.1

a) 4 feet **b)** 25 cm **c)** 8 miles

16.2

a) 4 feet **b)** $4\sqrt{10}$ meters **c)** 40 miles

16.3

a) 2.828 feet or $2\sqrt{2}$ feet **b)** 9.19 cm **c)** 1.414 mm

16.4

a) 54 inches **b)** 57.000 inches **c)** 68.750 m

16.5

a) $\frac{3}{4}\pi$ inches or ≈ 2.36 inches **b)** 50πmm or ≈ 157.08 mm

c) 12π cm or ≈ 37.70 cm

16.6 273 feet **16.7** 120 feet

16.8 88.5 inches **16.9** 112 mm

16.10

a) 1 square foot **b)** $39\frac{1}{16}$ cm^2 or 39.1625 cm^2

c) 4 square miles **d)** 9 square feet

16.11

a) $19.6875 \approx 19.69$ square meters **b)** 138.0 square feet

c) $.0375 \text{ mm}^2 \approx .04 \text{ mm}^2$

16.12

a) 19.69 square meters **b)** 138.0 square feet

c) 0.0375 square millimeters

16.13

a) 13.5 square meters **b)** 155.25 square feet

c) 2.625 square millimeters

16.14

a) 9.84 square meters **b)** 69.0 square feet

c) 0.02 square millimeters

16.15

a) 2.598 square feet **b)** 101.48 square centimeters

c) 23.38 square meters

16.16

a) $\pi \text{ ft}^2 \approx 3.14$ square feet **b)** 122.72 square centimeters

c) 28.27 square meters

16.17 8856 square inches **16.18** 6372.24 square yards

16.19 $8\sqrt{2}$ **16.20** 32,400 tiles **16.21** 976 square feet

16.22 12.79 gallons or 13 gallons. **16.23** $235.52

16.24

a) 260 square feet **b)** $682.50

16.25 2304.4 square centimeters **16.26** 4056 square centimeters

16.27 650 square centimeters

16.28 441.2 square meters

16.29 $\frac{1}{2}eh' + \frac{1}{2}eh$ or $\frac{1}{2}e\left(h' + h\right)$

16.30 75 square inches

16.31 0.750 inches

16.32 3.578 m^2 or 38.5 ft^2

16.33 2159052 square millimeters. 201,486,744 cubic millimeters

16.34 1054 cubic inches

16.35 168 cubic feet

16.36 336 cubic feet

16.37

a) Surface area = 12.57 square feet; Volume = 4.19 cubic feet
b) Surface area = 12.57 square meters; Volume = 4.19 cubic meters
c) Surface area = 6647.61 square millimeters; Volume = 50965.01 cubic millimeters
d) Surface area = 201.06 square inches; Volume = 268.08 cubic inches

16.38

a) Surface area = 4.712 square feet; Volume = 0.785 cubic feet
b) Surface area = 16474.5 square millimeters; Volume = 1.71 cubic meters
c) Surface area = 16464.5 square millimeters; Volume 151233 cubic millimeters
d) Surface area = 276.46 square inches; Volume = 351.86 cubic inches

16.39

a) 0.2618 cubic feet
b) 0.568 cubic meters
c) 50411.04 cubic millimeters
d) 117.29 cubic inches

16.40 1728 cubic inches
16.41 7200 cubic feet; 203.88 cubic meters
16.42 41472 cubic inches; 679604.32 cubic centimeter
16.43 135625271.3 cubic inches
16.44 960 cubic feet; 27.18 cubic meters

16.45

a) 54.675 gallons to completely fill the reservoir
b) 47.841 gallons to fill the reservoir

16.46 1536 cubic feet
41.62 square meters
$21\frac{1}{3}$ square yards

16.47 97.22 cubic yards

16.48 77625 cubic feet water
16.49 5500 cubic yards. 4205.05 cubic meters

16.50 240.83 cubic meters
16.51 87964.59 liters
16.52 400
16.53 66215 gallons/hour
16.54 40.968 cubic centimeters
16.55 96000 gallons

16.56 159.075 square feet

16.57 7590000 liters

16.58

a) 42.78 cubic feet

b) 1.21 cubic meters

16.59

a) 13,500 liters

b) 3566.3 gallons

16.60 1875 kiloliters and 495376.486 gallons

16.61

a) 13253.59 gallons

b) 4970.10 gallons

c) 21.32 kiloliters drained; 31.36 kiloliters added

16.62 93.61 lb

16.63 2748893.57 liters

16.64 2.243 cubic inches

16.65 62,445.7 cubic inches

16.66 198.61 cubic inches volume; 134.01 cubic inches removed

B.17 Chapter 17 Exercises

17.1

a) Commutative property of addition

b) Transitive property of equality

c) Euclid postulate #1

d) Euclid postulate #9

e) Euclid postulate #11

f) Euclid postulate #5

g) Euclid postulate #7

17.2 One. Postulate #8. **17.3** Line. Postulate #3. **17.4** One. Postulate #6.
17.5 One. Postulate #2. **17.6** Yes. Postulate #10.

17.7 $\angle B = 60^\circ$

17.8 $\angle A = 59^\circ$

17.9 $\angle B = 27^\circ$

17.10 $\angle C = 95^\circ$

17.11 65°

17.12 14°

17.13 23°

17.14 88°

17.15 57°

17.16 54°

17.17 $\alpha = 44^\circ$

17.18 $\beta = 27^\circ$

17.19 $\alpha = 30^\circ$
$\beta = 120^\circ$

17.20 $\measuredangle CBA = 36^\circ$
$\measuredangle CAB = 54^\circ$

17.21 $\angle ABC = \angle B = 50^\circ$
$\measuredangle BDC = 25^\circ$

17.22 $\measuredangle \beta = 29^\circ$

17.23 $\alpha = 31^\circ$

17.24 $\beta = 37.5^\circ$

B.18 Chapter 18 Exercises

18.1 An angle bisector is a line that divides a given angle into two equal angles.

18.2 True

18.3 False

18.4 It is a line segment from one vertex which intersects the opposite side at right angles.

18.5 $\overline{DE} = 3$

18.6 $\measuredangle DFE = 70°$

18.7 $\measuredangle BAD = 30°$

18.8 $\overline{EG} = 15$

18.9 $\overline{BD} = \overline{DC} = 3;$ Yes; SAS

18.10 $\measuredangle BEC = 30°$

18.11 $\measuredangle BDC = 50°$
$\measuredangle BCD = 80°$

18.12 $BC = 20$

18.13 $\measuredangle ABD = 45°$
$\measuredangle BCD = 30°$

18.14 $\measuredangle AEB = 100°$
$\measuredangle CED = 30°$

18.15 $DE = 11$
$BC = 33$

18.16 $\measuredangle ACB = 90°$

18.17 45 feet

18.18 8 meters

18.19 $\measuredangle ACD = 26°4'$
$\measuredangle DCB = 63°6'$
$\measuredangle CBD = 26°4'$

18.20

a) $a = 4,\ h = \sqrt{12}$ or $h \approx 3.46$

b) $c = 9,\ h = \sqrt{20}$ or 4.47

c) $q = 4,\ b = \sqrt{80}$ or 8.94

d) $p = 18,\ h = \sqrt{108}$ or 10.39

18.21 $CD = 5.08$

18.22 $\beta = 36°$

18.23

a) 20

b) 5.29

c) 3.46

d) 16

e) 60°

f) $a = 3x = 9$
$b = 4x = 12$

g) $c = \sqrt{98} \approx 9.90$

h) 13

i) 8

18.24 22.14 feet

18.25 0.384 inch

18.26 0.552 units

18.27 7.015 units

18.28 15 inches

18.29 $AD = 10$ cm

18.30 perimeter $= 37.7078$ or ≈ 37.71 inches

18.31 $x = 4.436$ inches
$y = 6.171$ inches

18.32 $\angle C' = 37°$
$b' = 5$ inches

18.33 $\Delta AEB \cong \Delta CED$

18.34 $\measuredangle ABC = 55^\circ$

$\measuredangle ACB = \measuredangle ABC = 55^\circ$ (isosceles base angles equal)

18.35 x = 19 inches
y = 8 inches

18.36 $x = 4$
$y = 12$

18.37 $\measuredangle DFE = 30^\circ$

18.38 p' = 5.25 inches
h = 7.31 inches

18.39 d = 3.56 cm
h = 6.60 cm

18.40 h = 6.01 mm

18.41 x = 11.65 units

18.42 $h = 46.73$

18.43 58.48 square inches

18.44 28.05 square cm

18.45 A = 37.24 square feet
h = 6.01 inches

18.46 61.11 square mm

18.47 1308.53 square inches

18.48 26.27 square mm

18.49 3114.44 square inches

B.19 Chapter 19 Exercises

19.1

a) Circle: Set of all points in a plane that are equidistant from a given point in that plane.
b) Chord: A line segment that has as its endpoints two points of the circle.
c) Semicircle: An arc that equals one half of total circle circumference.
d) Central angle: An angle having the center of the circle as its vertex and radii as legs.
e) Sector: Area bounded by arc and associated radii.
f) Radius: Line segment from center point to circumference.
g) Secant: A line that contains a chord.
h) Inscribed angle: Angle whose vertex lies on the circle and has two chords as legs.
i) Diameter: Line segment that includes center point and two points on the circumference.
j) Arc: Portion of a circle that ends at two points of the circle.
k) Tangent: A line that intersects the circle at exactly two points.
l) Intercepted arc: The portion of a circle that has as its endpoints the radii of a central angle.

19.2 An inscribed polygon is a polygon whose sides are all chords of a circle.
19.3 A circumscribed polygon is a polygon whose sides are all tangent to a circle.

19.4 2

19.5 A circle

19.6 $\alpha = 10^\circ$

19.7 $\beta = 139°$

19.8 $\angle A = 56°$
$\angle B = 28°$
$\angle C = 124°$
$\angle D = 28°$
$\angle E = 28°$

19.9 $y = 12$ inches

19.10 $\angle PAO = 20°$

19.11 $\overline{AB} = 15$ cm

19.12 $\angle\alpha = 70°$

19.13 4.25 inches

19.14 $\alpha = 19.5°$

19.15 $\angle ACO = 38°$

19.16 Diameter = 5.46 inches

19.17 $\angle ABC = 59°$

19.18 Diameter $= \left(1.735 + 3.262 - \sqrt{1.735^2 + 3.262^2}\right) \approx 1.30$ meters

19.19 $y = 1.326$ inches

19.20 $\overline{AB} = 4.907$ or 4.91mm

19.21 $x = 2.5$ inches
diameter = 3.54 inches

19.22 $x = 5.303$ inches

19.23 $\angle ABC = 32.5°$
$\angle BCA = 50°$

19.24 $\alpha = 50°$

19.25 $X = 1.666$ inches
$Y = 3.332$ inches

19.26 $\angle\alpha = 102°$
$\angle\beta = 41°$

19.27 $\angle\alpha = 62.5°$
$\angle\beta = 25°$
$\angle\omega = 86.5°$

19.28 Radius = 2.392 inches

19.29 $x = 1.2374$ inches

19.30 $D = 1.624$ inches

19.31 Diameter = 2.598 inches

19.32 Diameter = 2.52 mm

19.33 $\alpha = 32°$

B.20 Chapter 20 Exercises

20.1

a) The adjacent side is the side of a right triangle that is not the hypotenuse, but terminates at the vertex of the angle of interest.

b) The opposite side of a triangle is the side that does not terminate at the vertex of the angle of interest.

c) The hypotenuse is the side opposite the 90º angle of a right triangle.

20.2 $\sin = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\cos = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\tan = \dfrac{\text{opposite}}{\text{adjacent}}$

$\csc = \dfrac{\text{hypotenuse}}{\text{opposite}}$

$\sec = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

$\cot = \dfrac{\text{adjacent}}{\text{opposite}}$

20.3

a) $\sin 30^\circ = 0.5000$
b) $\sin 36.45^\circ = 0.5941$
c) $\cot 70.30^\circ = 0.3580$
d) $\sin 5.22^\circ = 0.0910$
e) $\tan 76^\circ = 4.0108$
f) $\cos 3.52^\circ = 0.9981$
g) $\tan 45^\circ = 1.0000$
h) $\tan 22.40^\circ = 0.4122$
i) $\tan 66.35^\circ = 2.283$
j) $\tan 66.66^\circ = 2.3175$
k) $\csc 22.22^\circ = 2.6444$
l) $\tan 19.22^\circ = 0.3486$
m) $\cos 45^\circ = 0.7071$
n) $\cos 55.10^\circ = 0.5722$
o) $\sec 15.5^\circ = 1.0377$
p) $\cot 45^\circ = 1.0000$
q) $\sin 60.75^\circ = 0.8725$

20.4

a) $\sin\alpha = \dfrac{3}{\sqrt{58}}$
b) $\csc\alpha = \dfrac{\sqrt{58}}{3}$
c) $\sin\beta = \dfrac{7}{\sqrt{58}}$
d) $\csc\beta = \dfrac{\sqrt{58}}{7}$
e) $\cos\alpha = \dfrac{7}{\sqrt{58}}$
f) $\sec\alpha = \dfrac{\sqrt{58}}{7}$
g) $\cos\beta = \dfrac{3}{\sqrt{58}}$
h) $\sec\beta = \dfrac{\sqrt{58}}{3}$
i) $\tan\alpha = \dfrac{3}{7}$
j) $\cot\alpha = \dfrac{7}{3}$
k) $\tan\beta = \dfrac{7}{3}$
l) $\cot\beta = \dfrac{3}{7}$

20.5

a) 0.8910 **b)** 0.8910 **c)** 1.2780 **d)** 0.78128 **e)** 1.3457

20.6

a) $\csc 82^\circ = \dfrac{1}{\sin 82^\circ}$
b) $\sec 19^\circ = \dfrac{1}{\cos 19^\circ}$
c) $\cot 7^\circ = \dfrac{1}{\tan 7^\circ}$

20.6

d) $\dfrac{1}{\cos 14^\circ} = \sec 14^\circ$ **e)** $\dfrac{1}{\cot 75^\circ} = \tan 75^\circ$ **f)** $\dfrac{1}{\sin 30^\circ} = \csc 30^\circ$

20.7

a) 31.5° **b)** 69.25° **c)** 34.59° **d)** 44.92°
e) 45.01° **f)** 2.00° **g)** 22.50° **h)** 68.50°
i) 38.58° **j)** 38.25° **k)** 14.75° **l)** 55.01°
m) 61.08° **n)** 35.01° **o)** 8.50° **p)** 32.67°
q) 22.52° **r)** 52.00°

20.8 $x = 39.72^\circ$, $\cos x = 0.76912$

20.9 $x = 70.76^\circ$, $\cot x = 0.34905$

20.10

a) $b = 10.392$
hypotenuse $= 12$
$A = 30^\circ,\ B = 60^\circ$

b) $a = 381.62$
hypotenuse $= 382.15$
$A = 87^\circ$

c) $a = 6.93$
$c = 8$
$\angle B = 30$

d) $b = 3978.69$
$c = 4100.50$
$\angle A = 14^\circ$

e) $a = 186.72$
$b = 71.67$
$\angle A = 69^\circ$

f) $a = 24.67$
$\angle A = 53.51^\circ$
$\angle B = 36.49^\circ$

g) $a = 45.41$
$b = 56.07$
$\angle B = 51^\circ$

h) $a = 1757.92$
$\angle A = 53.25^\circ$
$\angle B = 36.75^\circ$

i) $a = 54.63$
$c = 74.69$
$\angle A = 47^\circ$

j) $a = 3.45$
$c = 91.98$
$\angle B = 87.85^\circ$

k) $a = 6.652$
$c = 13.72$
$\angle B = 61^\circ$

l) $a = 0.59$
$b = 8.44$
$\angle A = 4^\circ$

m) $b = 21.54$
$c = 22.65$
$\angle B = 72^\circ$

n) $a = 13.34$
$c = 16.09$
$\angle A = 56^\circ$

20.11 30° **20.12** 60° **20.13** 45°

20.14 39.81° from the horizontal; 50.19° from the vertical

20.15 $b = 19.45$ inches

20.16 449.90 feet **20.17** 7.79 inches **20.18** 6 inches
20.19 8.40 inches **20.20** 8.03° **20.21** 148.68 feet
20.22 36.87°, $HC = 25$ feet **20.23** 39.09°
20.24 Angle is 3.43° if grade is 6%. Angle is 0.367° if grade is 0.64%.

20.25 428.4 feet **20.26** 123.64 feet **20.27** 95.54 feet
20.28 22.65 feet **20.29** 1369.04 feet **20.30** 26.11 feet
20.31 76.21° **20.32** 53.99 feet **20.33** 479.90 feet
20.34 0.9526 inch **20.35** 54.43° **20.36** 74.84 meters

20.37	8.12º	**20.38**	0.1164 inch	**20.39**	0.233 inch
20.40	85.70 mm, $\angle A = 6.34°$			**20.41**	19.60º
20.42	38.98 mm	**20.43**	1.759 inches	**20.44**	1.057 inches
20.45	0.562 inch	**20.46**	2.06 inches	**20.47**	2.4484 mm
20.48	6.203º	**20.49**	1.59 inches	**20.50**	15.03 mm

20.51

	Quadrant			
	I	II	III	IV
$\sin\theta$	+	+	–	–
$\cos\theta$	+	–	–	+
$\tan\theta$	+	–	+	–
$\csc\theta$	+	+	–	–
$\sec\theta$	+	–	–	+
$\cot\theta$	+	–	+	–

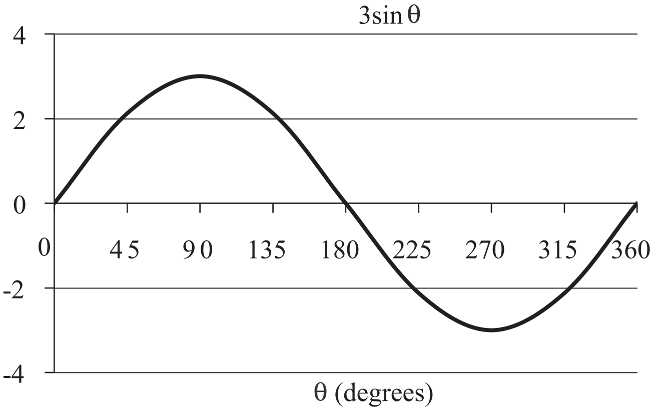
20.52

a) 0.08716 **b)** –7.1154 **c)** –1.66428 **d)** –0.90040
e) –0.89879 **f)** –1.66164 **g)** 0.57358 **h)** –3.07768
i) 1.01247 **j)** –0.98163 **k)** –1.55572 **l)** –0.08716

20.53

a)

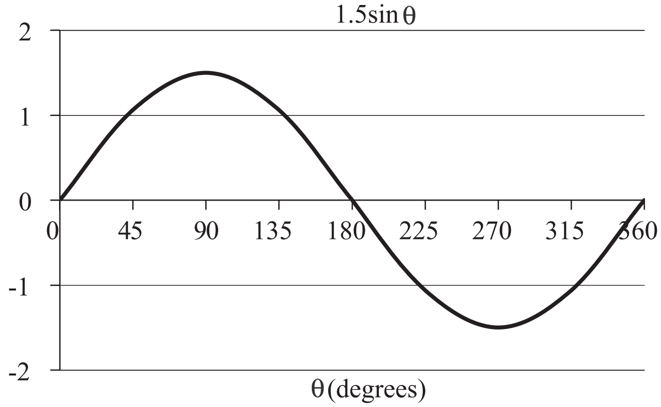
θ	sinθ	3sinθ
0	0	0
45	0.7071	2.1213
90	1	3
135	0.7071	2.1213
180	0	0
225	-0.7071	-2.1213
270	-1	-3
315	-0.7071	-2.1213
360	0	0



20.53

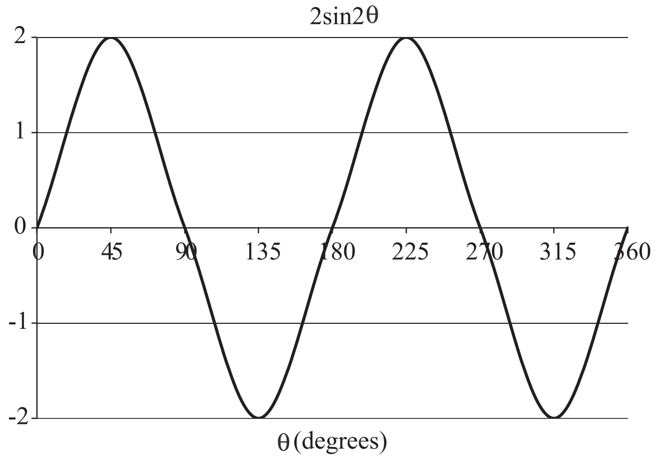
b)

θ	sinθ	1.5sinθ
0	0	0
45	0.7071	1.0607
90	1	1.5
135	0.7071	1.0607
180	0	0
225	-0.7071	-1.0607
270	-1	-1.5
315	-0.7071	-1.0607
360	0	0



c)

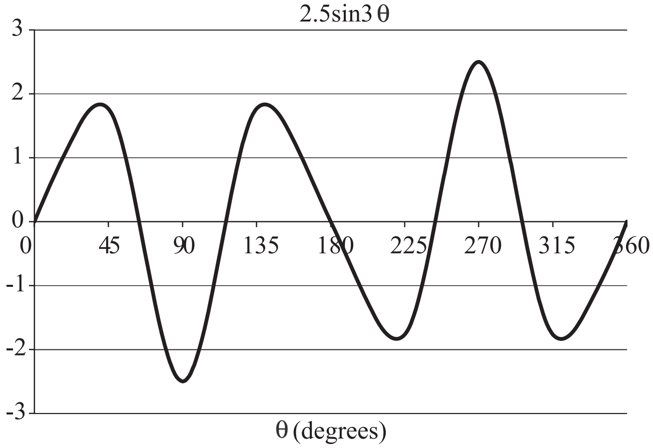
θ	sin2θ	2sin2θ
0	0	0
45	1	2
90	0	0
135	-1	-2
180	0	0
225	1	2
270	0	0
315	-1	-2
360	0	0



20.53

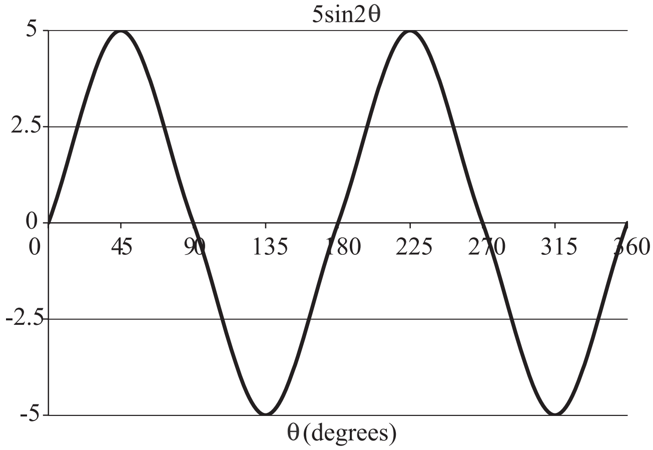
d)

θ	sin3θ	2.5sin3θ
0	0	0
45	0.7071	1.7678
90	-1	-2.5
135	0.7071	1.7678
180	0	0
225	-0.7071	-1.7678
270	1	2.5
315	-0.7071	-1.7678
360	0	0



e)

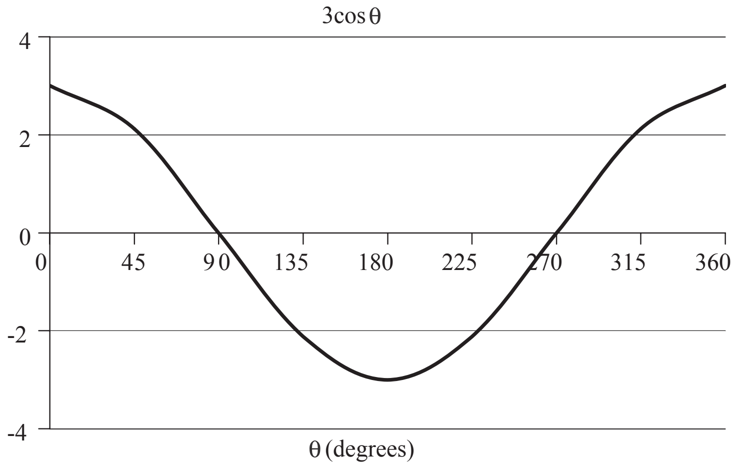
θ	sin2θ	5sin2θ
0	0	0
45	1	5
90	0	0
135	-1	-5
180	0	0
225	1	5
270	0	0
315	-1	-5
360	0	0



20.54

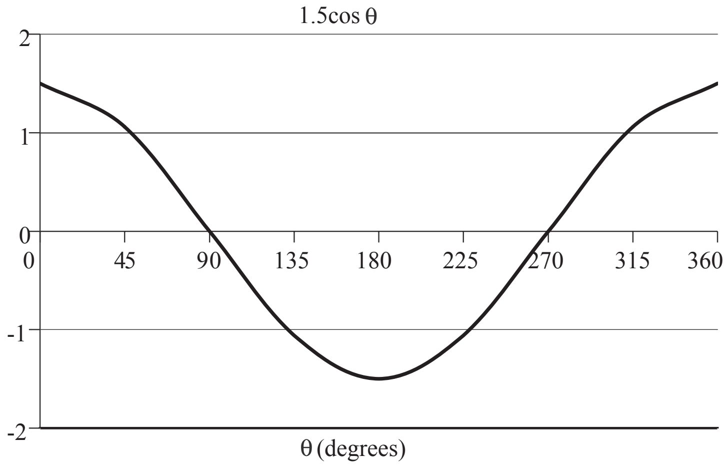
a)

radians	θ	cosθ	3cosθ
0	0	1	3
0.785398	45	0.7071	2.1213
1.570796	90	0	0
2.356194	135	-0.7071	-2.1213
3.141593	180	-1	-3
3.926991	225	-0.7071	-2.1213
4.712389	270	0	0
5.497787	315	0.7071	2.1213
6.283185	360	1	3



b)

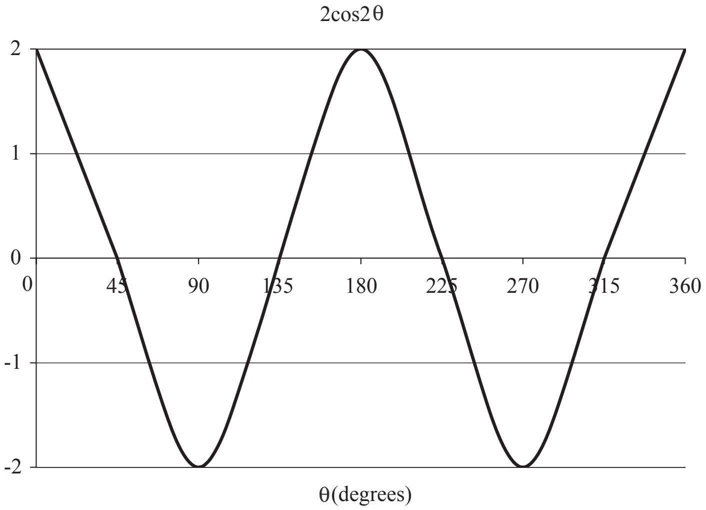
θ	cosθ	1.5cosθ
0	1	1.5
45	0.7071	1.0607
90	0	0
135	-0.7071	-1.0607
180	-1	-1.5
225	-0.7071	-1.0607
270	0	0
315	0.7071	1.0607
360	1	1.5



20.54

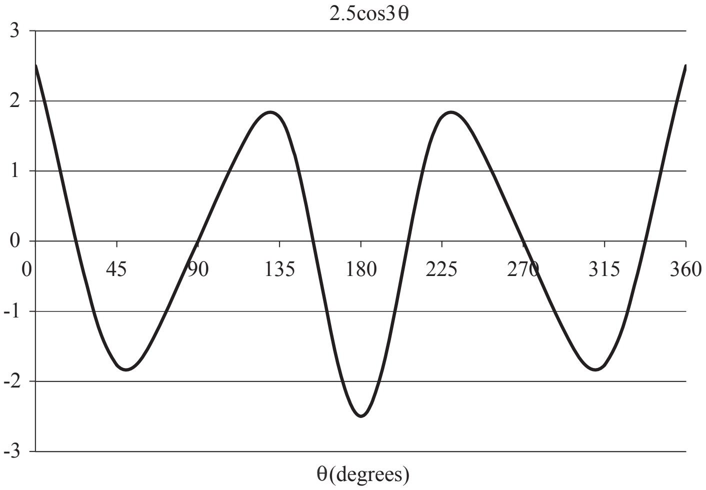
c)

θ	cos2θ	2cos2θ
0	1	2
45	0	0
90	-1	-2
135	0	0
180	1	2
225	0	0
270	-1	-2
315	0	0
360	1	2



d)

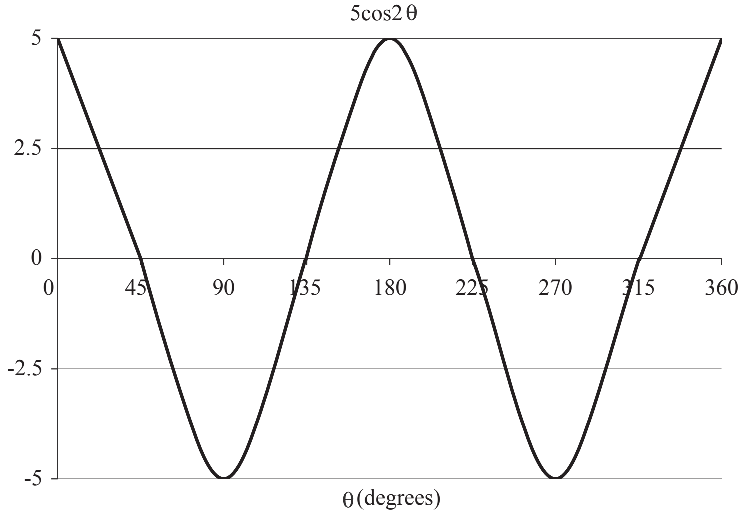
	cos3θ	2.5cos3θ
0	1	2.5
45	-0.7071	-1.7678
90	0	0
135	0.7071	1.7678
180	-1	-2.5
225	0.7071	1.7678
270	0	0
315	-0.7071	-1.7678
360	1	2.5



20.54

e)

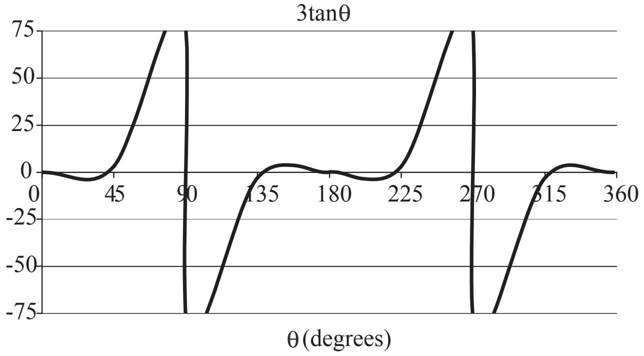
θ	cos2 θ	5cos2θ
0	1	5
45	0	0
90	-1	-5
135	0	0
180	1	5
225	0	0
270	-1	-5
315	0	0
360	1	5



20.55

a)

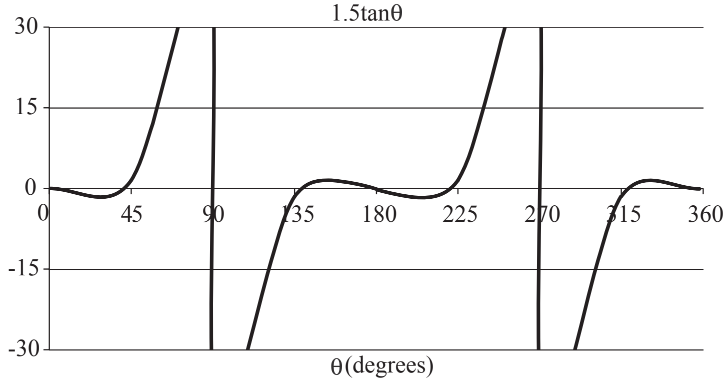
θ	tan	3tan
0	0	0
45	1	3
90	∞	∞
135	-1	-3
180	0	0
225	1	3
270	∞	∞
315	-1	-3
360	0	0



20.55

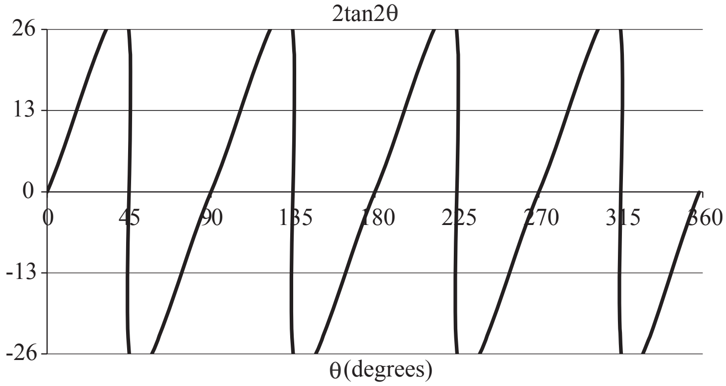
b)

θ	tanθ	1.5tanθ
0	0	0
45	1	1.5
90	∞	∞
135	-1	-1.5
180	0	0
225	1	1.5
270	∞	∞
315	-1	-1.5
360	0	0



c)

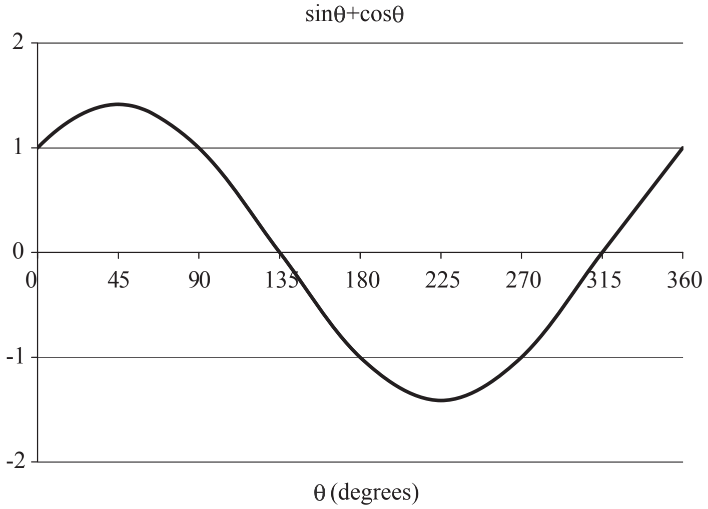
θ	tan2θ	2tan2θ
0	0	0
45	∞	∞
90	0	0
135	∞	∞
180	0	0
225	∞	∞
270	0	0
315	∞	∞
360	0	0



20.56

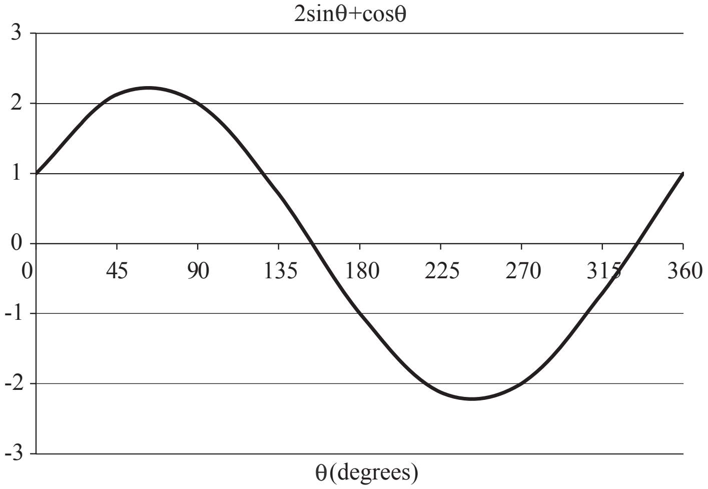
a)

θ	sinθ	cos θ	sinθ+cosθ
0	0	1	1
45	0.7071	0.7071	1.4142
90	1	6.12574E-17	1
135	0.7071	-0.7071	0
180	0	-1	-1
225	-0.7071	-0.7071	-1.4142
270	-1	0	-1
315	-0.7071	0.7071	0
360	0	1	1



b)

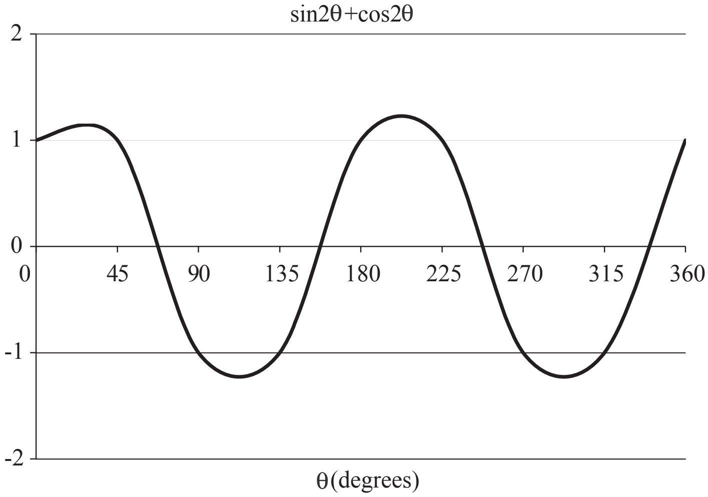
	sin θ	cosθ	2sinθ+cosθ
0	0	1	1
45	0.7071	0.7071	2.1213
90	1	0	2
135	0.7071	-0.7071	0.7071
180	0	-1	-1
225	-0.7071	-0.7071	-2.1213
270	-1	0	-2
315	-0.7071	0.7071	-0.7071
360	0	1	1



20.56

c)

θ	sin2θ	cos2θ	sin2θ+cos2θ
0	0	1	1
45	1	0	1
90	0	-1	-1
135	-1	0	-1
180	0	1	1
225	1	0	1
270	0	-1	-1
315	-1	0	-1
360	0	1	1



B.21 Chapter 21 Exercises

21.1 **a)** $\angle A = 22.47°$ **b)** $AC = 3.74$ cm

21.2 **a)** $g = 41.95$ mm **b)** $e = 42.31$ mm

21.3 **a)** Altitude = 2.01 inches **b)** AB = 5.95 inches

21.4 Radius = 1.15 inches

21.5 Radius = 1.41 units

21.6 **a)** AC = 4.28 meters **b)** BC = 2.36 meters

21.7 $X = 1.21$ inches

21.8 $\angle C = 100.98°$
$a = 89.26$ mm

21.9 $\angle B = 66.00°$
$\angle C = 70.75°$
$c = 21.70$ inches

21.10 1.18607 or 1.19 inches

21.11 6.98 inches

21.12 4.51 inches

21.13 109.65°

21.14 2.79 cm

21.15 $\angle B = 116.12°$
BC = 5.2 inches
AC = 7.497

21.16 **a)** $\angle A = 25.635°$ **b)** $\angle B = 90.97°$

21.17 **a)** $\angle A = 21.25°$ **b)** $\angle B = 135.40°$

21.18 0.692 inch **21.19** 4.942 inches
21.20 BD = 3.15 meters **21.21** Altitude = 33.52 mm
21.22 h = 8.945 mm **21.23** Diameter = 0.9769 inch
21.24 $P = 1.754$ inches **21.25** $A = 3.49$ inches
$B = 1.559$ inches

21.26 Included angle = 16.60º, Taper per foot = 3.52 inches

21.27 $\alpha = 24.15°$ **21.28** 0.4356 inch
21.29 $X = 3.305$ inches **21.30** $X = 1.596$ inches
21.31 $C = 2.81$ inches **21.32** A = 0.367 inch
21.33 D = 3.21 inches **21.34** X = 0.404 inch
α = 26.03°

21.35 $X = 3.280$ millimeters **21.36** $X = 3.66$ inches
$Y = 4.280$ inches

21.37 $X = 5.325$ inches

B.22 Chapter 22 Exercises

22.1 **a)** 23.499° **b)** 11.5000°

22.2 24.9249°

22.3 **a)** $H = 3.1895$ inches **b)** $H = 6.3791$ inches

22.4 32.6319° **22.5** 20.7591° **22.6** 31.4752°
22.7 2.6532 inches **22.8** 6.0425 inches

22.9 **a)** 21.000° **b)** 45.7860°

22.10 4.688 inches **22.11** H = 527.401 **22.12** M = 41.329 mm
22.13 θ = 8.584° **22.14** 147.0414 mm

22.15

	# Holes	**Diameter Hole-Circle**	**Nearest 1/100**	**Nearest 0.0001**
a)	5	8 inches	4.70 inches	4.7023
b)	9	6 inches	2.05 inches	2.0521
c)	14	12 inches	2.67 inches	2.6703
d)	17	10 inches	1.84 inches	1.8375
e)	22	20 inches	2.85 inches	2.8463
f)	8	7 inches	2.68 inches	2.6787
g)	4	9 inches	6.36 inches	6.3640

22.15

	# Holes	Diameter Hole-Circle	Nearest 1/100	Nearest 0.0001
h)	7	7 inches	3.04 inches	3.0373
i)	3	3 inches	2.60 inches	2.5981
j)	4	6 inches	3.53 inches	3.5267
k)	8	10 inches	3.83 inches	3.8268
l)	7	9 inches	3.91 inches	3.9050
m)	11	10 inches	2.82 inches	2.8173
n)	22	22 inches	3.13 inches	3.1309
o)	14	9 inches	2.0 inches	2.0027
p)	9	6 inches	2.05 inches	2.0521
q)	10	10 inches	3.09 inches	3.0902
r)	18	12 inches	2.08 inches	2.0838

22.16

a)

$A=(4,\ 0)$

$B=(7.804,\ 2.764)$

$C=(6.351,7.236)$

$D=(1.649,\ 7.236)$

$E=(0.196,\ 2.764)$

b)

$A=(6,0)$

$B=(9.527,1.146)$

$C=(11.706,4.146)$

$D=(11.706,\ 7.854)$

$E=(9.527,\ 10.854)$

$F=(6,\ 12)$

$G=(2.473,\ 10.854)$

$H=(0.294,\ 7.854)$

$I=(0.294,\ 4.146)$

$J=(2.473,\ 1.146)$

c)

$A=(4,\ 0)$

$B=(7.464,\ 2.000)$

$C=(7.464,\ 6.000)$

$D=(4.000,\ 8.000)$

$E=(0.536,\ 6.000)$

$F=(0.536,\ 2.000)$

22.16

d)

$A = (5,\ 0)$
$B = (8.9092,\ 1.8826)$
$C = (9.8746,\ 6.1127)$
$D = (7.1693,\ 9.504)$
$E = (2.8307,\ 9.504)$
$F = (0.1254,\ 6.1127)$
$G = (1.0908,\ 1.8826)$

e)

$A = (6,\ 0)$
$B = (9.000,\ 0.804)$
$C = (11.196,\ 3.000)$
$D = (12,\ 6)$
$E = (11.196,\ 9.000)$
$F = (9.000,\ 11.196)$
$G = (6,\ 12)$
$H = (3.000,\ 11.196)$
$I = (0.804,\ 9.000)$
$J = (0,\ 6)$
$K = (0.804,\ 3.000)$
$L = (3.000,\ 0.804)$

22.17

a)

$A = (5,\ 10)$
$B = (8.909,\ 8.117)$
$C = (9.875,\ 3.887)$
$D = (7.169,\ 0.496)$
$E = (2.831,\ 0.496)$
$F = (0.125,\ 3.887)$
$G = (1.091,\ 8.117)$

b)

$A = (2.5,\ 5.0)$
$B = (4.665,\ 1.25)$
$C = (0.335,\ 1.25)$

c)

$A = (4,\ 8)$
$B = (7.464,\ 6.000)$
$C = (7.464,\ 2.000)$
$D = (4.000,\ 0.000)$
$E = (0.536,\ 2.000)$
$F = (0.536,\ 6.000)$

22.17

d)	**e)**
$A=(6,\ 12)$	$A=(3,\ 6)$
$B=(9.000,\ 11.196)$	$B=(4.763,\ 5.427)$
$C=(11.196,\ 9.000)$	$C=(5.853,\ 3.927)$
$D=(12,\ 6)$	$D=(5.853,\ 2.073)$
$E=(11.196,\ 3.000)$	$E=(4.763,\ 0.573)$
$F=(9.000,\ 804)$	$F=(3,\ 0)$
$G=(6,\ 0)$	$G=(1.237,\ 0.573)$
$H=(3.000,\ 0.804)$	$H=(0.147,\ 2.073)$
$I=(0.804,\ 3.000)$	$I=(0.147,\ 3.927)$
$J=(0,\ 6)$	$J=(1.237,\ 5.427)$
$K=(0.804,\ 9.000)$	
$L=(3.000,\ 11.196)$	

22.18

a) $X=6.187$ and $Y=6.187$ **b)** $X=12.963$ and $Y=2.053$

c) $X=93.1604$ and $Y=218.412$

22.19

a) 2.437 inches **b)** 4.5799 inches **c)** 6.170 inches

22.20 $AB=45.593$ mm, $AC=89.363$ mm

22.21

a) $AB=3.782$ inches **b)** $AC=6.120$ inches

22.22 $M=3.944$ mm

22.23 $D=1.1856$ inches

22.24 $M=3.012$ inches

22.25 $A=0.221$ inch

22.26 $X=4.049$ inches

$Y=1.024$ inches

22.27 $X=3.366$ inches

$Y=0.6830$ inch

22.28 $X=3.7075$ inches

$Y=0.8538$ inch

22.29 $M=66.594$ mm

22.30 $\beta=53.278°$

22.31 $M=29.50$ mm

22.32 $M=5.853$ mm

22.33 $X=4.395$ inches

$Y=3.433$ inches

22.34 $D=2.474$ inches

22.35 $A=3.350$ inches

22.36 $X=3.036$ inches

22.37 $X = 0.218$ inch
$Y = 0.19$ inch

22.38 $X = 1.973$ inches

22.39 $X = 3.577$ inches
$Y = 3.193$ inches
$Z = 0.541$ inch

22.40 $X = 0.912$ inch

22.41 $X = 3.4375$ inches

22.42 $M_1 = 2.186$ inches
$\mathrm{M}_2 = 2.186$ inches

22.43 $M_1 = M_2 = 51.26$ mm

22.44 $\phi \max = 45°$

22.45 $E = 83.44\%$

22.46 $M = 0.5938$ inch

22.47 0.1083 inch

22.48 $W = 0.0384$ inch

22.49 $M = 1.8011$ inches

INDEX

A

AA theorem for similarity 345
Abscissa 169
Absolute error 74
Absolute pressure 137–138
Absolute temperature 133
Absolute value 20–22
Accuracy 75
Acre 120
Acute angles 282
Acute triangles
  oblique 365–366
Addition 10, 17–20, 22–24
  absolute value 21–22
  algebraic fractions 263–266
  common fractions 37–42
  decimal fractions 63–65
  measurement 73
  mixed numbers 43–47
Additive identity 6
Additive inverse 6, 20, 24, 157–159
Additive process 10
Adjacent angles 282
Air pressure 135–136
Algebra 1, 141, 155
Algebraic expressions 141–154
  expanding 243–248
Algebraic fractions 259–269
Algorithms 141
Altitude 343–344, 347
Amplitude 434
Angle bisector 341–343
Angles 281–284
  central 376
  complementary 326
  congruent 287
  inscribed 377
  measure 126–131
  supplementary 326–327
Angular measure 493
Antilog 111–112
Arabic numerals 3–5
Arcs 376, 381
Area 120–121
  polygons 301–312
  triangle 369–373
Arithmetic 1
  operations 10–15
ASA theorem for congruence 362
Associative property
  addition 6
  multiplication 6
Axioms 5, 321–322

B

Base 90–94, 99–100
Base 10 4
Binary operation 22, 25
Binomials 142, 245–248, 262
  factoring 251–255
Bisector 341–343, 385
Board feet 122–123
Board measure 122–126
Boyle's law 237–238
British thermal unit 134–135
Btu 134–135

C

Calorie 134
Cancellation 48–49, 55–56
Cartesian plane 169–172
Celsius 132
Central angle 376, 387–390
Charles' law 234
Chord 376, 381, 391
Circles 299–300, 375–400
  area 306
  circumscribed 342
  inscribed 342–343
Circular measure 493
Circumference 299–300, 376
Circumscribed circle 342
Closed intervals 165
Coefficient 142
Combined variation 239–241
Common denominator 37
Common factor 32
Common fractions 27–56
  converting to decimal 59–63
  converting to percent 83–86
  equivalent 58
  raising to higher terms 31–32
  reducing 32–36
Common logarithm 111
Common multiple 37
Commutative property
  addition 6
  multiplication 6, 141
Commutative property of multiplation 251
Complementary angles 283, 326
Complex fractions 55–56
Compound proportion 217–218
Compound ratio 217–218
Compound shapes
  area 306–308
  volume 317–318
Compressible fluids 136
Concave polygons 286
Concentric circles 378
Cones 296
  volume 316–317
Congruence 287–288, 325
Congruent angles 287, 326
Congruent circles 385
Congruent triangles 360–364
Conical tapers 226–229
Constant 141
Constant coefficient 253
Conversion factors 490–496
Converting
  common fractions 59–63
  decimal fractions 59–63
  decimal to common fractions 161–162
  improper fractions 29–30

  measurement units 117–118
  mixed numbers 29–30
  percent 83–90
  temperature 133–134
  to scientific notation 109–110
  weights and mass 131–132
Convex polygons 286
Coordinate distances
  trigonometry 470–476
Coordinate plane 169
Corresponding angles 284, 344–345
Corresponding sides 344–345
Cosecant 402–407
Cosine 402–407, 434
  law 447–450
Cotangent 402–407
  law 450–452
Coulomb's law 239
Cube 294
  surface area 313–314
Cubic measure 492–493
Cutting speed 198
Cylinders 296
  volume 316

D

Decimal 57–59
Decimal fractions 57–81
  addition 63–65
  converting to common fraction 59–63
  division 67–69
  equivalent 58
  multiplication 65–66
  subtraction 63–65
Decimal places 66
Decimal point 57
Decimal tolerances 76–81
Degree of a polynomial 142
Degrees 126
Degrees decimal 127
Denominator 27
  common 37
  rationalizing 151–152
Depth of cut 197–198
Diagonal 286, 299
Diagonals 290
Diameter 299, 376, 381
Diamond 290
Difference of squares 246, 256–257
Direct current electrical formulas 190–191
Direct variation 233–236
Discount 96–98
Distance formula 187–188
Distributive property 6
Division 24–25
  algebraic fractions 261–263
  common fractions 48–51
  decimal fractions 67–69
  measurement 74
  mixed numbers 52–54
  short and long 11–15
  with scientific notation 110
Dodecahedron 294

E

Electrical formulas 190–191, 204–205
Elements 6
Ellipse 379–380
Empty set 7
Energy 495
English system of measurement 115
Equality 322
Equations 143
  solving 155–163
Equiangular polygon 287, 290
Equilateral triangles 288, 342–343
Equilaterial polygon 287
Equivalent fractions 31, 89
Equivalent percents 89
Error
  measurement 74–75
Euclid's postulates 322–324
Expanded notation 4–5
Expanding
  algebraic expressions 243–248
  binomials 245–246
  polynomials 243
Exponents 4, 99–113
  operations 102–103
  operations on expressions 146–149
  rules 101–108
Expression 142
Expressions
  combining like terms 152–154
  evaluating 143–145
  exponents 146–149
  radicals 149–151
  rationalizing the denominator 151–152
Exterior angles 284
Exterior point
  circle 375

F

Factoring 273–274
  imaginary numbers 257–258
  polynomials 248–250
Factorization
  prime 33–34
Factor-label method 117–118, 217
Factors 9, 32
Fahrenheit 132
First degree polynomial 142
Flat tapers 226–229
Focus 379–380
FOIL method 245–246, 251–253, 256–257
Force 131
Formulas 144–145, 187–211
  trigonometry 485–488
Fraction bar 27
Fractions
  algebraic 259
  common 27–56
  decimal 57–81
  equivalent 31
  improper 28
  proper 28
  unity 28
Frustum
  volume 316–317

G

Gage blocks 79–80, 497–498
Gauge (see *Gage)*
Gauge pressure 137–138
GCF (see *Greatest common factor)*
Gear ratios 216
Geometry 1, 333–335
Gradians 126
Graphs
  inequalities 163–164
  linear equations 169–185
  slope-intercept form 180–185
Gravity 131
Greatest common factor 33–36, 38, 249
Greek letters 1–2, 282, 489
Grouping symbols 15

H

Heat 132–135, 496
Hectare 120
Hero's formula 369–373
Hexagons 286, 292, 345, 379
  area 305
Higher roots 105
Higher terms 31–32
Hole circle 468–469
Hooke's law 234
Horizontal lines 179–180
Horsepower 190, 193–197, 205
Hydraulic cylinders 202–204
Hydraulic formulas 200–202, 210–211
Hydraulic hoist 201–202
Hypotenuse 289, 343–344, 347

I

Icosahedron 294
Imaginary numbers 257–258
Improper fractions 28
  converting 29–30
Indexing 199
Indirect variation (see *Inverse variation)*
Inequalities
  graphing 164
  solving 155, 163–168
Inequality 143
Inscribed angle 377

Inscribed circle 342–343, 386
Integers 7–8
  operations 17
Interest
  simple 95–96
Interest rate 95
Interior angles 284, 287, 327–333
Interior point
  circle 375
International system of units (see *SI measurement)*
Intersection of sets
  7
Interval notation 163, 165–168
Inverse
  operations 20
Inverse logarithm 112
Inverse natural logarithm 112–113
Inverse operations 156
Inverse trigonometric functions 414–415
Inverse variation 233, 236–238
Irrational nubmers 7–8
Isosceles
  triangle 343
Isosceles triangles 288

J

Jig-boring 470–476
Joint variation 239–241
Joule 134, 190

K

Kelvin 133

L

Law of cosines 447–450
Law of sines 445–447
LCD (see *Least common denominator)*
Least common denominator 37–40, 43–45, 263
Least common multiple 263
Length 118–119
Like terms 152–154, 158–159
Limits
  dimensions 76–78
Line segments 281
  congruent 287
Linear equations
  graphing 169–185
  one variable 155–163
Linear measure 118–119, 490
Lines 279–281
  graphing points 172–174
  parallel 327
  perpendicular 327
  point-slope form 176–177
  slope 175–180
  slope-intercept form 177–178
List price 96–98
Literal equations 187–211
Log (see *Logarithms)*
Logarithms 111–113
Long division 11–13
Lowest terms 32–36, 48

M

Major axis 380
Mass 131–132, 494
Mean proportional 219, 347
Measurement 73–76, 115–140
  converting units 118–123
  systems 115–118
Metric system 116–118
Minor axis 381
Minuend 44
Mixed number percent 86–88
Mixed numbers 29
  addition 43–47
  converting 29–30
  division 52–54
  multiplication 52–54
  subtraction 43–47
Mixed operations 14–15
Mixture proportions 223–225
Monomials 142, 243–244
Motors 196–197, 206
Multiples 31
Multiplication 11, 24–25
  algebraic fractions 261–263
  common fractions 48–51
  decimal fractions 65–66
  measurement 74
  mixed numbers 52–54
  monomials 243–244
  polynomials 243–244, 266
  table 9
  with scientific notation 110
  zero property 6
Multiplicative identity 6, 31
Multiplicative inverse 6, 157

N

Natural logarithms 112
Natural numbers 8
Negative base 100
Negative error 74
Negative exponent 102
Negative numbers 17–20
Negative slope 176
Newton's law of gravitational force 239–240
Newton-meter 190
Null set 7
Number line 17–20
  7–8
Number system
  base 10 4
Numerals 3–5
Numerator 27

O

Oblique triangles 289, 344, 439–462
  acute 365–366
  obtuse 366–368
Obtuse angles 283, 289
Obtuse triangles 289
  oblique 366–368
Octagons 293, 345
Octahedron 294
Ohm's Law 144, 190–191
Open intervals 165
Operations
  inverse 20, 156
  order 14–15
  percent 83–98
  polynomials 243
  signed numbers 17–26
Order of operations 14–15
Ordered pair 169
Origin 169

P

Parallel lines 280, 327, 345
Parallelograms 290
  area 303
Partial root 149
Pascal's law 200–201
PEMDAS 15, 26, 143–144
Pentagons 286, 292
Percentage formula 188
Percents 83–98
  converting to common fraction 85–88
  converting to decimal fraction 85
  mixed numbers 86–88
  solving problems 90–94
Perimeter 297–301
  square 298
Period 434
Perpendicular lines 280, 327
Phase shift 434
Pi 1, 299–300
Place value 4, 69–73
Plane geometry 279–296
Planes 284
Plotting points 170–172
Points 169, 279
  graphing 172–174
  plotting 170–172
Point-slope form 176–177
Polygons 286–294, 329–333
  area 301–312
  perimeter 297–298
Polyhedrons 294–295
Polynomial 142
Polynomials 243
  expanding 247–248
  factoring 248–250
  multiplication 266

  operations 243
Positive base 100
Positive error 74
Positive numbers 17–20
Positive slope 176
Postulates 321–325
Pound force 135
Power 4, 193, 496
  of 10 99–101
  of a power 103
  power of 10 4
Power formula 190–191
Power of 10 61
Powers of x 146–148
Precision 75
Precision gage blocks 79–80
Pressure 135–139, 200, 495
Price 96–98
Prime factorization 33–36
Prime number 7, 33
Principal 95
Principle root 104
Prisms 295
  surface area 312–314
  volume 314–315
Product 9
Projection 348
Projection formula 365–368
Proper fractions 28
Proportion 213–241
  compound 217–218
  mixture 223–225
  simple 214–215
  tapers 225–233
Proportionality constant 233, 238
Pumps 196–197, 206
Pyramids 295
  volume 315
Pythagorean theorem 353–360

Q

Quadrants 170, 429–432
Quadratic equations 268–278
Quadratic formula 276–278
Quadrilaterals 286, 289
Quotient 12

R

Radians 126–131
Radical equations 162–163
Radicals 104
  operations on expressions 149–151
  simplifying 106–107, 150
Radicand 104
Radius 299, 376, 383
Raising
  common fractions 31–32
Rankine 133
Rate 90–95
Ratio 213–241
  compound 217–218
  gears 216
  simple 213
Rational expression 143
Rational expressions 259
Rational numbers 7–8, 161
Rays 281
Real numbers 5–9
  number line 7–8
  properties 5–6, 141, 321
Reciprocal 49, 56, 158–159
Rectangles 290
  area 302–303
Rectilinear sides 301
Reducing
  common fractions 32–36
  mixed numbers 36
Reducing mixed numbers 36
Reflexive property 6
Reflexivity 321
Regular polygon 287
Regular polyhedrons 294
Relative pressure 137–138
Relative temperature 132
Remainder 14, 29
Rhombi 290
Right angles 282, 289
Right triangle 347–348, 353–360, 401, 410–437, 439–443
Right triangles 289
Rise 175
Rod 297
Roman numerals 3
Roots 99–100, 271
  higher 105
  of negative numbers 105–106
  square 104–105
Rounding 69–73
Run 175

S

SAS theorem for congruence 361
SAS theorem for similarity 345
Scalene triangles 288, 343–344
Scientific notation 108–111
Secant 376, 402–407
Second degree polynomial 142–143
Sector 376
Segment
  circle 376
Semicircles 376
Set notation 165–168
Sets 6–9
  defined 6
  real number 6–7
Short division 11–12
SI measurement 116–118
SI units 489
Signed numbers
  operations 17–26
Significant digits 70–72
Similar triangles 291, 344–353
Simple interest 95–96
Simple proportion 214–215
Simplifying
  algebraic fractions 259–261
  radicals 106–107
Sine 402–407, 434
Sine bars 463–488
Sine plates 463–468
Sines
  law 445–447
Skew lines 280
Slope 175–180
Slope-intercept form 177–178
  graphing 180–185
Solids
  surface area 312–314
Solution set 163
Special exponents 101–102
Special products 256–258
Spheres 296
  volume 316
Spindle speed 198–199
Square measure 491
Square root 149
Square roots 104–105
Squares 291
  area 301–302
  perimeter 298–299
SSS theorem for congruence 362
Standard atmosphere 136
Statistics 1
Straigth angles 282
Strain 140, 234–235
Stress 234–235, 495
Subsets 6–9
  real number 7–9
Subtraction 10, 17–20, 22–24
  absolute value 21–22
  algebraic fractions 263
  common fractions 37–42
  decimal fractions 63–65
  measurement 73
  mixed numbers 43–47
Subtrahend 44
Sum of squares 257–258
Supplementary angles 283, 326–327
Surface area
  solids 312–314
Surface area formula 188–189
Symbols 1–5
Symmetric property 6
Symmetry 321
Systems of measurement 115–118

T

Tangent 376, 383–385, 402–407, 434
Tangent circles 378
Tapers 225–233
Temperature 132–135, 496
Term 142

Tessellations 292
Tetrahedron 294
Theorems 325–335, 361–362
  circle 381–400
Thread translation 191–192
Tolerances 76–81
Torque 196–197
Transitive property 6
Transitivity 321–322
Translation 191–192
Transversal 327
Transversal line 283–284
Trapezoids 291
  area 303–304
Triangles 286–289, 327–330, 341–373
  area 304–305
  congruent 304, 360–364
  equilateral 342–343
  isosceles 343
  oblique 344
  right 347–348
  scalene 343–344
  special 427–429
Triangulation 341
Trigonometry 1
  cofunction identities 407–408
  complementary identities 407
  functions 402–407
  graphing 433–437
  oblique angle 439–462
  reciprocal identities 408
  right triangle 401–437
  shop 463–488
  solving sides 410–427
Trinomials 142, 247–248
  factoring 251–255

U

Undefined slope 176, 179
Union of sets
  7
Unit circle 429–432
Unity 28
Universal set 7

V

Vaccums 138–139
Variable 141
Variables 155–163
  applying exponent rules 146–148
Variation 233–241
Velocity 195
Vertex 287
Vertical angles 283–284, 325
Vertical lines 179–180
Volume 121
  solids 314–318
Volume mixture 223–224

W

Water flow rate 202
Watt 190
Weight 131–132, 494
Weight mixture 224–225
Whole numbers 8
Worm gear ratios 192–193

X

X-axis 169, 179
X-intercept 179
X-values 173–174

Y

Y-axis 169, 180
Y-intercept 178
Y-values 173–174

Z

Zero degree polynomial 142–143
Zero property of multiplication 6
Zero slope 176